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A NOTE ON THE SQUARE SUBGROUPS OF DECOMPOSABLE TORSION-FREE ABELIAN GROUPS OF RANK THREE

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Abstract. A hypothesis stated in [16] is confirmed for the case of associative rings. The answers to some questions posed in the mentioned paper are also given. The square subgroup of a completely decomposable torsion-free abelian group is described (in both cases of associative and general rings). It is shown that for any such a group A, the quotient group modulo the square subgroup of A is a nil-group. Some results listed in [16] are generalized and corrected. Moreover, it is proved that for a given abelian group A, the square subgroup of A considered in the class of associative rings, is a characteristic subgroup of A.

1. Introduction

Probably the most natural issue concerning the abelian groups in the context of defining the ring structure on them, is the following question: given an abelian group A, does there exist a ring $(A, *)$ satisfying $A{*}A \neq \{0\}$? If the answer is negative, then A is called a nil-group. Nil-groups were studied for a long time by many authors (see, e.g., [13, 15]) and there are several generalizations. One of them is the concept of the square subgroup of an abelian group. Given an abelian group A, the square subgroup $\Box A$ of A can be understood as the subgroup of A generated by squares of all possible rings defined on A (see, [6]).

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That notion has been originally introduced by A.E. Stratton and M.C. Webb in [18] as a result of the reformulation of Feigelstock's problem posed in [12]. Feigelstock has asked whether the fact that squares of all possible rings defined on A are contained in some subgroup H of A implies that A/H is a nil group. A.E. Stratton and M.C. Webb have shown that the answer to Feigelstock's question is positive if A is a torsion group or if H is a direct summand of A. Moreover, the referee of [18] have pointed that the answer is negative in the general case. A counterexample is surprisingly simple. Namely, it is sufficient to consider a torsion-free nil group which is not divisible (for details, we refer the reader to [18, Example]). However, the replacement of the subgroup H by $\Box A$ in Feigelstock's problem made it much harder. The problem was unsolved for 35 years although it appeared in papers related to this issue (see, [1,3–5]). Negative answers for mixed and torsion-free abelian groups were given by A. Najafizadeh, R.R. Andruszkiewicz and M. Woronowicz in 2015 and 2016 (see, [6, 7, 9, 17]). Previously, the square subgroup of a torsion-free abelian group was investigated only in some special cases. Namely, A.M. Aghdam and A. Najafizadeh have proved that for every indecomposable torsion-free abelian group A of rank two which is not homogeneous, the quotient group $A/\Box A$ is a nil group (see, [4,5]). In particular, they have described the square subgroups of these groups.

Recently, it turned out that knowledge related to the square subgroup of an abelian group is useful for describing additive groups of commutative rings (see, [8, Lemma 2.15 and Theorem 2.16]). Some new examples of non-splitting groups and mixed SI-groups are also closely related to the topic (see, [9, Lemma 4.2, Theorem 4.5] and compare with [9, Theorem 4.8]). For all these reasons, it seems interesting to continue the study on the square subgroup of an abelian group.

The direct inspiration to write this note was the paper entitled 'On the square subgroups of decomposable torsion-free abelian groups of rank three' written by F. Hasani, F. Karimi, A. Najafizadeh and M.Y. Sadeghi (see, [16]). The authors have studied there the square subgroup of a torsion-free abelian group $A = A_1 \oplus A_2$ of rank three, assuming that A_i is a group of rank i, A_2 is not a nil group and either $t(A_1) \in T(A_2)$ or $t(A_1)$ is incomparable to any type belonging to $T(A_2)$. They have achieved interesting results in this field which, in some cases, contribute to this area of research. However, the authors left open descriptions of the square subgroup $\Box A$ of A in the following cases:

(Q_1) $t(A_1) = t_0$ and $T(A_2) = \{t_0, t_1, t_2\}$, where $t_0 < t_1$, $t_0 < t_2$, $t_1^2 = t_1$ and $t_2^2 \neq t_2$;

(Q_2) $t(A_1) = t_0$ and $T(A_2) = \{t_0, t_1, t_2\}$, where $t_0 < t_1$, $t_0 < t_2$, $t_1 \neq t_2$, $t_1^2 = t_1$ and $t_2^2 = t_2$;

(Q_3) $T(A_2) = \{t_1, t_2\}$ and either $t(A_1) = t_1$ or $t(A_1) = t_2$.

(compare with comments listed after the proof of [16, Theorem 3.9]). We investigate all these situations especially for the case of associative rings, where the square subgroup of an abelian group A is denoted by $\square_a A$. We present complete descriptions of $\square_a A$ and $\square_{(a)} A$ for the cases (Q_1) and (Q_2), respectively, and a partial description of $\square_{(a)} A$ for the case (Q_3) (see, Theorems 4.3, 4.5–4.7). In particular, for the associative case, we confirm a hypothesis stated in [16] which is connected with (Q_1). Furthermore, we give a description of the square subgroup of a completely decomposable torsion-free abelian group (in both cases of associative and general rings). As a consequence, we present generalizations of some results from [16] (see, e.g., Theorem 4.9) and we show that for every completely decomposable torsion-free abelian group A, the quotient group modulo $A/\square A$ is a nil-group. Moreover, we make a little correction of [16, Theorem 3.7] (see, Remark 4.10) and we prove that $\square_a A$ is a characteristic subgroup of A.

2. Notation and preliminaries

Symbols $\mathcal{D}(A)$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{P}$, $\mathbb{N}$ and $\mathbb{N}_0$ stand for the divisible hull of an abelian group A, the field of rationals, the ring of integers, the sets of all prime numbers, positive integers and nonnegative integers, respectively. If $\{A_i : i \in I\}$, where $I \neq \emptyset$, is a family of abelian groups and $i \in I$, then $\overline{A_i}$ denotes the subgroup of $\bigoplus_{i\in I} A_i$ such that the support of an arbitrary element of $\overline{A_i}$ is contained in the set $\{i\}$. If $x \in \bigoplus_{i\in I} A_i$, then the support of x is denoted by $\operatorname{supp}(x)$. The additive group of a ring R is denoted by R^+. If $X \subseteq R$, then the symbol $[X]$ stands for the subring of R generated by X. The notation $I \lhd R$ means that I is a two-sided ideal in R. The sign function is denoted by sgn. All other designations are consistent with generally accepted standards (see, e.g., [14, 15]).

Complete preliminary knowledge of all main issues related to this paper is contained in [16] and in the second section of [10]. However, for the transparency of the paper, we remind the reader of the most important facts. If A is a torsion-free abelian group of rank three with a maximal independent system $\{x, y, z\}$, then the symbols U_0, V_0 and W_0 stand for the subgroups of $\mathbb{Q}^+$ such that $\langle x\rangle_* = U_0 x$, $\langle y\rangle_* = V_0 y$ and $\langle z\rangle_* = W_0 z$ (see, [16, Preliminaries]). The following formula greatly simplifies the considerations related to the square subgroups of an abelian group: that is,

$$\square_{(a)} A = \sum_{* \in \mathrm{Mult}_{(a)} A} A * A \tag{2.1}$$

where $\mathrm{Mult}_{(a)} A$ means the set of all (associative) ring multiplications on the group A (compare with [6, Remarks 1.2 and 1.10]). It follows from [6, Corollary 2.6] that if there exists an abelian group A satisfying $\square_a A \subsetneq \square A$, then A is reduced and non-torsion. For more detailed information on torsion-free groups of rank two and the square subgroup of various abelian groups we refer the reader to [1–3, 6, 7, 17]. The most basic properties of types which will be used often throughout the paper are listed in the following lemma.

LEMMA 2.1. *Let A, B and C be torsion-free abelian groups.*

(i) *If a and b are dependent elements of A, then $t(a) = t(b)$.*
(ii) *$t(a+b) \geq t(a) \wedge t(b)$ for all $a, b \in A$.*
(iii) *If $A = B \oplus C$, $b \in B$ and $c \in C$, then $t(b+c) = t(b) \wedge t(c)$.*
(iv) *If $f \in \mathrm{Hom}(A, B)$, then $t\big(f(a)\big) \geq t(a)$ for every $a \in A$.*
(v) *$t(a) \cdot t(b) \geq t(a)$ for all $a, b \in A$.*
(vi) *If $R = (A, \star)$ is a ring, then $t(a \star b) \geq t(a) \cdot t(b)$ for all $a, b \in A$.*

The proofs of (i)–(v) can be found in [15, p. 109 and 110]. Property (vi) is listed in [2, Lemma 1] (if $a \star b = 0$, then the assertion is obvious).

It is easily seen that [16, Proposition 3.2] remains true in the associative case. Thus, we have the following:

PROPOSITION 2.2. *Let $\{A_i \colon i \in I\}$ be a family of fully invariant subgroups of an abelian group A such that $A = \bigoplus_{i \in I} A_i$. Then $\square_{(a)} A = \bigoplus_{i \in I} \square_{(a)} A_i$.*

Proposition 2.2 together with [2, Theorem 4] implies that Theorems 3.4–3.9 of [16] remain true in the associative case. Generalizations of Theorems 3.5–3.7 of [16] are presented as Theorem 4.9. In particular, some inconsistencies of [16, Theorem 3.7] are noted and corrected there (see, Remark 4.10). Slightly generalized results stated in [16, Theorems 3.4, 3.8 and 3.9] are presented below.

THEOREM 2.3. *Let A_2 be a torsion-free abelian group of rank two with $\square A_2 \neq \{0\}$ and $T(A_2) = \{t_0, t_1, t_2\}$ where $t_0 < t_1$, $t_0 < t_2$, $t_1^2 = t_1$ and $t_2^2 \neq t_2$. Let A_1 be a torsion-free abelian group of rank one and let $A = A_1 \oplus A_2$.*

(i) *If the type $t(A_1)$ of A_1 is non-idempotent and incomparable with all members of $T(A_2)$, then $\square_{(a)} A = \{0\} \oplus \langle y \rangle_*$ for some $y \in A_2$ such that $t(y) = t_1$.*
(ii) *If $t(A_1) = t_1$, then $\square_{(a)} A = A_1 \oplus \langle y \rangle_*$ for some $y \in A_2$ such that $t(y) = t_1$.*
(iii) *If $t(A_1) = t_2$, then $\square_{(a)} A = \{0\} \oplus \langle y \rangle_*$ for some $y \in A_2$ such that $t(y) = t_1$.*

3. On the square subgroup of a completely decomposable torsion-free abelian group

It turns out that an important role in research related to the questions mentioned in the Introduction, is played by knowledge concerning the square subgroup of a completely decomposable torsion-free abelian group. It is studied in this section. Since every abelian torsion-free group of rank one can be embedded into the group of rationals, we restrict our main considerations to the direct sums of these subgroups. It is easily seen that if G is a nontrivial subgroup of the group $\mathbb{Q}^+$, $m = \min G \cap \mathbb{N}$ and $A = \frac{1}{m} \cdot G$, then A is a subgroup of $\mathbb{Q}^+$ satisfying $1 \in A$ and $A \cong G$. Therefore, it is sufficient to consider only direct sums of subgroups of $\mathbb{Q}^+$ containing the number one.

The following concept plays a key role in the elementary classification of nil subgroups of the group $\mathbb{Q}^+$ (see, [19]) and will be very useful in our next considerations.

Definition 3.1. Let A be a subgroup of $\mathbb{Q}^+$ such that $1 \in A$. Then we define $\Omega_A = \{p \in \mathbb{P} \colon p^{-1} \in A\}$, $\Omega_A^\infty = \{p \in \Omega_A \colon A = pA\}$ and $\Omega_A^0 = \Omega_A \setminus \Omega_A^\infty$.

Remark 3.2. It follows from [19, Remark 4.1] that if A is a subgroup of $\mathbb{Q}^+$ containing $\mathbb{Z}$ as a proper subset, then $A = \left\langle \frac{1}{p^{\alpha_p}} : p \in \Omega_A^0 \right\rangle + \left[\frac{1}{q} : q \in \Omega_A^\infty\right]^+$ where $\alpha_p = \max\{\alpha \in \mathbb{N} \colon p^{-\alpha} \in A\}$. If either $\Omega_A^0 = \emptyset$ or $\Omega_A^\infty = \emptyset$, then the group associated with the empty set is trivial. Thus, it is easily seen that if X and Y are subgroups of $\mathbb{Q}^+$ such that $1 \in X \cap Y$, then $\langle xy : x \in X, y \in Y\rangle = \{xy \colon x \in X, y \in Y\}$. In other words, the subgroup XY of $\mathbb{Q}^+$ understood as the group generated by all products xy where $x \in X$ and $y \in Y$, is the set of all these products.

Lemma 3.3. *Let n be a positive integer and let A, B, C be subgroups of $\mathbb{Q}^+$ satisfying $1 \in A \cap B \cap C$ and $\Box B = \{0\}$. If $nAC \subseteq B$, then $\{0\} \oplus \{0\} \oplus B \subseteq \Box_a(A \oplus C \oplus B)$. If $nA^2 \subseteq B$ or $nAB \subseteq B$, then $\{0\} \oplus B \subseteq \Box_a(A \oplus B)$.*

Proof. Since $nAC \subseteq B$, we get $\Omega_{AC}^\infty \subseteq \Omega_B^\infty$. Consequently, $P_0 = \Omega_{AC} \cap \Omega_B^0$ and $P_1 = \Omega_{AC} \setminus \Omega_B$ are subsets of Ω_{AC}^0 satisfying $P_0 \cap P_1 = \emptyset$ and $|P_1| < \infty$. If $\Omega_{AC}^0 \neq \emptyset$, then for each $p \in \Omega_{AC}^0$ there exists the maximal $M_p \in \mathbb{N}$ such that $p^{-M_p} \in AC$. Let $P_2 = \{p \in P_0 \colon p^{-M_p} \notin B\}$. Then $|P_2| < \infty$ because $nAC \subseteq B$. For $i = 1, 2$ define:

$$\lambda_i = \begin{cases} \prod\limits_{p \in P_i} p^{M_p} & \text{if } P_i \neq \emptyset, \\ 1 & \text{if } P_i = \emptyset. \end{cases}$$

Let $\lambda = \lambda_1\lambda_2$. For the case $\Omega^0_{AC} = \emptyset$, we put $\lambda = 1$. Definition of λ implies that $\lambda AC \subseteq B$ and, consequently, the multiplication $(a_1, c_1, b_1) * (a_2, c_2, b_2) = (0, 0, \lambda a_1 c_2)$ for all $a_1, a_2 \in A$, $c_1, c_2 \in C$ and $b_1, b_2 \in B$, induces a nontrivial ring structure on $G = A \oplus C \oplus B$. It is easy to check that the ring $R_1 = (G, *)$ is associative. Take any $p \in \mathbb{P}$ and $\alpha \in \mathbb{N}$ such that $p^{-\alpha} \in B$. Remark 3.2 together with (2.1) implies that it is sufficient to show that there exists an associative ring R on G for which $(0, 0, p^{-\alpha}) \in R^2$. We need only consider three cases:

(i) $p \in P_0 \setminus P_2$. If $\alpha \leq M_p$, then $p^{-\alpha} \in AC \cap B$. Moreover, $\lambda^{-1} \in AC$, by [19, (ii) of Remark 4.1], and $p \nmid \lambda$. We apply [19, (ii) of Remark 4.1] again to obtain $\lambda^{-1}p^{-\alpha} \in AC$. Consequently, in view of Remark 3.2, we get $(0, 0, p^{-\alpha}) = (0, 0, \lambda\lambda^{-1}p^{-\alpha}) \in R_1^2$. Now suppose that $\alpha > M_p$ and define $\beta = \alpha - M_p$. Then $\beta > 0$ and it follows from [19, (i) and (ii) of Remark 4.1] that the multiplication $(a_1, c_1, b_1) \circledast (a_2, c_2, b_2) = \big(0, 0, p^{-\beta}\lambda a_1 c_2\big)$ provides a nontrivial ring structure on G. Let $R_2 = (G, \circledast)$. Then R_2 is an associative ring. Since $\lambda^{-1}p^{-M_p} \in AC$, we get $(0, 0, p^{-\alpha}) = \big(0, 0, p^{-(\beta + M_p)}\big) = \big(0, 0, p^{-\beta}\lambda\lambda^{-1}p^{-M_p}\big) \in R_2^2$.

(ii) $p \in P_2$. As $\lambda^{-1} \in AC$, the definition of λ implies that it is sufficient to consider the multiplication $(a_1, c_1, b_1) \star (a_2, c_2, b_2) = (0, 0, p^{-\alpha}\lambda a_1 c_2)$.

(iii) $p \in \Omega_B \setminus P_0$. Since $\lambda^{-1} \in AC$, it follows from [19, (i) and (ii) of Remark 4.1] that it is sufficient to consider the multiplication given by the same formula as in (ii).

For the case $nA^2 \subseteq B$ we define $P_0 = \Omega_A \cap \Omega^0_B$, $P_1 = \Omega_A \setminus \Omega_B$, $P_2 = \{p \in P_0 \colon p^{-2M_p} \notin B\}$ where $M_p = \max\{m \in \mathbb{N} \colon p^{-m} \in A\}$, and we proceed analogously to the previous part of that proof.

Now suppose that $nAB \subseteq B$. As $1 \in B$, we get $nA \subseteq B$. Thus $n^2A^2 = n\big((nA)A\big) \subseteq nBA \subseteq B$ and the assertion follows from the previous considerations. $\square$

The next proposition follows partially from the proof of Proposition 4.9 stated in [19]. However, the mentioned proof is quite technical, so we present the complete reasoning for the transparency of the paper.

Proposition 3.4. *Let $I \neq \emptyset$, let A_i be a subgroup of $\mathbb{Q}^+$ such that $1 \in A_i$ for each $i \in I$, and let $A = \bigoplus_{i \in I} A_i$. Then $\square A = \bigoplus_{i \in X} \overline{A_i}$ where X is the subset of I containing all elements i for which there exist $k, l \in I$ and $n \in \mathbb{N}$ satisfying $nA_kA_l \subseteq A_i$. Moreover, $\square_a A = \square A$.*

Proof. If A is a nil group, then [19, Proposition 4.9] implies that $X = \emptyset$ and the assertion follows. Now suppose that A is not a nil group. Take any $* \in \mathrm{Mult}(A)$ such that $A * A \neq \{0\}$. Then, there exist $a, c \in A$ and $k, l \in I$ satisfying $\pi_k(a) * \pi_l(c) \neq 0$ where π_j is the natural projection of the group

A onto its subgroup $\overline{A_j}$ for $j = k, l$. It follows from [15, Theorem 119.1] that there exists a ring $R = \big(\mathcal{D}(A), \circledast\big)$ such that $(A, *)$ is a subring of R. Take any $i \in \operatorname{supp}\big(\pi_k(a) * \pi_t(c)\big)$. For $j = k, l$, let φ_j be the natural injection of $\mathbb{Q}^+$ into $\mathcal{D}(A)$ such that $\varphi_j(\mathbb{Q}^+)$ is the j-th direct summand of $\mathcal{D}(A)$, let ψ_i be the natural projection of $\mathcal{D}(A)$ onto its i-th direct summand Q_i and let $\phi\colon Q_i \to \mathbb{Q}^+$ be the natural isomorphism. Define $\vartheta = \phi \circ \psi_i$ and $q_1 \odot q_2 = \vartheta\big(\varphi_k(q_1) \circledast \varphi_l(q_2)\big)$ for all $q_1, q_2 \in \mathbb{Q}^+$. Since ϑ, φ_k, φ_l are additive homomorphisms and $\circledast$ is a nonzero ring multiplication, we infer that the multiplication $\odot$ induces a nontrivial ring structure on $\mathbb{Q}^+$. Hence, by [19, Remark 4.2], there exists $q \in \mathbb{Q} \setminus \{0\}$ such that $q_1 \odot q_2 = q_1 \cdot q \cdot q_2$ for all $q_1, q_2 \in \mathbb{Q}^+$. Thus $q \cdot x \cdot y = x \odot y = \vartheta\big(\varphi_k(x) * \varphi_l(x)\big) \in A_i$ for all $x \in A_k$ and $y \in A_l$, whence $qA_kA_l \subseteq A_i$. Consequently, there exists $n \in \mathbb{N}$ such that $nA_kA_l \subseteq A_i$. Therefore $i \in X$. Thus $\pi_k(a) * \pi_l(c) \in \bigoplus_{i\in Y} \overline{A_i}$, where $Y = X \cap \operatorname{supp}\big(\pi_k(a) * \pi_l(c)\big)$. Consequently, $A * A \subseteq \bigoplus_{i\in X} \overline{A_i}$. Hence, by the arbitrary choice of $* \in \operatorname{Mult}(A)$ and by (2.1), we get $\square A \subseteq \bigoplus_{i\in X} \overline{A_i}$.

Take any $i \in X$. If A_i is not a nil group then $\overline{A_i} \subseteq \square_a A$, by [19, Theorem 4.8] and [6, Proposition 1.4 and Remark 1.10]. Next suppose that A_i is a nil group. Since $i \in X$, there exist $k, l \in I$ and $n \in \mathbb{N}$ such that $nA_kA_l \subseteq A_i$. It follows from [19, Theorem 4.8] that we can exclude the case $k = l = i$. In all other cases we get either $\{0\} \oplus A_i \subseteq \square_a(A_k \oplus A_i)$ or $\{0\} \oplus \{0\} \oplus A_i \subseteq \square_a(A_k \oplus A_l \oplus A_i)$, by Lemma 3.3. Thus, in view of [6, Proposition 1.4 and Remark 1.10], we get $\overline{A_i} \subseteq \square_a A$. Consequently, $\bigoplus_{i\in X} \overline{A_i} \subseteq \square_a A$. Finally, $\square A = \square_a A = \bigoplus_{i\in X} \overline{A_i}$, by [6, Corollary 1.9]. $\square$

Remark 3.5. Let A, B, C be subgroups of $\mathbb{Q}^+$ such that $1 \in A \cap B \cap C$. Notice that if there exists a positive integer n such that $nAC \subseteq B$, then $t(A) \cdot t(C) \leq t(B)$ (compare with the first section of [2]). Conversely, if G_1, G_2, G_3 are nontrivial torsion-free abelian groups of rank one, then there exist subgroups A_1, A_2, A_3 of $\mathbb{Q}^+$ such that $1 \in A_1 \cap A_2 \cap A_3$ and $A_i \cong G_i$ for each $i = 1, 2, 3$. Hence, the condition $t(G_1) \cdot t(G_2) \leq t(G_3)$ implies that $t(A_1) \cdot t(A_2) \leq t(A_3)$. Thus $sA_1A_2 \subseteq A_3$ for some positive integer s.

The following result is a direct consequence of Remark 3.5 and Proposition 3.4.

Theorem 3.6. *If $A = \bigoplus_{i\in I} A_i$ is a completely decomposable torsion-free abelian group, then $\square_{(a)} A = \bigoplus_{i\in X} \overline{A_i}$ where X is the subset of I containing all elements i for which there exist $k, l \in I$ such that $t(A_k) \cdot t(A_l) \leq t(A_i)$.*

The foregoing theorem implies at once the following:

Corollary 3.7. *If A is a completely decomposable torsion-free abelian group, then $A/\square_{(a)} A$ is a nil-group.*

4. Some answers to (Q_1)–(Q_3)

The following lemma will be useful in describing the square subgroup of an abelian group in the case (Q_1).

LEMMA 4.1. *Let $A = A_1 \oplus A_2$ where A_1 is a torsion-free abelian group of rank one with $t(A_1) = t_0$ satisfying $t_0^2 \neq t_0$ and A_2 is a torsion-free abelian group of rank two such that $T(A_2) = \{t_0, t_1, t_2\}$ where $t_0 < t_1$, $t_0 < t_2$, $t_1^2 = t_1$ and $t_2^2 \neq t_2$. Then $T(A) = T(A_2)$. Moreover, if $x \in A_1 \setminus \{0\}$ and $y, z \in A_2$ satisfy $t(y) = t_1$ and $t(z) = t_2$, then all rings on A satisfy the following multiplication table:*

$$z^2 = zy = yz = 0,\ y^2 = ry,\ xy = sy,\ yx = s'y,\ xz = gz,\ zx = g'z \tag{4.1}$$

and

$$x^2 = qy \quad \text{or} \quad x^2 = qz \tag{4.2}$$

where $r, s, s', g, g', q \in \mathbb{Q}$.

PROOF. The equality $T(A_2) = T(A)$ follows at once from (iii) of Lemma 2.1. By the same argument, we get $A(t_1), A(t_2) \subseteq A_2$. If $a \in A(t_1) \cap A(t_2)$, then $t(a) \geq t_1 > t_0$ and $t(a) \geq t_2 > t_0$. Moreover, $t_1 \neq t_2$, so [11, Proposition 1] implies that $a = 0$. Thus $A(t_1) + A(t_2) = A(t_1) \oplus A(t_2)$. Suppose, contrary to our claim, that $A(t_1)$ is a rank two group. Then $A(t_1)$ is an essential subgroup of A_2, so $A(t_1) \cap A(t_2) \neq \{0\}$, a contradiction. Therefore $A(t_1)$ is a rank one group. The analogical reasoning shows that $A(t_2)$ is a rank one group too. Consider an arbitrary ring $R = (A, \cdot)$. Then $A(t_1), A(t_2) \lhd R$, so $yz = zy = 0$, $y^2 = ry$, $xy = sy$, $yx = s'y$, $xz = gz$, $zx = g'z$ and $z^2 = r'z$ for some $r, r', s, s', g, g' \in \mathbb{Q}$. Since $t_2^2 \neq t_2$, it follows from (vi) of Lemma 2.1 that $r' = 0$ and, consequently, $z^2 = 0$. Next, $t_0^2 \neq t_0$, so if $x^2 \neq 0$, then either $t(x^2) = t_1$ or $t(x^2) = t_2$, by (vi) of Lemma 2.1. Thus, in view of $A(t_1) + A(t_2) = A(t_1) \oplus A(t_2)$, we obtain either $x^2 \in A(t_1)$ or $x^2 \in A(t_2)$. As both these groups are of rank one, we get $x^2 = qy$ or $x^2 = qz$ for some $q \in \mathbb{Q}$. If $x^2 = 0$, then it is sufficient to put $q = 0$. □

COROLLARY 4.2. *If we replace the condition $t_2^2 \neq t_2$ with the conjunction $t_2^2 = t_2$ and $t_2 \neq t_1$ in the foregoing lemma, then $T(A) = T(A_2)$ and every ring $R = (A, \cdot)$ satisfies $zy = yz = 0$ $y^2 = ry$, $z^2 = r'z$, $xy = sy$, $yx = s'y$, $xz = gz$ and $zx = g'z$ for some $r, r', s, s', g, g' \in \mathbb{Q}$. Moreover, there exists $q \in \mathbb{Q}$ for which $x^2 = qy$ or $x^2 = qz$ in the ring R.*

The answer to (Q_1) for the associative case is the following.

THEOREM 4.3. *Let $A = A_1 \oplus A_2$ where A_1 is a torsion-free abelian group of rank one with $t(A_1) = t_0$ satisfying $t_0^2 \neq t_0$ and A_2 is a torsion-free abelian group of rank two such that $T(A_2) = \{t_0, t_1, t_2\}$ where $t_0 < t_1$, $t_0 < t_2$, $t_1^2 = t_1$ and $t_2^2 \neq t_2$. If $\square A_2 \neq \{0\}$, then either $\square_a A = \langle y\rangle_*$ or $\square_a A = \langle y\rangle_* \oplus \langle z\rangle_*$ for some $y, z \in A_2$ of types t_1 and t_2, respectively.*

PROOF. Take any $x \in A_1 \setminus \{0\}$ and $y, z \in A_2$ such that $t(y) = t_1$ and $t(z) = t_2$. Since A_2 is a direct summand of A, [2, Theorem 4], [3, the proof of Lemma 3.1] and [6, Proposition 1.4 and Remark 1.10] imply that $\langle y\rangle_* \subseteq \square_a A$. Moreover, it follows form Lemma 4.1 that every ring R with $R^+ = A$ satisfies the conditions listed in (4.1) and (4.2). Consider an arbitrary associative ring $S = (A, \cdot)$. Then $0 = (x \cdot x) \cdot z = x \cdot (x \cdot z) = g^2 z$, so $g = 0$. Similarly, $g' = 0$ and, consequently, $x \cdot z = z \cdot x = 0$. Hence $A_1 \cdot A_2 = \langle x\rangle_* \cdot \langle y, z\rangle_* \subseteq \langle y\rangle_*$, $A_2 \cdot A_1 \subseteq \langle y\rangle_*$, $A_2 \cdot A_2 \subseteq \langle y\rangle_*$ and either $A_1 \cdot A_1 \subseteq \langle y\rangle_*$ or $A_1 \cdot A_1 \subseteq \langle z\rangle_*$. Thus $S^2 \subseteq \langle y\rangle_* \oplus \langle z\rangle_*$. In particular, $S^2 \subseteq \langle y\rangle_*$ if $x^2 = qy$. Hence, by (2.1), we get $\square_a A \subseteq \langle y\rangle_*$ if $x^2 = qy$ in every associative ring on A, or $\square_a A \subseteq \langle y\rangle_* \oplus \langle z\rangle_*$ if $x^2 = qz$ and $q \neq 0$ is possible in some associative ring on A. First suppose that $x^2 = qz$ and $q \neq 0$ (in the ring S). Then the multiplication

$$(ux, a) \circledcirc (u'x, a') = (0, uu'qz)$$

for all $u, u' \in U_0$ and $a, a' \in A_2$, provides a nontrivial associative ring structure on the group A (we use the notation related to the external direct sums for the transparency of the formula). In particular, $U_0^2 q \subseteq W_0$. Obviously, $q = \frac{k}{n}$ for some $k \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{N}$ satisfying $\mathrm{GCD}(k, n) = 1$. Define $\alpha = |k|$ and $m = \mathrm{sgn}(k) \cdot n$. Then $\alpha \in \mathbb{N}$ and $\alpha U_0^2 \subseteq W_0$. Moreover, the function $\star\colon A \times A \to A$ given by

$$(ux, a) \star (u'x, a') = (0, \alpha uu'z) \tag{4.3}$$

for all $u, u' \in U_0$ and $a, a' \in A_2$, is the m-th multiple of $\circledcirc$ in the group $\mathrm{Mult}(A)$. Therefore $(A, \star)$ is a nontrivial associative ring. Moreover, $W_0 z = \langle z\rangle_*$. Thus, in view of (4.3) and by the proof of Lemma 3.3, we get $\langle z\rangle_* \subseteq \square_a A$. Hence, by (2.1) and by the already proven inclusion $\langle y\rangle_* \subseteq \square_a A$, we obtain $\langle y\rangle_* \oplus \langle z\rangle_* \subseteq \square_a A$. Finally, $\square_a A = \langle y\rangle_* \oplus \langle z\rangle_*$.

Now suppose that $x^2 = qy$ in every associative ring with the additive group A. Then, for any such a ring P we have $P^2 \subseteq \langle y\rangle_*$ and, consequently, $\square_a A = \langle y\rangle_*$. $\square$

Corollary 4.4. *Let A be an abelian group satisfying all the assumptions of Theorem 4.3 except for $\Box A_2 \neq \{0\}$. If $x^2 = qz$ with $q \neq 0$ in some associative ring on A, then $\Box_a A = \langle z \rangle_*$.*

Proof. We retain all designations from the proof of Theorem 4.3. Notice that the conditions listed in (4.1) and (4.2) imply that $A_2 \lhd S$. Moreover, $\Box A_2 = \{0\}$, so $y^2 = 0$ in the ring S. Next, $x \cdot (x \cdot y) = s^2 y$ and $(x \cdot x) \cdot y = q(z \cdot y) = 0$, by (4.1), so $s = 0$. Consequently, $xy = 0$. Analogously, $yx = 0$. Moreover, $x \cdot z = z \cdot x = 0$, by the proof of Theorem 4.3. Combining this with (4.1) we get $S^2 = \left(U_0^2 q\right) z$. Hence $\Box_a A \subseteq \langle z \rangle_*$ and $U_0^2 q \subseteq W_0$. Therefore, just as in the proof of Theorem 4.3 we obtain $\langle z \rangle_* \subseteq \Box_a A$. Finally, $\Box_a A = \langle z \rangle_*$. $\Box$

The next result is an answer to (Q_2).

Theorem 4.5. *Let $A = A_1 \oplus A_2$ where A_1 is a torsion-free abelian group of rank one with $t(A_1) = t_0$ satisfying $t_0^2 \neq t_0$ and A_2 is a torsion-free abelian group of rank two such that $T(A_2) = \{t_0, t_1, t_2\}$ where $t_0 < t_1$, $t_0 < t_2$, $t_1 \neq t_2$, $t_1^2 = t_1$ and $t_2^2 = t_2$. If $\Box A_2 \neq \{0\}$, then $\Box_{(a)} A = A_2$.*

Proof. It follows from [11, Proposition 1] that types t_1 and t_2 are incomparable. Hence, by [3, the proof of Lemma 3.3] and [2, Theorem 4] we get $\Box_{(a)} A_2 = A_2$. Moreover, A_2 is a direct summand of A, so $A_2 \subseteq \Box_{(a)} A$, by [6, Proposition 1.4 and Remark 1.10]. On the other hand, $\Box_{(a)} A \subseteq \langle y, z \rangle_*$, by Corollary 4.2 and (2.1). Furthermore, $A_2 = \langle y, z \rangle_*$, so $\Box_{(a)} A = A_2$. $\Box$

The next two theorems describe the square subgroup of an abelian group in some cases related to (Q_3).

Theorem 4.6. *Let $A = A_1 \oplus A_2$ where A_1 is a torsion-free abelian group of rank one and A_2 is an indecomposable torsion-free abelian group of rank two such that $T(A_2) = \{t_1, t_2\}$ and $t_1 < t_2$. If $\Box A_2 \neq \{0\}$, $t(A_1) = t_1$ and $t_1^2 \neq t_1$, then $\Box_{(a)} A = \langle z \rangle_*$ for some $z \in A_2$ with $t(z) = t_2$.*

Proof. Take any $x \in A_1 \setminus \{0\}$ and $y, z \in A_2$ such that $t(y) = t_1$ and $t(z) = t_2$. It follows from [3, the proof of Theorem 4.2] and [2, Theorem 4] that $\Box_{(a)} A_2 = \langle z \rangle_*$. Hence, by [6, Proposition 1.4 and Remark 1.10] we obtain $\langle z \rangle_* \subseteq \Box_{(a)} A$. Moreover, $A(t_2) \subseteq A_2$, by (iii) of Lemma 2.1. Suppose, contrary to our claim, that $A(t_2)$ is a group of rank two. Then it is an essential subgroup of A_2. Hence $ny \in A(t_2)$ for some $n \in \mathbb{N}$. Since y and ny are dependent elements of A_2, (i) of Lemma 2.1 implies that $t(ny) = t_1$. Therefore $t_1 \geq t_2$, a contradiction. Thus $A(t_2)$ is a rank one group and, consequently, $A(t_2) = \langle z \rangle_*$. Moreover, $t_1^2 \neq t_1$, so (vi), (v) and (iii) of Lemma 2.1 imply that in an arbitrary (associative) ring $R = (A, \cdot)$ we have $x^2, x \cdot y, y \cdot x, y^2 \in \langle z \rangle_*$.

Furthermore, $z \cdot A, A \cdot z \subseteq \langle z\rangle_*$, because $A(t_2) \lhd R$. Thus $R^2 \subseteq \langle z\rangle_*$ and, consequently, $\Box_{(a)}A \subseteq \langle z\rangle_*$. Finally, $\Box_{(a)}A = \langle z\rangle_*$. $\Box$

THEOREM 4.7. *Let $A = A_1 \oplus A_2$ where A_1 is a torsion-free abelian group of rank one and A_2 is an indecomposable torsion-free abelian group of rank two such that $T(A_2) = \{t_1, t_2\}$ and $t_1 < t_2$. If $t_1^2 \neq t_1$, $t_2^2 = t_2$, $\Box A_2 \neq \{0\}$ and $t(A_1) = t_2$, then $\Box_{(a)}A = A_1 \oplus \langle z\rangle_*$ for some $z \in A_2$ with $t(z) = t_2$.*

PROOF. Take any $x \in A_1 \setminus \{0\}$ and $y, z \in A_2$ such that $t(y) = t_1$ and $t(z) = t_2$. Just as in the proof of Theorem 4.6 we obtain $\langle z\rangle_* \subseteq \Box_{(a)}A$. Moreover, $A_1 \subseteq \Box_{(a)}A$, by [19, Theorem 4.8] and [6, Proposition 1.4 and Remark 1.10], so $A_1 \oplus \langle z\rangle_* \subseteq \Box_{(a)}A$. Using methods similar to those of the preceding proofs, we infer that $A(t_2)$ is a group of rank two with a maximal independent system $\{x, z\}$. Thus $A(t_2) = A_1 \oplus \langle z\rangle_*$. Consider an arbitrary (associative) ring $R = (A, \cdot)$. Since $t_1^2 \neq t_1$, (vi) and (iii) of Lemma 2.1 imply that $y^2 \in A(t_2)$. Furthermore, $x^2, x \cdot y, y \cdot x \in A(t_2)$ and $z \cdot A, A \cdot z \subseteq A(t_2)$, because $A(t_2) \lhd R$. Therefore $R^2 \subseteq A(t_2)$ and, consequently, $\Box_{(a)}A \subseteq A(t_2)$. Finally, $\Box_{(a)}A = A_1 \oplus \langle z\rangle_*$. $\Box$

REMARK 4.8. If $t(A_1) = t_0$ is incomparable to neither t_1 nor t_2, then $\Box_{(a)}A = \Box_{(a)}A_1 \oplus \Box_{(a)}A_2$, by [16, Proposition 3.2 and Lemma 3.3] and Remark 2.2. Moreover, it follows from [3, the proof of Theorem 4.1] and [2, Theorem 4] that $\Box_{(a)}A_2 = \langle z\rangle_*$. Hence, by [19, Theorem 4.8], we obtain $\Box_{(a)}A = A_1 \oplus \langle z\rangle_*$ if $t_0^2 = t_0$, or $\Box_{(a)}A = \langle z\rangle_*$ if $t_0^2 \neq t_0$.

Notice that Theorem 3.6 together with Proposition 2.2 and [16, Lemma 3.3] gives some generalizations of [16, Theorems 3.5–3.7]. Namely, we have the following:

THEOREM 4.9. *Let B be a torsion-free abelian group of rank two with $T(B) = \{t_0, t_1, t_2\}$ where $t_0 < t_1$, $t_0 < t_2$, $t_1^2 = t_1$, $t_2^2 \neq t_2$, and let $\{A_i \colon i \in I\}$ be a nonempty family of nontrivial torsion-free abelian groups of rank one such that the type of each A_i is incomparable to every member of $T(B)$. If $\Box B \neq \{0\}$, then $\Box_{(a)}\Big(\big(\bigoplus_{i\in I} A_i\big) \oplus B\Big) = \big(\bigoplus_{i\in X} \overline{A_i}\big) \oplus \langle y\rangle_*$ where y is some element of B with $t(y) = t_1$ and $X = \big\{i \in I \colon t(A_k) \cdot t(A_l) \leq t(A_i) \text{ for some } k, l \in I\big\}$.*

REMARK 4.10. In view of Theorem 4.9, Proposition 2.2 and [16, Lemma 3.3], the assumption that $t(A_i)$ and $t(A_j)$ are incomparable for all distinct $i, j \in I$, is necessary in [16, Theorem 3.7].

5. Other results

It follows from [6, Lemma 1.8] that for a given abelian group A, the square subgroup $\square A$ is a fully invariant subgroup of A. Now we prove the complementary result for the associative case.

PROPOSITION 5.1. *If A is an abelian group, then the square subgroup $\square_a A$ is a characteristic subgroup of A.*

PROOF. Take any $a, b \in A$. Let f be an automorphism of A and let $*$ be any associative ring multiplication on A. An easy computation shows that the function $\star\colon A \times A \to A$ given by

$$x \star y = f\Big(f^{-1}(x) * f^{-1}(y)\Big) \text{ for all } x, y \in A,$$

is an associative ring multiplication on A. Therefore $f\Big(f^{-1}(a) * f^{-1}(b)\Big) \in \square_a A$. Hence, by (2.1), we get $f(\square_a A) \subseteq \square_a A$. $\square$

REMARK 5.2. We remind the reader that the set

$$I(A) = \Big\{\varphi(A)\colon \varphi \in \mathrm{Hom}\,\big(A, \mathrm{End}(A)\big)\Big\}$$

plays an important role in studying subgroups that are always ideals (cf. [15, p. 279]). It follows from (2.1) that for any abelian group A, the square subgroup $\square_{(a)} A$ is an ideal in every (associative) ring R with the additive group A. Hence, by [15, Theorem 117.2], we conclude that $\square A$ is $I(A)$-admissible subgroup of A, i.e., $I(A)\square A \subseteq \square A$.

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