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SOME NEW OSTROWSKI'S INEQUALITIES FOR FUNCTIONS WHOSE n^{th}DERIVATIVES ARE LOGARITHMICALLY CONVEX

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Abstract. Some new Ostrowski's inequalities for functions whose n^{th}derivative are logarithmically convex are established.

1. Introduction

In 1938, A.M. Ostrowski proved an interesting integral inequality, estimating the absolute value of the derivative of a differentiable function by its integral mean as follows

Theorem 1 ([2]). *Let $f\colon I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a differentiable mapping in the interior I° of I, and $a, b \in I^{\circ}$, with $a < b$. If $|f'| \leq M$ for all $x \in [a, b]$, then*

$$(1.1)\quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq M(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right], \quad \forall x \in [a, b].$$

This is well-known as Ostrowski inequality. In recent years, a number of authors have written about generalizations, extensions and variants of inequality (1.1).

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In [1], Cerone et al. proved the following identity

Lemma 1 ([1, Lemma 2.1]). *Let $f\colon [a,b]\to\mathbb{R}$ be a differentiable mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then for all $x\in[a,b]$ we have the identity*

$$\int_a^b f(t)\,dt=\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)$$
$$+(-1)^n\int_a^b K_n(x,t)\,f^{(n)}(t)\,dt,$$

where the kernel $K_n\colon [a,b]^2\to\mathbb{R}$ is given by

$$K_n(x,t)=\begin{cases}\frac{(t-a)^n}{n!} & \text{if } t\in[a,x],\\ \frac{(t-b)^n}{n!} & \text{if } t\in(x,b],\end{cases}\qquad x\in[a,b],$$

and n is natural number, $n\geq 1$.

Wang et al. [4], proved the following identities

Lemma 2 ([4, Lemma 2.2]). *For $\alpha>0$ and $k>0$, $z>0$:*

$$J(\alpha,k):=\int_0^1(1-t)^{\alpha-1}k^t\,dt=\sum_{i=1}^{\infty}\frac{(\ln k)^{i-1}}{(\alpha)_i}<\infty,\tag{1.2}$$

$$H(\alpha,k,z):=\int_0^1 t^{\alpha-1}k^t\,dt=z^\alpha k^z\sum_{i=1}^{\infty}\frac{(-z\ln k)^{i-1}}{(\alpha)_i}<\infty,\tag{1.3}$$

where $(\alpha)_i=\prod_{j=0}^{i-1}(\alpha+j)$.

We also recall that a positive function $f\colon I\to\mathbb{R}$ is said to be logarithmically convex, if

$$f(tx+(1-t)y)\ \leq [f(x)]^t\,[f(y)]^{1-t}$$

holds for all $x,y\in I$ and $t\in[0,1]$ (see [3]).

In this paper, by using the identity given in Lemma 1, we establish some new Ostrowski's inequalities for functions whose n^{th}derivatives are logarithmically convex.

2. Main results

In what follows, we assume that $n \in \mathbb{N}$, and $I \subset \mathbb{R}$ is an interval where $[a,b] \subset I$.

THEOREM 2. *Let $f\colon I \to \mathbb{R}$ be n times differentiable mapping on $[a,b]$ such that $f^{(n)} \in L([a,b])$ and $f^{(n)}(x) \neq 0$ for all $x \in [a,b]$. If $\left|f^{(n)}\right|$ is logarithmically convex, then the following inequality*

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$

$$\leq \begin{cases} \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right| & \text{if } \lambda = \tau = 1 \\ \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{n!}\left|f^{(n)}(x)\right| \sum_{i=1}^{\infty}\frac{(\ln\tau)^{i-1}}{(n+1)_i} & \text{if } \lambda = 1 \neq \tau \\ \frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right| + \frac{(x-a)^{n+1}}{n!}\left|f^{(n)}(a)\right| \lambda \sum_{i=1}^{\infty}\frac{(-\ln\lambda)^{i-1}}{(n+1)_i} & \text{if } \lambda \neq 1 = \tau \\ \frac{(x-a)^{n+1}}{n!}\left|f^{(n)}(a)\right| \lambda \sum_{i=1}^{\infty}\frac{(-\ln\lambda)^{i-1}}{(n+1)_i} & \\ \qquad + \frac{(b-x)^{n+1}}{n!}\left|f^{(n)}(x)\right| \sum_{i=1}^{\infty}\frac{(\ln\tau)^{i-1}}{(n+1)_i} & \text{if } \lambda \neq 1 \text{ and } \tau \neq 1 \end{cases}$$

holds for all $x \in [a,b]$, where

$$\lambda = \frac{\left|f^{(n)}(x)\right|}{\left|f^{(n)}(a)\right|}, \tag{2.1}$$

$$\tau = \frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(x)\right|}, \tag{2.2}$$

and $(n+1)_i = \prod_{j=0}^{i-1}(n+1+j)$.

PROOF. From Lemma 1 and the properties of modulus, we have

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$

$$\leq \int_a^x \frac{(u-a)^n}{n!}\left|f^{(n)}(u)\right| du + \int_x^b \frac{(b-u)^n}{n!}\left|f^{(n)}(u)\right| du$$

$$= \frac{(x-a)^{n+1}}{n!} \int_0^1 t^n \left| f^{(n)}((1-t)\, a + tx) \right| dt \tag{2.3}$$

$$+ \frac{(b-x)^{n+1}}{n!} \int_0^1 (1-t)^n \left| f^{(n)}((1-t)\, x + tb) \right| dt.$$

Since $\left|f^{(n)}\right|$ is logarithmically convex, (2.3) becomes

$$\left| \int_a^b f(t)\, dt - \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \tag{2.4}$$

$$\leq \frac{(x-a)^{n+1}}{n!} \int_0^1 t^n \left| f^{(n)}(a) \right|^{1-t} \left| f^{(n)}(x) \right|^t dt$$

$$+ \frac{(b-x)^{n+1}}{n!} \int_0^1 (1-t)^n \left| f^{(n)}(x) \right|^{1-t} \left| f^{(n)}(b) \right|^t dt$$

$$= \frac{(x-a)^{n+1}}{n!} \left| f^{(n)}(a) \right| \int_0^1 t^n \lambda^t\, dt$$

$$+ \frac{(b-x)^{n+1}}{n!} \left| f^{(n)}(x) \right| \int_0^1 (1-t)^n \tau^t\, dt,$$

where λ and τ are defined as in (2.1) and (2.2) respectively.

Now, we proceed to the discussion of possible cases.

If $\lambda = \tau = 1$, then (2.4) gives

$$\left| \int_a^b f(t)\, dt - \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \tag{2.5}$$

$$\leq \frac{(x-a)^{n+1}}{n!} \left| f^{(n)}(a) \right| \int_0^1 t^n dt + \frac{(b-x)^{n+1}}{n!} \left| f^{(n)}(x) \right| \int_0^1 (1-t)^n\, dt$$

$$= \frac{(x-a)^{n+1}}{(n+1)!} \left| f^{(n)}(a) \right| + \frac{(b-x)^{n+1}}{(n+1)!} \left| f^{(n)}(x) \right|.$$

If $\lambda = 1$ and $\tau \neq 1$, using (1.2), from (2.4) we obtain

$$\left| \int_a^b f(t)\, dt - \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \tag{2.6}$$

$$\leq \frac{(x-a)^{n+1}}{(n+1)!} |f^{(n)}(a)| + \frac{(b-x)^{n+1}}{n!} |f^{(n)}(x)| \sum_{i=1}^{\infty} \frac{(\ln \tau)^{i-1}}{(n+1)_i}.$$

If $\lambda \neq 1$ and $\tau = 1$, we can use (1.3) with $z = 1$. Then, from (2.4) we obtain

$$(2.7)\quad \left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\leq \frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right| + \frac{(x-a)^{n+1}}{n!}\left|f^{(n)}(a)\right| \lambda \sum_{i=1}^{\infty}\frac{(-\ln\lambda)^{i-1}}{(n+1)_i}.$$

In the case $\lambda \neq 1$ and $\tau \neq 1$, using Lemma 2 for (2.4), we get

$$(2.8)\quad \left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\leq \frac{(x-a)^{n+1}}{n!}\left|f^{(n)}(a)\right| \lambda \sum_{i=1}^{\infty}\frac{(-\ln\lambda)^{i-1}}{(n+1)_i} + \frac{(b-x)^{n+1}}{n!}\left|f^{(n)}(x)\right| \sum_{i=1}^{\infty}\frac{(\ln\tau)^{i-1}}{(n+1)_i}.$$

The desired result follows from (2.5)–(2.8). □

Theorem 3. *Let $f\colon I \to \mathbb{R}$ be n times differentiable mapping on $[a,b]$ such that $f^{(n)} \in L([a,b])$ and $f^{(n)}(x) \neq 0$ for all $x \in [a,b]$, and let $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $|f^{(n)}|^q$ is logarithmically convex, then the following inequality*

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\leq \begin{cases} \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}} n!}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}} n!}\left|f^{(n)}(x)\right| & \text{if } \lambda = \tau = 1 \\ \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}} n!}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}} n!}\left(\frac{\tau^q-1}{q\ln\tau}\right)^{\frac{1}{q}}\left|f^{(n)}(x)\right| & \text{if } \lambda = 1 \neq \tau \\ \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}} n!}\left(\frac{\lambda^q-1}{q\ln\lambda}\right)^{\frac{1}{q}}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}} n!}\left|f^{(n)}(x)\right| & \text{if } \lambda \neq 1 = \tau \\ \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}} n!}\left(\frac{\lambda^q-1}{q\ln\lambda}\right)^{\frac{1}{q}}\left|f^{(n)}(a)\right| & \\ \quad + \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}} n!}\left(\frac{\tau^q-1}{q\ln\tau}\right)^{\frac{1}{q}}\left|f^{(n)}(x)\right| & \text{if } \lambda \neq 1 \text{ and } \tau \neq 1 \end{cases}$$

holds for all $x \in [a,b]$, where λ and τ are defined by (2.1) and (2.2) respectively.

Proof. From Lemma 1, properties of modulus, and Hölder's inequality, we have

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|\\
&\le \int_a^x \frac{(u-a)^n}{n!}\left|f^{(n)}(u)\right|du+\int_x^b \frac{(b-u)^n}{n!}\left|f^{(n)}(u)\right|du\\
&=\frac{(x-a)^{n+1}}{n!}\int_0^1 t^n\left|f^{(n)}((1-t)\,a+tx)\right|dt\\
&\quad+\frac{(b-x)^{n+1}}{n!}\int_0^1 (1-t)^n\left|f^{(n)}((1-t)\,x+tb)\right|dt\\
&\le \frac{(x-a)^{n+1}}{n!}\left(\int_0^1 t^{np}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}((1-t)\,a+tx)\right|^q dt\right)^{\frac{1}{q}}\\
&\quad+\frac{(b-x)^{n+1}}{n!}\left(\int_0^1 (1-t)^{np}\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}((1-t)\,x+tb)\right|^q dt\right)^{\frac{1}{q}}\\
&=\frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left(\int_0^1\left|f^{(n)}((1-t)\,a+tx)\right|^q dt\right)^{\frac{1}{q}}\\
&\quad+\frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left(\int_0^1\left|f^{(n)}((1-t)\,x+tb)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|f^{(n)}\right|^q$ is logarithmically convex function, we deduce

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|\\
&\le \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left|f^{(n)}(a)\right|\left(\int_0^1 \lambda^{qt}\,dt\right)^{\frac{1}{q}}\\
&\quad+\frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left|f^{(n)}(x)\right|\left(\int_0^1 \tau^{qt}\,dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.9}$$

Similarly to the proof of Theorem 2, if $\lambda = \tau = 1$, then (2.9) gives

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \leq \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}\, n!} \left| f^{(n)}(a) \right| + \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}\, n!} \left| f^{(n)}(x) \right|. \tag{2.10}$$

If $\lambda = 1$ and $\tau \neq 1$, then (2.9) becomes

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \tag{2.11}$$
$$\leq \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}\, n!} \left| f^{(n)}(a) \right| + \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}\, n!} \left(\int_0^1 \tau^{qt} dt \right)^{\frac{1}{q}} \left| f^{(n)}(x) \right|$$
$$= \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}\, n!} \left| f^{(n)}(a) \right| + \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}\, n!} \left(\frac{\tau^q - 1}{q \ln \tau} \right)^{\frac{1}{q}} \left| f^{(n)}(x) \right|.$$

If $\lambda \neq 1$ and $\tau = 1$, then (2.9) becomes

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \tag{2.12}$$
$$\leq \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}\, n!} \left| f^{(n)}(x) \right| + \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}\, n!} \left(\int_0^1 \lambda^{qt} dt \right)^{\frac{1}{q}} \left| f^{(n)}(a) \right|$$
$$= \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}\, n!} \left| f^{(n)}(x) \right| + \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}\, n!} \left(\frac{\lambda^q - 1}{q \ln \lambda} \right)^{\frac{1}{q}} \left| f^{(n)}(a) \right|.$$

In the case $\lambda \neq 1$ and $\tau \neq 1$, (2.9) gives

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \tag{2.13}$$
$$\leq \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}\, n!} \left(\frac{\lambda^q - 1}{q \ln \lambda} \right)^{\frac{1}{q}} \left| f^{(n)}(a) \right| + \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}\, n!} \left(\frac{\tau^q - 1}{q \ln \tau} \right)^{\frac{1}{q}} \left| f^{(n)}(x) \right|.$$

The desired result follows from (2.10)–(2.13). $\square$

THEOREM 4. *Let $f : I \to \mathbb{R}$ be n times differentiable mapping on $[a,b]$ such that $f^{(n)} \in L([a,b])$ and $f^{(n)}(x) \neq 0$ for all $x \in [a,b]$, and let $q > 1$. If $\left|f^{(n)}\right|^q$ is logarithmically convex, then the following inequality*

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$

$$\leq \begin{cases} \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right| & \text{if } \lambda = \tau = 1 \\ \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right| + \frac{(n+1)^{\frac{1}{q}}(b-x)^{n+1}}{(n+1)!}\left(\sum_{i=1}^{\infty}\frac{(\ln\tau^q)^{i-1}}{(n+1)_i}\right)^{\frac{1}{q}}\left|f^{(n)}(x)\right| & \\ & \text{if } \lambda = 1 \neq \tau \\ \frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+1)!}\lambda\left(\sum_{i=1}^{\infty}\frac{(-\ln\lambda^q)^{i-1}}{(n+1)_i}\right)^{\frac{1}{q}}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right| & \\ & \text{if } \lambda \neq 1 = \tau \\ \frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+1)!}\lambda\left(\sum_{i=1}^{\infty}\frac{(-\ln\lambda^q)^{i-1}}{(n+1)_i}\right)^{\frac{1}{q}}\left|f^{(n)}(a)\right| & \\ \quad + \frac{(n+1)^{\frac{1}{q}}(b-x)^{n+1}}{(n+1)!}\left(\sum_{i=1}^{\infty}\frac{(\ln\tau^q)^{i-1}}{(n+1)_i}\right)^{\frac{1}{q}}\left|f^{(n)}(x)\right| & \text{if } \lambda \neq 1 \text{ and } \tau \neq 1 \end{cases}$$

holds for all $x \in [a,b]$, where λ and τ are defined by (2.1) and (2.2) respectively.

PROOF. From Lemma 1, properties of modulus, and Hölder's inequality, we have

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$

$$\leq \int_a^x \frac{(u-a)^n}{n!}\left|f^{(n)}(u)\right| du + \int_x^b \frac{(b-u)^n}{n!}\left|f^{(n)}(u)\right| du$$

$$= \frac{(x-a)^{n+1}}{n!}\int_0^1 (t^n)^{1-\frac{1}{q}} \cdot (t^n)^{\frac{1}{q}}\left|f^{(n)}((1-t)\,a + tx)\right| dt$$

$$+ \frac{(b-x)^{n+1}}{n!}\int_0^1 ((1-t)^n)^{1-\frac{1}{q}} \cdot ((1-t)^n)^{\frac{1}{q}}\left|f^{(n)}((1-t)\,x + tb)\right| dt$$

$$\leq \frac{(x-a)^{n+1}}{n!}\left(\int_0^1 t^n dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^n\left|f^{(n)}((1-t)\,a + tx)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{n!}\left(\int_0^1 (1-t)^n\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)^n \left|f^{(n)}((1-t)\,x+tb)\right|^q dt\right)^{\frac{1}{q}}$$

$$=\frac{(n+1)^{\frac{1}{q}}\,(x-a)^{n+1}}{(n+1)!}\left(\int_0^1 t^n \left|f^{(n)}((1-t)\,a+tx)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\ \frac{(n+1)^{\frac{1}{q}}\,(b-x)^{n+1}}{(n+1)!}\left(\int_0^1 (1-t)^n \left|f^{(n)}((1-t)\,x+tb)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|f^{(n)}\right|^q$ is logarithmically convex, we deduce that

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k\,(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \tag{2.14}$$

$$\leq \frac{(n+1)^{\frac{1}{q}}\,(x-a)^{n+1}}{(n+1)!}\left(\left|f^{(n)}(a)\right|^q \int_0^1 t^n\lambda^{qt}dt\right)^{\frac{1}{q}}$$

$$+\ \frac{(n+1)^{\frac{1}{q}}\,(b-x)^{n+1}}{(n+1)!}\left(\left|f^{(n)}(x)\right|^q \int_0^1 (1-t)^n\,\tau^{qt}dt\right)^{\frac{1}{q}}.$$

In the case $\lambda=\tau=1$, (2.14) gives

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k\,(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \tag{2.15}$$

$$\leq \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right|+\frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right|.$$

If $\lambda=1$ and $\tau\neq 1$ then, applying (1.2) to (2.14), we get

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k\,(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \tag{2.16}$$

$$\leq \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right|+\frac{(n+1)^{\frac{1}{q}}\,(b-x)^{n+1}}{(n+1)!}\left(\sum_{i=1}^{\infty}\frac{(\ln\tau^q)^{i-1}}{(n+1)_i}\right)^{\frac{1}{q}}\left|f^{(n)}(x)\right|.$$

If $\lambda\neq 1$ and $\tau=1$ then, applying equality (1.3) with $z=1$ to the integral $\int_0^1 t^n\,(\lambda^q)^t\,dt$, we get

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \tag{2.17}$$
$$\leq\frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right|\lambda\left(\sum_{i=1}^{\infty}\frac{(-\ln\lambda^q)^{i-1}}{(n+1)_i}\right)^{\frac{1}{q}}+\frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right|.$$

For $\lambda\neq 1$ and $\tau\neq 1$, using Lemma 2 for (2.14), we obtain

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \tag{2.18}$$
$$\leq\frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right|\lambda\left(\sum_{i=1}^{\infty}\frac{(-\ln\lambda^q)^{i-1}}{(n+1)_i}\right)^{\frac{1}{q}}$$
$$+\frac{(n+1)^{\frac{1}{q}}(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right|\left(\sum_{i=1}^{\infty}\frac{(\ln\tau^q)^{i-1}}{(n+1)_i}\right)^{\frac{1}{q}}.$$

The desired result follows from (2.15)–(2.18). The proof is thus completed. □

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