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GENERALIZATIONS OF SOME INTEGRAL INEQUALITIES FOR FRACTIONAL INTEGRALS

Ghulam Farid, Atiq ur Rehman

Abstract. In this paper we give generalizations of the Hadamard-type inequalities for fractional integrals. As special cases we derive several Hadamard type inequalities.

1. Introduction

For the last few years many researchers have considered the field of inequalities; many extensions and generalizations of several well known inequalities have been found so far. Convexity and theory of inequalities provide fundamental assistance in various branches of mathematics, especially mathematical analysis, functional analysis etc. The Hadamard and Fejér–Hadamard inequalities are of great interest for the researchers, and their various extensions and generalizations have been found (see, [1–3, 6, 8, 11–14, 18–20] and references therein).

Convex functions play a vital role in various fields of mathematics, science and engineering.

A function $f\colon [a,b] \to \mathbb{R}$ is said to be convex if

$$f\left(\lambda x + (1-\lambda)y\right) \leq \lambda f(x) + (1-\lambda) f(y)$$

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holds for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. If $-f$ is convex, then f is called concave function and vice versa.

In literature double integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

where $f \colon I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, is known as the Hadamard inequality. If f is concave then the above inequalities hold in the reverse direction.

In [10] Fejér gave the following generalization of the Hadamard inequality.

THEOREM 1.1. [5] *Let $f \colon [a, b] \to \mathbb{R}$ be a convex function and $g \colon [a, b] \to \mathbb{R}$ is a nonnegative, integrable and symmetric to $\frac{a+b}{2}$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x)dx.$$

In literature above inequality is known as the Fejér-Hadamard inequality.

Nowadays the Hadamard and Fejér–Hadamard inequalities came into focus of many researchers via fractional calculus. Recently a lot of papers have been dedicated to this subject (see, [12, 17, 19] and references therein).

In [17] k-fractional Riemann–Liouville integrals are defined.

Let $f \in L_1[a, b]$. Then k-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as:

$$I_{a+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a$$

and

$$I_{b-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad x < b,$$

where $\Gamma_k(\alpha)$ is the k-Gamma function defined in [4] as:

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt,$$

also

$$\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha).$$

For $k = 1$, k-fractional integrals give well-known Riemann–Liouville fractional integrals.

In [18] Sarikaya et al. proved the following results of Fejér–Hadamard type inequalities for convex functions.

THEOREM 1.2. *Let $f\colon I^o \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$ and let $g\colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds with $\alpha > 0$ and $\|g\|_\infty = \sup |g(t)|$:*

$$\left| \left(\int_a^b g(s)ds \right)^{\alpha} [f(a) + f(b)] - \alpha \int_a^b \left(\int_a^t g(s)ds \right)^{\alpha-1} g(t)f(t)dt \right.$$

$$\left. - \alpha \int_a^b \left(\int_t^b g(s)ds \right)^{\alpha-1} g(t)f(t)dt \right| \le \frac{(b-a)^{\alpha+1}\|g\|_\infty^\alpha}{\alpha+1} [|f'(a)| + |f'(b)|].$$

THEOREM 1.3. *Let $f\colon I^o \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$ and let $g\colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $|f'|^q$ is convex on $[a, b]$, where $q > 1$, then the following inequality holds with $\alpha > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\|g\|_\infty = \sup |g(t)|$:*

$$\left| \left(\int_a^b g(s)ds \right)^{\alpha} [f(a) + f(b)] - \alpha \int_a^b \left(\int_a^t g(s)ds \right)^{\alpha-1} g(t)f(t)dt \right.$$

$$\left. -\alpha \int_a^b \left(\int_t^b g(s)ds \right)^{\alpha-1} g(t)f(t)dt \right| \le \frac{2(b-a)^{\alpha+1}\|g\|_\infty^\alpha}{(\alpha p+1)^{\frac{1}{p}}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}.$$

In this paper we generalize Theorem 1.2 and Theorem 1.3 via k-fractional integrals. We give also some refinements of results shown in [21]. We deduce the results of [5, 7, 18, 21] as special cases of ours.

2. Main results

In this section we give generalizations of Theorem 1.2 and Theorem 1.3. We also obtain results of [7, 18]. First, we need the following lemma.

LEMMA 2.1. *Let $f\colon I^o \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$ and let $g\colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $f' \in L_1[a, b]$, then* *the following equality holds for k-fractional integrals:*

$$\left(\int_a^b g(s)ds\right)^{\frac{\alpha}{k}}[f(a)+f(b)] - \frac{\alpha}{k}\int_a^b \left(\int_a^t g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt \tag{2.1}$$
$$- \frac{\alpha}{k}\int_a^b \left(\int_t^b g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt$$
$$= \int_a^b \left(\int_a^t g(s)ds\right)^{\frac{\alpha}{k}} f'(t)dt - \int_a^b \left(\int_t^b g(s)ds\right)^{\frac{\alpha}{k}} f'(t)dt.$$

Proof. One can have

$$\int_a^b \left(\int_a^t g(s)ds\right)^{\frac{\alpha}{k}} f'(t)dt \tag{2.2}$$
$$= \left(\int_a^b g(s)ds\right)^{\frac{\alpha}{k}} f(b) - \frac{\alpha}{k}\int_a^b \left(\int_a^t g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt$$

and

$$\int_a^b \left(\int_t^b g(s)ds\right)^{\frac{\alpha}{k}} f'(t)dt \tag{2.3}$$
$$= -\left(\int_a^b g(s)ds\right)^{\frac{\alpha}{k}} f(a) + \frac{\alpha}{k}\int_a^b \left(\int_t^b g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt.$$

Subtracting (2.3) from (2.2) we get (2.1). □

Using above lemma we obtain the following generalization of Theorem 1.2.

Theorem 2.2. *Let $f\colon I^o \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$ and let $g\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds for k-fractional integrals:*

$$\Bigg|\left(\int_a^b g(s)ds\right)^{\frac{\alpha}{k}}[f(a)+f(b)] - \frac{\alpha}{k}\int_a^b \left(\int_a^t g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt$$
$$- \frac{\alpha}{k}\int_a^b \left(\int_t^b g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt\Bigg| \le \frac{(b-a)^{\frac{\alpha}{k}+1}\|g\|_\infty^{\frac{\alpha}{k}}}{\frac{\alpha}{k}+1}[|f'(a)|+|f'(b)|].$$

PROOF. Obviously f' has Darboux property (as a derivative). Moreover, since $|f'|$ is convex, $|f'|$ has a bounded variation on $[a,b]$. Consequently, by [16, Proposition 3.2], f' is continuous and therefore $f' \in L_1[a,b]$. Hence, from (2.1) of Lemma 2.1, we have

$$\begin{aligned}&\Bigg|\left(\int_a^b g(s)ds\right)^{\frac{\alpha}{k}}[f(a)+f(b)] - \frac{\alpha}{k}\int_a^b \left(\int_a^t g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt\\&-\frac{\alpha}{k}\int_a^b \left(\int_t^b g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt\Bigg|\\&\le \int_a^b \left|\int_a^t g(s)ds\right|^{\frac{\alpha}{k}}|f'(t)|dt + \int_a^b \left|\int_t^b g(s)ds\right|^{\frac{\alpha}{k}}|f'(t)|dt.\end{aligned}$$

As $g(t) \le \|g\|_\infty$, so we have

$$\begin{aligned}&\Bigg|\left(\int_a^b g(s)ds\right)^{\frac{\alpha}{k}}[f(a)+f(b)] - \frac{\alpha}{k}\int_a^b \left(\int_a^t g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt\\&-\frac{\alpha}{k}\int_a^b \left(\int_t^b g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt\Bigg|\\&\le \|g\|_\infty^{\frac{\alpha}{k}}\left[\int_a^b (t-a)^{\frac{\alpha}{k}}|f'(t)|dt + \int_a^b (b-t)^{\frac{\alpha}{k}}|f'(t)|dt\right]\\&= \|g\|_\infty^{\frac{\alpha}{k}}\Bigg[\int_a^b (t-a)^{\frac{\alpha}{k}}\left|f'\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b\right)\right|dt\\&+\int_a^b (b-t)^{\frac{\alpha}{k}}\left|f'\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b\right)\right|dt\Bigg].\end{aligned}$$

Using convexity of $|f'|$ we get

$$\begin{aligned}&\Bigg|\left(\int_a^b g(s)ds\right)^{\frac{\alpha}{k}}[f(a)+f(b)] - \frac{\alpha}{k}\int_a^b \left(\int_a^t g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt\\&-\frac{\alpha}{k}\int_a^b \left(\int_t^b g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt\Bigg|\end{aligned}$$

$$\leq \|g\|_{\infty}^{\frac{\alpha}{k}} \left[\int_a^b (t-a)^{\frac{\alpha}{k}} \left(\frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| \right) dt \right.$$

$$\left. + \int_a^b (b-t)^{\frac{\alpha}{k}} \left(\frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| \right) dt \right],$$

from which, after a little computation, we get the required result. □

COROLLARY 2.3. *For $g(s) \equiv 1$ we have the following Hadamard-type inequality for k-fractional integrals:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[I_{a+}^{\alpha,k} f(b) + I_{b-}^{\alpha,k} f(a) \right] \right|$$

$$\leq \frac{b-a}{2(\frac{\alpha}{k}+1)} \left[|f'(a)| + |f'(b)| \right].$$

REMARK 2.4. For $k = 1$ in Theorem 2.2 we get Theorem 1.2. For $g(s) \equiv 1$ along with $k = 1$ we get [18, Corollary 2]. Further, if $\alpha = 1$ along with $k = 1$ we get [18, Corollary 3].

Now we give the generalization of Theorem 1.3.

THEOREM 2.5. *Let $f\colon I^o \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$ and let $g\colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $|f'|^q$ is convex on $[a, b]$, where $q > 1$, then the following inequality for k-fractional integrals holds with $\frac{1}{p} + \frac{1}{q} = 1$ and $\|g\|_\infty = \sup |g(t)|$:*

$$\left| \left(\int_a^b g(s)ds \right)^{\frac{\alpha}{k}} [f(a)+f(b)] - \frac{\alpha}{k} \int_a^b \left(\int_a^t g(s)ds \right)^{\frac{\alpha}{k}-1} g(t)f(t)dt \right. \tag{2.4}$$

$$\left. - \frac{\alpha}{k} \int_a^b \left(\int_t^b g(s)ds \right)^{\frac{\alpha}{k}-1} g(t)f(t)dt \right|$$

$$\leq \frac{2(b-a)^{\frac{\alpha}{k}+1} \|g\|_{\infty}^{\frac{\alpha}{k}}}{(\frac{\alpha p}{k}+1)^{\frac{1}{p}}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}.$$

PROOF. Obviously f' has Darboux property (as a derivative). Moreover, since $|f'|^q$ is convex, $|f'|^q$ and hence also $|f'|$ has a bounded variation on

$[a,b]$. Consequently, by [16, Proposition 3.2], f' is continuous and therefore $f'\in L_1[a,b]$. Hence, from (2.1) of Lemma 2.1, we have

$$\left|\left(\int_a^b g(s)ds\right)^{\frac{\alpha}{k}}[f(a)+f(b)]-\frac{\alpha}{k}\int_a^b\left(\int_a^t g(s)ds\right)^{\frac{\alpha}{k}-1}g(t)f(t)dt\right.$$
$$\left.-\frac{\alpha}{k}\int_a^b\left(\int_t^b g(s)ds\right)^{\frac{\alpha}{k}-1}g(t)f(t)dt\right|$$
$$\le\int_a^b\left|\int_a^t g(s)ds\right|^{\frac{\alpha}{k}}|f'(t)|dt+\int_a^b\left|\int_t^b g(s)ds\right|^{\frac{\alpha}{k}}|f'(t)|dt.$$

Using Hölder's inequality on R.H.S of above inequality we have

$$\left|\left(\int_a^b g(s)ds\right)^{\frac{\alpha}{k}}[f(a)+f(b)]-\frac{\alpha}{k}\int_a^b\left(\int_a^t g(s)ds\right)^{\frac{\alpha}{k}-1}g(t)f(t)dt\right.$$
$$\left.-\frac{\alpha}{k}\int_a^b\left(\int_t^b g(s)ds\right)^{\frac{\alpha}{k}-1}g(t)f(t)dt\right|$$
$$\le\left(\int_a^b\left|\int_a^t g(s)ds\right|^{\frac{\alpha p}{k}}dt\right)^{\frac{1}{p}}\left(\int_a^b|f'(t)|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_a^b\left|\int_t^b g(s)ds\right|^{\frac{\alpha p}{k}}dt\right)^{\frac{1}{p}}\left(\int_a^b|f'(t)|^q dt\right)^{\frac{1}{q}}.$$

As $g(t)\le\|g\|_\infty$, so we get

$$\left|\left(\int_a^b g(s)ds\right)^{\frac{\alpha}{k}}[f(a)+f(b)]-\frac{\alpha}{k}\int_a^b\left(\int_a^t g(s)ds\right)^{\frac{\alpha}{k}-1}g(t)f(t)dt\right.$$
$$\left.-\frac{\alpha}{k}\int_a^b\left(\int_t^b g(s)ds\right)^{\frac{\alpha}{k}-1}g(t)f(t)dt\right|$$
$$\le\|g\|_\infty^{\frac{\alpha}{k}}\left[\left(\int_a^b|t-a|^{\frac{\alpha p}{k}}dt\right)^{\frac{1}{p}}+\left(\int_a^b|b-t|^{\frac{\alpha p}{k}}dt\right)^{\frac{1}{p}}\right]\left(\int_a^b|f'(t)|^q dt\right)^{\frac{1}{q}}.$$

Using convexity of $|f'|^q$ we obtain

$$\begin{aligned}
&\Bigg|\left(\int_a^b g(s)ds\right)^{\frac{\alpha}{k}}[f(a)+f(b)] - \frac{\alpha}{k}\int_a^b\left(\int_a^t g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt \\
&\quad - \frac{\alpha}{k}\int_a^b\left(\int_t^b g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt\Bigg| \\
&\le \|g\|_\infty^{\frac{\alpha}{k}}\left[\left(\int_a^b |t-a|^{\frac{\alpha p}{k}}dt\right)^{\frac{1}{p}} + \left(\int_a^b |b-t|^{\frac{\alpha p}{k}}dt\right)^{\frac{1}{p}}\right] \\
&\left(\int_a^b\left(\frac{b-t}{b-a}|f'(a)|^q + \frac{t-a}{b-a}|f'(b)|^q\right)dt\right)^{\frac{1}{q}},
\end{aligned}$$

from which one can get inequality (2.4). □

Corollary 2.6. *For $g(s) \equiv 1$ we have the following Hadamard-type inequality for k-fractional integrals:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k}f(b) + I_{b-}^{\alpha,k}f(a)\right]\right| \le \frac{(b-a)}{(\frac{\alpha p}{k}+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Remark 2.7. For $k=1$ in Theorem 2.5 we get Theorem 1.3. If we take $\alpha = 1$ along with $k=1$, then we get [18, Corollary 5]. If we take $\alpha = 1$, $g(s) \equiv 1$ and with $k=1$, then we get [7, Theorem 2.3].

3. Refinements of Hadamard-type inequalities via *k*-fractional integrals

In this section we will give refinements of results proved in [21]. Then we will deduce some of the results proven in [5, 21].

For $\alpha, k > 0$ we define a function $H_\alpha^k\colon [0,1] \to \mathbb{R}$ as follows:

$$\text{(3.1)} \quad H_\alpha^k(t) = \frac{\alpha}{2k(b-a)^{\frac{\alpha}{k}}} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)\left[(b-x)^{\frac{\alpha}{k}-1} + (x-a)^{\frac{\alpha}{k}-1}\right] dx.$$

In the next theorem we investigate properties of this function. This result is also a generalization of [21, Theorem 2.2]. We use the following lemma [21].

Lemma 3.1. *Let $f\colon [a,b] \to \mathbb{R}$ be a convex function and let h be defined by*

$$h(t) = \frac{1}{2}\left[f\left(\frac{a+b}{2} - \frac{t}{2}\right) + f\left(\frac{a+b}{2} + \frac{t}{2}\right)\right].$$

Then h is increasing and convex on $[0, b-a]$ and

$$f\left(\frac{a+b}{2}\right) \le h(t) \le \frac{f(a)+f(b)}{2}$$

for all $t \in [0, b-a]$.

Theorem 3.2. *Let $f\colon [a,b] \to \mathbb{R}$ be a positive function with $a < b$. If f is convex on $[a,b]$, then the function H_α^k defined by (3.1) is convex and monotonically increasing on $[0,1]$ and*

$$\text{(3.2)} \quad f\left(\frac{a+b}{2}\right) = H_\alpha^k(0) \le H_\alpha^k(t) \le H_\alpha^k(1) = \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k} f(b) + I_{b-}^{\alpha,k} f(a)\right]$$

for all $t \in [0,1]$.

Proof. First we prove that H_α^k is convex on $[0,1]$. Let $t_1, t_2, \beta \in [0,1]$. Then

$$H_\alpha^k((1-\beta)t_1 + \beta t_2) = \frac{\alpha}{2k(b-a)^{\frac{\alpha}{k}}} \int_a^b f\left(\Big((1-\beta)t_1 + \beta t_2\Big)x + \left(1 - \big((1-\beta)t_1 + \beta t_2\big)\frac{a+b}{2}\right)\right) \left[(b-x)^{\frac{\alpha}{k}-1} + (x-a)^{\frac{\alpha}{k}-1}\right] dx$$

$$= \frac{\alpha}{2k(b-a)^{\frac{\alpha}{k}}} \int_a^b f\Bigg((1-\beta)\Bigg(\Big(x - \frac{a+b}{2}\Big)t_1 + \frac{a+b}{2}\Bigg)$$
$$+ \beta\Bigg(\Big(x - \frac{a+b}{2}\Big)t_2 + \frac{a+b}{2}\Bigg)\Bigg) \left[(b-x)^{\frac{\alpha}{k}-1} + (x-a)^{\frac{\alpha}{k}-1}\right] dx.$$

Using convexity of f we have

$$H_\alpha^k((1-\beta)t_1 + \beta t_2)$$
$$\leq \frac{\alpha(1-\beta)}{2k(b-a)^{\frac{\alpha}{k}}} \int_a^b f\Bigg(\Big(x - \frac{a+b}{2}\Big)t_1 + \frac{a+b}{2}\Bigg) \left[(b-x)^{\frac{\alpha}{k}-1} + (x-a)^{\frac{\alpha}{k}-1}\right] dx$$
$$+ \frac{\alpha\beta}{2k(b-a)^{\frac{\alpha}{k}}} \int_a^b f\Bigg(\Big(x - \frac{a+b}{2}\Big)t_2 + \frac{a+b}{2}\Bigg) \left[(b-x)^{\frac{\alpha}{k}-1} + (x-a)^{\frac{\alpha}{k}-1}\right] dx$$
$$= (1-\beta)H_\alpha^k(t_1) + \beta H_\alpha^k(t_2).$$

Next we prove that H_α^k is increasing on $[0,1]$.

$$H_\alpha^k(t) = \frac{\alpha}{2k(b-a)^{\frac{\alpha}{k}}} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) \left[(b-x)^{\frac{\alpha}{k}-1} + (x-a)^{\frac{\alpha}{k}-1}\right] dx$$
$$= \frac{\alpha}{2k(b-a)^{\frac{\alpha}{k}}} \int_a^{\frac{a+b}{2}} f\left(tx + (1-t)\frac{a+b}{2}\right) \left[(b-x)^{\frac{\alpha}{k}-1} + (x-a)^{\frac{\alpha}{k}-1}\right] dx$$
$$+ \frac{\alpha}{2k(b-a)^{\frac{\alpha}{k}}} \int_{\frac{a+b}{2}}^b f\left(tx + (1-t)\frac{a+b}{2}\right) \left[(b-x)^{\frac{\alpha}{k}-1} + (x-a)^{\frac{\alpha}{k}-1}\right] dx.$$

After a change of variables we obtain

$$H_\alpha^k(t) = \frac{\alpha}{2k(b-a)^{\frac{\alpha}{k}}} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)\left[(b-x)^{\frac{\alpha}{k}-1} + (x-a)^{\frac{\alpha}{k}-1}\right]dx$$
$$= \frac{\alpha}{4k(b-a)^{\frac{\alpha}{k}}} \int_0^{b-a} f\left(\frac{a+b}{2} - \frac{tx}{2}\right)\left[\left(\frac{b-a}{2} + \frac{x}{2}\right)^{\frac{\alpha}{k}-1} + \left(\frac{b-a}{2} - \frac{x}{2}\right)^{\frac{\alpha}{k}-1}\right]dx$$
$$+ \frac{\alpha}{4k(b-a)^{\frac{\alpha}{k}}} \int_0^{b-a} f\left(\frac{a+b}{2} + \frac{tx}{2}\right)\left[\left(\frac{b-a}{2} + \frac{x}{2}\right)^{\frac{\alpha}{k}-1} + \left(\frac{b-a}{2} - \frac{x}{2}\right)^{\frac{\alpha}{k}-1}\right]dx$$

$$= \frac{\alpha}{4k(b-a)^{\frac{\alpha}{k}}} \int_0^{b-a} \left(f\left(\frac{a+b}{2} - \frac{tx}{2}\right) + f\left(\frac{a+b}{2} + \frac{tx}{2}\right)\right) \left[\left(\frac{b-a}{2} + \frac{x}{2}\right)^{\frac{\alpha}{k}-1} + \left(\frac{b-a}{2} - \frac{x}{2}\right)^{\frac{\alpha}{k}-1}\right] dx.$$

Using Lemma 3.1 and nonnegativity of $\left(\frac{b-a}{2} + \frac{x}{2}\right)^{\frac{\alpha}{k}-1}$, $\left(\frac{b-a}{2} - \frac{x}{2}\right)^{\frac{\alpha}{k}-1}$ and $\frac{\alpha}{4k(b-a)^{\frac{\alpha}{k}}}$ we get H_α^k is increasing on $[0,1]$.

Finally note that $f\left(\frac{a+b}{2}\right) = H_\alpha^k(0)$ and

$$H_\alpha^k(1) = \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[I_{a+}^{\alpha,k} f(b) + I_{b-}^{\alpha,k} f(a)\right].$$

Therefore for $0 \le t \le 1$ we have $H_\alpha^k(0) \le H_\alpha^k(t) \le H_\alpha^k(1)$, that is (3.2) holds. □

Remark 3.3. For $k = 1$ in above theorem we get [21, Theorem 2.2]. For $\alpha = 1$ along with $k = 1$ in above theorem we get [5, Theorem 1].

To give refinement of [21, Theorem 2.3] we define another function $J_\alpha^k : [0,1] \to \mathbb{R}$ for $\alpha, k > 0$ as follows:

$$J_\alpha^k(t) = \frac{\alpha}{4k(b-a)^{\frac{\alpha}{k}}} \int_a^b \Bigg[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) \left(\left(\frac{2b-a-x}{2}\right)^{\frac{\alpha}{k}-1} + \left(\frac{x-a}{2}\right)^{\frac{\alpha}{k}-1}\right) + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right)\left(\left(\frac{b-x}{2}\right)^{\frac{\alpha}{k}-1} + \left(\frac{x+b-2a}{2}\right)^{\frac{\alpha}{k}-1}\right)\Bigg] dx. \tag{3.3}$$

We investigate properties of this function in the following theorem. This result is also a generalization of [21, Theoren 2.3].

Theorem 3.4. *Let $f\colon [a,b] \to \mathbb{R}$ be a positive function with $a < b$. If f is convex on $[a,b]$, then the function J_α^k defined by (3.3) is convex and monotonically increasing on $[0,1]$ and*

$$J_\alpha^k(0) = \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[I_{a+}^{\alpha,k} f(b) + I_{b-}^{\alpha,k} f(a)\right] \le J_\alpha^k(t) \le \frac{f(a)+f(b)}{2} = J_\alpha^k(1) \tag{3.4}$$

for all $t \in [0,1]$.

Proof. From some basic properties of convex functions it follows that the function J_α^k is convex.

We prove that J_α^k is increasing on $[0,1]$.

$$J_\alpha^k(t) = \frac{\alpha}{4k(b-a)^{\frac{\alpha}{k}}} \int_a^b \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) \right.$$
$$\left(\left(\frac{2b-a-x}{2}\right)^{\frac{\alpha}{k}-1} + \left(\frac{x-a}{2}\right)^{\frac{\alpha}{k}-1}\right)$$
$$\left. + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right)\left(\left(\frac{b-x}{2}\right)^{\frac{\alpha}{k}-1} + \left(\frac{x+b-2a}{2}\right)^{\frac{\alpha}{k}-1}\right)\right] dx$$
$$= \frac{\alpha}{4k(b-a)^{\frac{\alpha}{k}}} \int_a^{\frac{a+b}{2}} \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) \right.$$
$$\left(\left(\frac{2b-a-x}{2}\right)^{\frac{\alpha}{k}-1} + \left(\frac{x-a}{2}\right)^{\frac{\alpha}{k}-1}\right)$$
$$\left. + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right)\left(\left(\frac{b-x}{2}\right)^{\frac{\alpha}{k}-1} + \left(\frac{x+b-2a}{2}\right)^{\frac{\alpha}{k}-1}\right)\right] dx$$
$$+ \frac{\alpha}{4k(b-a)^{\frac{\alpha}{k}}} \int_{\frac{a+b}{2}}^b \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) \right.$$
$$\left(\left(\frac{2b-a-x}{2}\right)^{\frac{\alpha}{k}-1} + \left(\frac{x-a}{2}\right)^{\frac{\alpha}{k}-1}\right)$$
$$\left. + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right)\left(\left(\frac{b-x}{2}\right)^{\frac{\alpha}{k}-1} + \left(\frac{x+b-2a}{2}\right)^{\frac{\alpha}{k}-1}\right)\right] dx.$$

After a change of variables we obtain

$$J_\alpha^k(t) = \frac{\alpha}{4k(b-a)^{\frac{\alpha}{k}}} \int_a^b \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) \right.$$
$$\left(\left(\frac{2b-a-x}{2}\right)^{\frac{\alpha}{k}-1} + \left(\frac{x-a}{2}\right)^{\frac{\alpha}{k}-1}\right)$$
$$\left. + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right)\left(\left(\frac{b-x}{2}\right)^{\frac{\alpha}{k}-1} + \left(\frac{x+b-2a}{2}\right)^{\frac{\alpha}{k}-1}\right)\right] dx$$

$$= \frac{\alpha}{8k(b-a)^{\frac{\alpha}{k}}} \int_0^{b-a} f\left(a + \left(\frac{1-t}{2}\right)x\right)\left(\left(\frac{2b-2a-x}{2}\right)^{\frac{\alpha}{k}-1} + \left(\frac{x}{2}\right)^{\frac{\alpha}{k}-1}\right)dx$$

$$+ \frac{\alpha}{8k(b-a)^{\frac{\alpha}{k}}} \int_0^{b-a} f\left(b - \left(\frac{1-t}{2}\right)x\right)\left(\left(\frac{x}{2}\right)^{\frac{\alpha}{k}-1} + \left(\frac{2b-2a-x}{2}\right)^{\frac{\alpha}{k}-1}\right)dx$$

$$= \frac{\alpha}{8k(b-a)^{\frac{\alpha}{k}}} \int_0^{b-a} \left[f\left(a + \left(\frac{1-t}{2}\right)x\right) + f\left(b - \left(\frac{1-t}{2}\right)x\right)\right]$$

$$\left(\left(\frac{2b-2a-x}{2}\right)^{\frac{\alpha}{k}-1} + \left(\frac{x}{2}\right)^{\frac{\alpha}{k}-1}\right)dx.$$

As $q(t) = b - a - (1-t)x$ is increasing in $[0,1]$, so using Lemma 3.1 we get $h(q(t))$ is increasing on $[0,1]$ and nonnegativity of $\left(\frac{2b-2a-x}{2}\right)^{\frac{\alpha}{k}-1} + \left(\frac{x}{2}\right)^{\frac{\alpha}{k}-1}$ and $\frac{\alpha}{8k(b-a)^{\frac{\alpha}{k}}}$ gives that J_α^k is increasing on $[0,1]$.

Finally note that

$$J_\alpha^k(0) = \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k} f(b) + I_{b-}^{\alpha,k} f(a)\right]$$

and $J_\alpha^k(1) = \frac{f(a)+f(b)}{2}$. Therefore for $0 \le t \le 1$ we have $J_\alpha^k(0) \le J_\alpha^k(t) \le J_\alpha^k(1)$, that is (3.4) holds. □

REMARK 3.5. For $k = 1$ in above theorem we get [21, Theorem 2.3]. For $\alpha = 1$ along with $k = 1$ we get [21, Theorem 1.2].

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COMSATS
Institute of Information Technology
Attock Campus
Pakistan
e-mail: faridphdsms@hotmail.com, ghlmfarid@ciit-attock.edu.pk
e-mail: atiq@mathcity.org