NUMERICAL SOLUTION OF TIME FRACTIONAL SCHRÖDINGER EQUATION BY USING QUADRATIC B-SPLINE FINITE ELEMENTS

Alaattin Esen, Orkun Tasbozan*

Abstract. In this article, quadratic B-spline Galerkin method has been employed to solve the time fractional order Schrödinger equation. Numerical solutions and error norms L_2 and L_∞ are presented in tables.

1. Introduction

Consider the time fractional nonlinear Schrödinger equation [15]

$$(1.1) \qquad i\frac{\partial^\gamma U(x,t)}{\partial t^\gamma} + \frac{\partial^2 U(x,t)}{\partial x^2} + |U(x,t)|^2 U(x,t) = f(x,t)$$

with the boundary conditions

$$(1.2) \qquad U(a,t) = h_1(t), \quad U(b,t) = h_2(t), \quad t \geq 0$$

and with the initial condition

$$(1.3) \qquad U(x,0) = g(x), \quad a \leq x \leq b,$$

where the fractional derivative is in the Caputo's sense [11, 21], $0 < \gamma < 1$ and $i = \sqrt{-1}$.

Equation (1.1) is solved numerically by using quadratic B-spline Galerkin method with given initial and boundary conditions. In the rest of the article we use $L1$ approximation for $0 < \gamma \leq 1$

$$(1.4) \qquad \left. \frac{\partial^\gamma f(t)}{\partial t^\gamma} \right|_{t_m} = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{m-1} b_k^\gamma \left[f(t_{m-k}) - f(t_{m-1-k}) \right],$$

where $b_k^\gamma = (k+1)^{1-\gamma} - k^{1-\gamma}$.

Fractional calculus has taken part in literature in 1695 with letters which were written by two famous mathematicians G.W. Leibnitz and L' Hospital to each other. After that time, a lot of famous mathematicians such as Euler, Laplace, Fourier, Lacroix, Abel, Riemann, Liouville, Caputo have made contribution to the development of arbitrary order differentiation and integration [14]. Fractional order differentiation concept paves way for various applications and it expresses the many physical problems [19]. Recent applications(studies) show that the fractional order differential equations are very effective tools to articulate complex events and modelling many physical, engineering phenomenons [21]. There are many applications of these type equations various fields such as viscoelastic, biology, signal process, electromagnetic, chaos and fractals, traffic system, chemistry, control system, economics, finance and etc. [13].

Riemann–Liouville approximation which is frequently seen in the literature, is composed of fractional integral and fractional derivative that are mode by two famous mathematicians Riemann and Liouville. In this approximation, the initial conditions which are given with fractional order equations composed of limit values of Rieman-Liouville derivative at the initial point. This is the biggest disadvantage of this approximation. These initial points don't have any physical meaning. In 1967, M. Caputo presented a fractional order derivative which involves the limit values at the initial points of integer order derivatives with initial values that are given with fractional order equation. Due to the advantage of Caputo's definition, Caputo fractional derivative is preferred as fractional derivative operator in many fractional order differential equations [21].

In recent years, many scientists have been searching the analytical and numerical solutions of fractional order differential equations by using Laplace transform method [21, 14], power series method [21], Adomian decomposition method [23, 17], variational iteration method [16, 30], differential transform method [18, 2], homotopy perturbation method [10, 27], homotopy analysis method [1, 9], finite difference methods [26, 25], finite element method [8, 24, 7, 3, 4, 6, 5, 28, 29] and etc.

Finite elements method first arose in 1960 and after that time, it has been used commonly in various fields of physics and engineering. Argyris, Clough and Zienkiewicz made contribution to this method [20]. With the development of computer in the last recent fifty years, it is seen that this method has a great importance in solving many problems which arise in physics and engineering [12].

$U(x,t)$ function which is given in fractional order Eq. (1.1) is a complex function. So $U(x,t)$ can be written

$$(1.5) \qquad U(x,t) = R(x,t) + iS(x,t),$$

where $R(x,t)$ is the real part and $S(x,t)$ is the imaginary part of the function $U(x,t)$. Substituting Eq. (1.5) in problem (1.1)-(1.3), we obtain a fractional partial differential equation system

$$(1.6) \qquad \begin{aligned} \frac{\partial^\gamma S}{\partial t^\gamma} - \frac{\partial^2 R}{\partial x^2} - \left(R^2 + S^2\right) R &= -f_r(x,t), \\ \frac{\partial^\gamma R}{\partial t^\gamma} + \frac{\partial^2 S}{\partial x^2} + \left(R^2 + S^2\right) S &= f_I(x,t), \end{aligned}$$

where

$$f_r(x,t) = -\frac{2t^{2-\gamma}}{\Gamma(3-\gamma)} \cos(2\pi x) + (t^6 - 4\pi^2 t^2)\sin(2\pi x),$$

$$f_I(x,t) = \frac{2t^{2-\gamma}}{\Gamma(3-\gamma)} \sin(2\pi x) + (t^6 - 4\pi^2 t^2)\cos(2\pi x)$$

with the boundary conditions

$$(1.7) \qquad \begin{aligned} R(0,t) = 0, \quad R(1,t) = 0, \quad t \geq 0, \\ S(0,t) = t^2, \quad S(1,t) = t^2, \quad t \geq 0, \end{aligned}$$

with the initial conditions

$$(1.8) \qquad R(x,0) = 0, \quad S(x,0) = 0, \quad 0 \leq x \leq 1.$$

Exact solutions of this system can be given as

$$(1.9) \qquad R(x,t) = t^2 \sin(2\pi x), \quad S(x,t) = t^2 \cos(2\pi x).$$

2. Quadratic B-spline finite element Galerkin solutions

In this part, to apply the method, we multiply the Eq. system (1.6) with W weighted function and integrate the system over the region, then we obtain the weighted integral function of the system

(2.1)
$$\int_0^1 W \left[\frac{\partial^\gamma S}{\partial t^\gamma} - \frac{\partial^2 R}{\partial x^2} - (R^2 + S^2)R \right] dx = - \int_0^1 W f_r(x,t)dx,$$

$$\int_0^1 W \left[\frac{\partial^\gamma R}{\partial t^\gamma} + \frac{\partial^2 S}{\partial x^2} + (R^2 + S^2)S \right] dx = \int_0^1 W f_I(x,t)dx.$$

Applying partial integration, we get the weak form of the system over only one finite element $[x_m, x_{m+1}]$ as

(2.2)
$$\int_{x_m}^{x_{m+1}} (W \frac{\partial^\gamma S}{\partial t^\gamma} + \frac{\partial W}{\partial x} \frac{\partial R}{\partial x} - (R^2 + S^2)WR)dx$$

$$= W \frac{\partial R}{\partial x} \Big|_{x_m}^{x_{m+1}} - \int_{x_m}^{x_{m+1}} W f_r(x,t)dx,$$

$$\int_{x_m}^{x_{m+1}} bigl(W \frac{\partial^\gamma R}{\partial t^\gamma} - \frac{\partial W}{\partial x} \frac{\partial S}{\partial x} + (R^2 + S^2)WS)dx$$

$$= -W \frac{\partial S}{\partial x} \Big|_{x_m}^{x_{m+1}} + \int_{x_m}^{x_{m+1}} W f_I(x,t)dx.$$

Now, let us define the quadratic B-spline base functions. Partitioning the interval $[a,b]$ into N finite elements of uniformly equal length by knots x_m, $m = 0, 1, 2, \ldots, N$ such that $a = x_0 < x_1 \cdots < x_N = b$ and $h = x_{m+1} - x_m$. We define the quadratic B-splines $Q_m(x)$, $(m = -1(1)N)$, at the knots x_m are defined over the interval $[a,b]$ by [22]

(2.3)
$$Q_m(x) = \frac{1}{h^2} \begin{cases} (x_{m+2} - x)^2 - 3(x_{m+1} - x)^2 + 3(x_m - x)^2, & x \in [x_{m-1}, x_m], \\ (x_{m+2} - x)^2 - 3(x_{m+1} - x)^2, & x \in [x_m, x_{m+1}], \\ (x_{m+2} - x)^2, & x \in [x_{m+1}, x_{m+2}], \\ 0, \text{ otherwise.} \end{cases}$$

A basis for the functions defined over $[a, b]$ is formed by the set of splines $\{Q_{-1}(x), Q_0(x), \ldots, Q_N(x)\}$. Hence, a solution approximation $R_N(x, t)$ and $S_N(x, t)$ can be written in terms of the quadratic B-splines trial functions as:

$$R_N(\xi, t) = \sum_{j=-1}^{N} \delta_j(t) Q_j(\xi), \quad S_N(\xi, t) = \sum_{j=-1}^{N} \sigma_j(t) Q_j(\xi),$$

where $\delta_m(t)$'s and $\sigma_m(t)$'s are unknown, time dependent parameters which are going to be determined from the boundary and weighted residual conditions. Each quadratic B-spline covers three elements, thus each element $[x_m, x_{m+1}]$ is covered by three quadratic B-splines. For problem, the finite elements are identified with the interval $[x_m, x_{m+1}]$ and the elements knots x_m, x_{m+1}. When we use the nodal values R_m, S_m, R'_m, S'_m which are given in terms of the parameters $\delta_m(t)$, $\sigma_m(t)$

$$R_N(x_m) = R_m = \delta_{m-1} + \delta_m,$$

$$S_N(x_m) = S_m = \sigma_{m-1} + \sigma_m,$$

(2.4)

$$R'_N(x_m) = R'_m = 2(-\delta_{m-1} + \delta_m)/h,$$

$$S'_N(x_m) = S'_m = 2(-\sigma_{m-1} + \sigma_m)/h,$$

the variations of $R_N(x, t)$, $S_N(x, t)$ over the typical element $[x_m, x_{m+1}]$ is given by

(2.5) $$R_N(\xi, t) = \sum_{j=m-1}^{m+1} \delta_j(t) Q_j(\xi), \quad S_N(\xi, t) = \sum_{j=m-1}^{m+1} \sigma_j(t) Q_j(\xi).$$

If we take quadratic B-spline functions (2.3) instead of weighted functions and substituting the approximations (2.5) into equation system (2.2), we get

$$\sum_{j=m-1}^{m+1} \left(\int_0^h Q_i Q_j d\xi \right) \dot{\sigma}_j^e + \sum_{j=m-1}^{m+1} \left(\int_0^h Q'_i Q'_j d\xi \right) \delta_j^e - Z_m \sum_{j=m-1}^{m+1} \left(\int_0^h Q_i Q_j d\xi \right) \delta_j^e$$

(2.6) $$- \sum_{j=m-1}^{m+1} \left. (Q_i Q'_j) \right|_0^h \delta_j^e = - \int_0^h Q_i \tilde{f}_r(\xi, t) d\xi, \quad i = m-1, m, m+1,$$

$$\sum_{j=m-1}^{m+1} (\int_0^h Q_i Q_j d\xi) \dot{\delta}_j^e - \sum_{j=m-1}^{m+1} (\int_0^h Q_i' Q_j' d\xi) \sigma_j^e + Z_m \sum_{j=m-1}^{m+1} (\int_0^h Q_i Q_j d\xi) \sigma_j^e$$

(2.7)
$$+ \sum_{j=m-1}^{m+1} (Q_i Q_j')\Big|_0^h \sigma_j^e = \int_0^h Q_i \tilde{f}_I(\xi, t) d\xi, \quad i = m-1, m, m+1,$$

where $Z_m = R^2 + S^2$ and $\dot{\delta}, \dot{\sigma}$ denote the $\gamma-$ order fractional derivatives dependent to t. When we take

$$A_{ij}^e = \int_0^h Q_i Q_j d\xi, \quad B_{ij}^e = \int_0^h Q_i' Q_j' d\xi, \quad C_{ij}^e = Q_i Q_j'\Big|_0^h,$$

$$D_i^e = - \int_0^h Q_i \tilde{f}_r(\xi, t) d\xi, \quad E_i^e = \int_0^h Q_i \tilde{f}_I(\xi, t) d\xi,$$

equation system (2.6)-(2.7) can be written in the matrix form

(2.8)
$$A^e \dot{\sigma}^e + B^e \delta^e - Z_m A^e \delta^e - C^e \delta^e = D^e,$$
$$A^e \dot{\delta}^e - B^e \dot{\sigma}^e + Z_m A^e \sigma^e + C^e \sigma^e = E^e,$$

where $\delta^e = (\delta_{m-1}, \delta_m, \delta_{m+1})$ and $\sigma^e = (\sigma_{m-1}, \sigma_m, \sigma_{m+1})$. Evaluating the above integrals for $i, j, k = m-1, m, m+1$, by using quadratic B-spline functions A_{ij}^e, B_{ij}^e and C_{ij}^e, element matrices can be found as

$$A_{ij}^e = \int_0^h Q_i Q_j d\xi = \frac{h}{30} \begin{bmatrix} 6 & 13 & 1 \\ 13 & 54 & 13 \\ 1 & 13 & 6 \end{bmatrix},$$

$$B_{ij}^e = \int_0^h Q_i' Q_j' d\xi = \frac{2}{3h} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

$$C_{ij}^e = Q_i Q_j'\Big|_0^h = \frac{2}{h} \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Also, D_i^e and E_i^e element matrixes can be evaluated as

$$D_i^e = -\int_0^h Q_i \tilde{f}_r(\xi, t) d\xi = \begin{bmatrix} -\int_0^h Q_{m-1} \tilde{f}_r(\xi, t) d\xi \\ -\int_0^h Q_m \tilde{f}_r(\xi, t) d\xi \\ -\int_0^h Q_{m+1} \tilde{f}_r(\xi, t) d\xi \end{bmatrix},$$

$$E_i^e = \int_0^h Q_i \tilde{f}_I(\xi, t) d\xi = \begin{bmatrix} \int_0^h Q_{m-1} \tilde{f}_I(\xi, t) d\xi \\ \int_0^h Q_m \tilde{f}_I(\xi, t) d\xi \\ \int_0^h Q_{m+1} \tilde{f}_I(\xi, t) d\xi \end{bmatrix}.$$

From equation $Z_m = R_N^2 + S_N^2$, by substituting the nodal values

$$Z_m = (\delta_{m-1} + \delta_m)^2 + (\sigma_{m-1} + \sigma_m)^2$$

into equation (2.8), we obtain

(2.9)
$$A\dot{\sigma} + B\delta - A(Z_m)\delta - C\delta = D,$$

$$A\dot{\delta} - B\dot{\sigma} + A(Z_m)\sigma + C\sigma = E,$$

where $\delta = (\delta_{-1}, \delta_0, \ldots, \delta_{N-1}, \delta_N)$ and $\sigma = (\sigma_{-1}, \sigma_0, \ldots, \sigma_{N-1}, \sigma_N)$. For $m = 1(1)N - 2$, the generalized rows of matrices A, B, C can be written as

$$A : \frac{h}{30}(1, 26, 66, 26, 1), \quad B : \frac{2}{3h}(-1, -2, 6, -2, -1), \quad C : (0, 0, 0, 0, 0),$$

and that of $A(Z_m)$ as

$$\frac{h}{30}(Z_{m1}, 13Z_{m1} + 13Z_{m2}, 6Z_{m1} + 54Z_{m2} + 6Z_{m3}, 13Z_{m2} + 13Z_{m3}, Z_{m3}),$$

where

(2.10)
$$Z_{m1} = (\delta_{m-2} + \delta_{m-1})^2 + (\sigma_{m-2} + \sigma_{m-1})^2,$$

$$Z_{m2} = (\delta_{m-1} + \delta_m)^2 + (\sigma_{m-1} + \sigma_m)^2,$$

$$Z_{m3} = (\delta_m + \delta_{m+1})^2 + (\sigma_m + \sigma_{m+1})^2.$$

In system (2.9), if we write

(2.11)
$$\delta = \frac{\delta^n + \delta^{n+1}}{2}, \quad \sigma = \frac{\sigma^n + \sigma^{n+1}}{2},$$

Crank-Nicolson finite difference approximations instead of δ, σ and

$$(2.12) \quad \dot{\delta} = \frac{d^\gamma \delta}{dt^\gamma} = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{n-1} \left[(k+1)^{1-\gamma} - k^{1-\gamma} \right] \left[\delta^{n-k} - \delta^{n-k-1} \right],$$

$$(2.13) \quad \dot{\sigma} = \frac{d^\gamma \sigma}{dt^\gamma} = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{n-1} \left[(k+1)^{1-\gamma} - k^{1-\gamma} \right], \left[\sigma^{n-k} - \sigma^{n-k-1} \right]$$

$L1$ approximation instead of $\dot{\delta}_m$, $\dot{\sigma}_m$, we obtain $(2N+4) \times (2N+4)$ algebraic equation system as follows:

$$\frac{A}{(\Delta t)^\gamma \Gamma(2-\gamma)} \sigma^{n+1} + \frac{1}{2} \left[B - A(Z_m) - C \right] \delta^{n+1}$$

$$(2.14) \quad = \frac{A}{(\Delta t)^\gamma \Gamma(2-\gamma)} \sigma^n - \frac{1}{2} \left[B - A(Z_m) - C \right] \delta^n$$

$$- \frac{A}{(\Delta t)^\gamma \Gamma(2-\gamma)} \sum_{k=1}^{n} \left[(k+1)^{1-\gamma} - k^{1-\gamma} \right] \left[\sigma^{n-k} - \sigma^{n-k-1} \right] + D,$$

$$\frac{A}{(\Delta t)^\gamma \Gamma(2-\gamma)} \delta^{n+1} - \frac{1}{2} \left[B - A(Z_m) - C \right] \sigma^{n+1}$$

$$(2.15) \quad = \frac{A}{(\Delta t)^\gamma \Gamma(2-\gamma)} \delta^n + \frac{1}{2} \left[B - A(Z_m) - C \right] \sigma^n$$

$$- \frac{A}{(\Delta t)^\gamma \Gamma(2-\gamma)} \sum_{k=1}^{n} \left[(k+1)^{1-\gamma} - k^{1-\gamma} \right] \left[\delta^{n-k} - \delta^{n-k-1} \right] + E.$$

By vanishing the parameters δ_{-1}, δ_N, σ_{-1}, σ_N with the help of boundary conditions given with problem, we handle $(2N \times 2N)$ square matrix algebraic system. To start the iteration, we have to know the initial parameters δ^0, σ^0. After evaluating the initial parameters δ^0 and σ^0, the numerical solutions can be obtained by using these parameters in the equation system (2.14)–(2.15). We can attain numerical solutions at the intended time by writing these values in the R_N and S_N approximations. To enhance the approximation R_N and S_N solutions, we apply the iteration formulas

$$\delta_m^* = \delta_m^n + \frac{1}{2}(\delta_m^{n+1} - \delta_m^n), \quad \sigma_m^* = \sigma_m^n + \frac{1}{2}(\sigma_m^{n+1} - \sigma_m^n)$$

to nonlinear terms in Eq. system (2.14)–(2.15) for each time step.

Initial parameters can be obtained as follows: The initial vectors

(2.16)
$$\mathbf{d}^0 = (\delta_{-1}, \delta_0, \delta_1, \ldots, \delta_{N-2}, \delta_{N-1}, \delta_N)^T,$$
$$\tilde{\mathbf{d}}^0 = (\sigma_{-1}, \sigma_0, \sigma_1, \ldots, \sigma_{N-2}, \sigma_{N-1}, \sigma_N)^T$$

are determined with the aid of the initial and boundary conditions. It is necessary to have the initial numerical approximations $R_N(x, 0)$ and $S_N(x, 0)$ which satisfy the following conditions:

(2.17)
$$R_N(x, 0) = R(x_m, 0), \quad m = 0, 1, \ldots, N,$$
$$S_N(x, 0) = S(x_m, 0), \quad m = 0, 1, \ldots, N,$$
$$R'_N(x_0, 0) = R'(x_0, 0), \quad S'_N(x_0, 0) = S'(x_0, 0).$$

Hence, using these conditions lead to a two-diagonal systems of matrix of the forms

(2.18)
$$W\mathbf{d}^0 = \mathbf{b}, \quad W\tilde{\mathbf{d}}^0 = \tilde{\mathbf{b}},$$

where

$$W = \begin{bmatrix} \frac{-2}{h} & \frac{2}{h} & & & & & \\ 1 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & 1 & \\ & & & & & 1 & 1 \end{bmatrix}$$

and

$$\mathbf{b} = (R'(x_0, 0), R(x_0, 0), R(x_1, 0), \ldots, R(x_{N-2}, 0), R(x_{N-1}, 0), R(x_N, 0))^T,$$
$$\tilde{\mathbf{b}} = (S'(x_0, 0), S(x_0, 0), S(x_1, 0), \ldots, S(x_{N-2}, 0), S(x_{N-1}, 0), S(x_N, 0))^T.$$

2.1. Numerical Results

In this section, considered problem is solved numerically by using Galerkin finite element method. Now, consider the time fractional nonlinear Schrödinger equation (1.1) with the boundary conditions

$$U(0,t) = it^2, \quad U(1,t) = it^2, \quad t \geq 0$$

and the initial condition

$$U(x,0) = 0, \quad 0 \leq x \leq 1.$$

Additionally, the $f(x,t)$ is the form

$$f(x,t) = -\frac{2t^{2-\gamma}}{\Gamma(3-\gamma)} \cos(2\pi x) + (t^6 - 4\pi^2 t^2)\sin(2\pi x)$$
$$+ i\left(\frac{2t^{2-\gamma}}{\Gamma(3-\gamma)} \sin(2\pi x) + (t^6 - 4\pi^2 t^2)\cos(2\pi x)\right)$$

and the exact solution of this problem is given by [15]

$$U(x,t) = t^2(\sin(2\pi x) + i\cos(2\pi x)).$$

L_2 and L_∞ error norms which are evaluated with the numerical solutions of real and imaginary parts are given for $\Delta t = 0.005$, $\gamma = 0.50$, $t = 1$ and different values of N in the Tables 1–2, respectively.   Both of two tables show

Table 1. Error norms and numerical solutions of real part of the problem for $\gamma = 0.50$, $\Delta t = 0.005$, $t = 1$

x	$N = 25$	$N = 30$	$N = 35$	$N = 40$	Exact
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.2	0.942844	0.947022	0.949524	0.951150	0.951057
0.4	0.581600	0.584759	0.586665	0.587916	0.587785
0.6	-0.585850	-0.586937	-0.587577	-0.587970	-0.587785
0.8	-0.945695	-0.948541	-0.950228	-0.951298	-0.951057
1.0	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	5.338936	2.598532	0.966275	0.170855	
$L_\infty \times 10^3$	9.117853	4.239642	1.331924	0.562147	

that the error norms decrease as the partition number N increases in the numerical solutions obtained by Galerkin method. In Tables 3–4, the numerical of real and imaginary parts of the solutions and L_2, L_∞ error norms are given, respectively for $\gamma = 0.50$, $N = 40$, $t = 1$ and different values of Δt. From the tables, it is seen that the error norms decrease for the increasing values of Δt

Table 2. Error norms and numerical solutions of imaginary part of the problem for $\gamma = 0.50$, $\Delta t = 0.005$, $t = 1$

x	$N = 25$	$N = 30$	$N = 35$	$N = 40$	Exact
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
0.2	0.314710	0.311708	0.309935	0.308792	0.309017
0.4	-0.795690	-0.802782	-0.807021	-0.809772	-0.809017
0.6	-0.795791	-0.802899	-0.807144	-0.809898	-0.809017
0.8	0.314552	0.311525	0.309739	0.308586	0.309017
1.0	1.000000	1.000000	1.000000	1.000000	1.000000
$L_2 \times 10^3$	8.643806	4.248708	1.583606	0.311933	
$L_\infty \times 10^3$	14.545977	6.790519	2.133489	0.911772	

Table 3. Error norms and numerical solutions of real part of the problem for $\gamma = 0.50$, $N = 40$, $t = 1$

x	$\Delta t = 0.0125$	$\Delta t = 0.01$	$\Delta t = 0.008$	$\Delta t = 0.005$	Exact
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.2	0.959018	0.956395	0.954297	0.951150	0.951057
0.4	0.594085	0.592028	0.590383	0.587916	0.587785
0.6	-0.590194	-0.589454	-0.588861	-0.587970	-0.587785
0.8	-0.956777	-0.954952	-0.953491	-0.951298	-0.951057
1.0	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	5.305685	3.575290	2.191628	0.170855	
$L_\infty \times 10^3$	10.472901	7.168726	4.524940	0.562147	

Table 4. Error norms and numerical solutions of imaginary part of the problem for $\gamma = 0.50$, $N = 40$, $t = 1$

x	$\Delta t = 0.0125$	$\Delta t = 0.01$	$\Delta t = 0.008$	$\Delta t = 0.005$	Exact
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
0.2	0.303001	0.304932	0.306476	0.308792	0.309017
0.4	-0.824177	-0.819376	-0.815534	-0.809772	-0.809017
0.6	-0.824501	-0.819635	-0.815741	-0.809898	-0.809017
0.8	0.302458	0.304500	0.306134	0.308586	0.309017
1.0	1.000000	1.000000	1.000000	1.000000	1.000000
$L_2 \times 10^3$	8.706987	5.860150	3.582478	0.311933	
$L_\infty \times 10^3$	16.864538	11.544108	7.286787	0.911772	

time step. Lastly the numerical solutions of real and imaginary parts, L_2 and L_∞ error norms for the considered problem are given for $\Delta t = 0.005$, $N = 40$, $t = 1$ and different values of γ in Tables 5–6. The L_∞ error norms obtained by radial basis functions given in Ref. [15] and L_2, L_∞ error norms of the numerical solutions of real and imaginary parts of the considered problem obtained by B-spline Galerkin method for $N = 30$, $t = 1$ are given in Tables 7–8 for $\gamma = 0.10$, $\gamma = 0.30$, respectively. The L_∞ error norms of the numerical results that obtained by this method with the choice $\Delta t = 0.008$ are smaller than the L_∞ error norms that are given in Ref. [15]. In Figures 1–2, the absolute errors

Table 5. Error norms and numerical solutions of real part of the problem for $\Delta t = 0.005$, $N = 40$, $t = 1$

x	$\gamma = 0.10$	$\gamma = 0.30$	$\gamma = 0.70$	$\gamma = 0.90$	Exact
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.2	0.951334	0.951257	0.951010	0.950787	0.951057
0.4	0.588199	0.588084	0.587685	0.587291	0.587785
0.6	-0.587683	-0.587799	-0.588207	-0.588610	-0.587785
0.8	-0.951110	-0.951187	-0.951445	-0.951679	-0.951057
1.0	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	0.240379	0.186763	0.271229	0.531091	
$L_\infty \times 10^3$	0.468496	0.499439	0.665903	0.844600	

Table 6. Error norms and numerical solutions of imaginary part of the problem for $\Delta t = 0.005$, $N = 40$, $t = 1$

x	$\gamma = 0.10$	$\gamma = 0.30$	$\gamma = 0.70$	$\gamma = 0.90$	Exact
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
0.2	0.308882	0.308856	0.308677	0.308454	0.309017
0.4	-0.809636	-0.809678	-0.809933	-0.810222	-0.809017
0.6	-0.809770	-0.809814	-0.810029	-0.810235	-0.809017
0.8	0.308659	0.308632	0.308524	0.308445	0.309017
1.0	1.000000	1.000000	1.000000	1.000000	1.000000
$L_2 \times 10^3$	0.436366	0.499439	0.485480	0.846714	
$L_\infty \times 10^3$	0.776350	0.328332	1.061380	1.314998	

Table 7. Comparison of error norms of problem with the results from Ref. [15] for $N = 30$, $\gamma = 0.10$, $t = 1$

	Real		Imaginary	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
$\Delta t = 0.008$	0.5442	0.4981	0.8560	0.7526
[15]	———	2.8536	———	2.1753

Table 8. Comparison of error norms of problem with the results from Ref. [15] for $N = 30$, $\gamma = 0.30$, $t = 1$

	Real		Imaginary	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
$\Delta t = 0.008$	0.5254	0.4382	0.8846	0.6685
[15]	———	2.8610	———	2.1771

of real and imaginary parts solutions are presented for $\Delta t = 0.005$, $N = 40$, $t = 1$ and different values of γ.

(a) $\gamma = 0.10$

(b) $\gamma = 0.30$

(c) $\gamma = 0.50$

(d) $\gamma = 0.70$
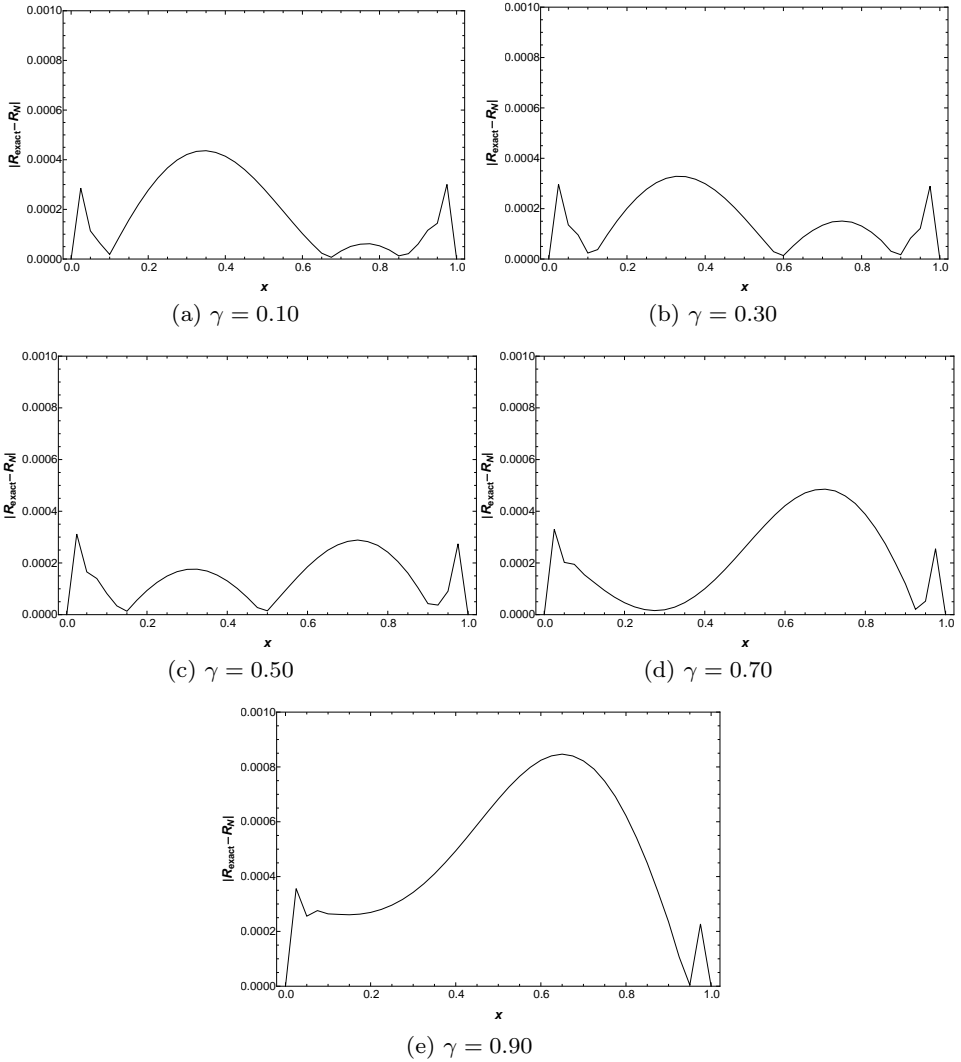
(e) $\gamma = 0.90$

Figure 1. Error distributions of real part of Problem for $\Delta t = 0.005$, $N = 40$, $t = 1$

3. Conclusion

In this paper, quadratic B-spline Galerkin method has been applied to achieve the numerical solutions of the time fractional Schrödinger equation. The time fractional derivative operators are made allowance for the Caputo fractional derivatives. It can be easily viewed from the numerical solutions
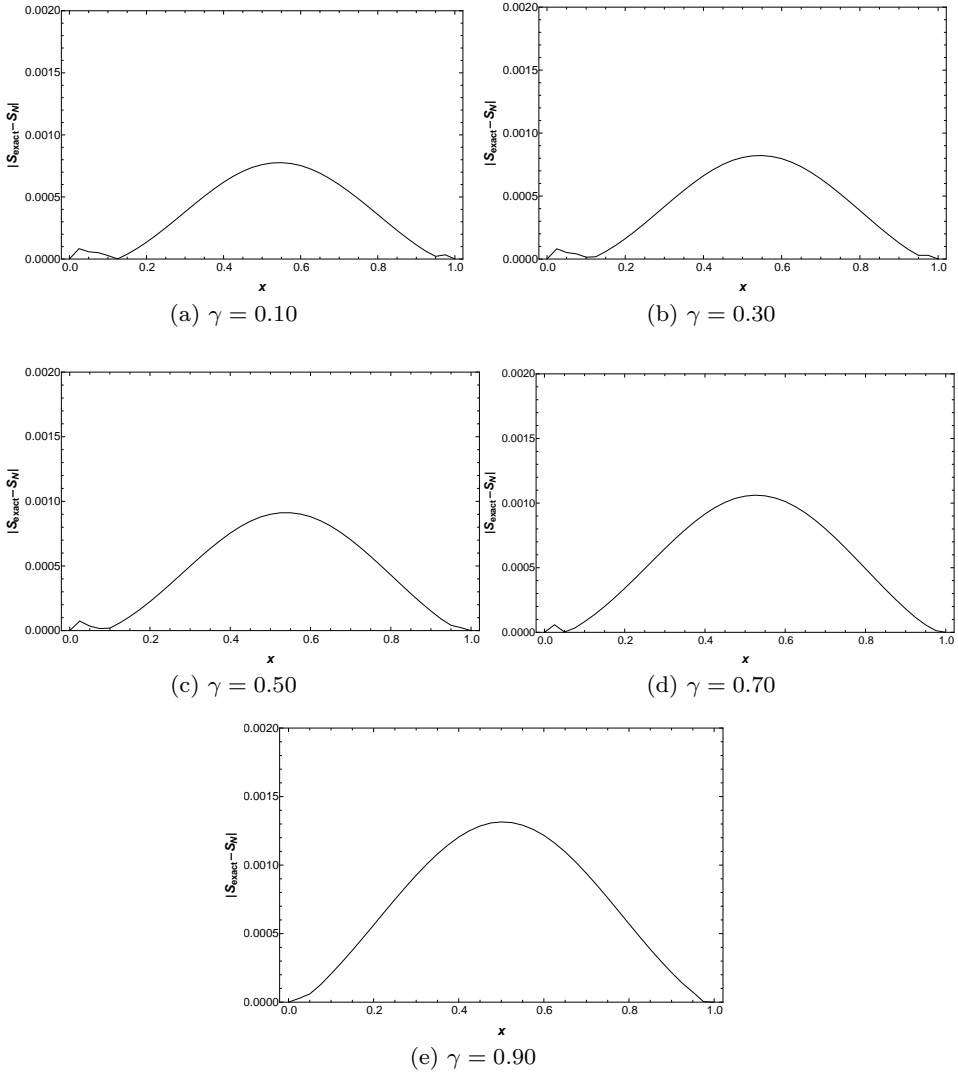
Figure 2. Error distributions of imaginary part of Problem for $\Delta t = 0.005$, $N = 40$, $t = 1$

and error norms in tables obtained that this method is an extremely good to achieve numerical solutions of time fractional partial differential equations arising in physics and engineering.

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Department of Mathematics   Department of Mathematics
Inönü University   Mustafa Kemal University
Malatya, 44280   Hatay, 31000
Turkey   Turkey
                                             e-mail: orkun.tasbozan@inonu.edu.tr