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UNIVERSALLY KURATOWSKI–ULAM SPACES AND OPEN-OPEN GAMES

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Abstract. We examine the class of spaces in which the second player has a winning strategy in the open-open game. We show that this spaces are not universally Kuratowski–Ulam. We also show that the games G and G_7 introduced by P. Daniels, K. Kunen, H. Zhou [Fund. Math. **145** (1994), no. 3, 205–220] are not equivalent.

1. Introduction

First we shall recall some game introduced in [2] called G_2. Let X be a topological space equipped with a topology $\mathcal{T}$ and let $\mathcal{B} \subseteq \mathcal{T}$ be its base. The length of the game is ω. Two players I and II take turns playing. At the n-th move II chooses a family $\mathcal{P}_n$ consisting of open non-empty subset of X such that $\operatorname{cl} \bigcup \mathcal{P}_n = X$, then I picks a $V_n \in \mathcal{P}_n$. I wins iff $\operatorname{cl} \bigcup_{n\in\omega} V_n = X$. Otherwise player II wins. Denote by D_{cov} a collection of families $\mathcal{F}$ consisting of open sets with $\operatorname{cl} \bigcup \mathcal{F} = X$. We say that $\sigma_{cov} \colon (\bigcup D_{cov})^{<\omega} \to D_{cov}$ is a *winning strategy for player II* in the game G_2 whenever, for any sequence $U_0, U_1, \ldots$ consisting of non-empty open subsets with $U_0 \in \sigma_{cov}(\emptyset) = \mathcal{P}_0 \in D_{cov}$ and $U_n \in \sigma_{cov}(U_0, U_1, \ldots, U_{n-1}) = \mathcal{P}_n \in D_{cov}$, for all $n \in \omega$, there holds $\operatorname{cl} \bigcup_{n\in\omega} U_n \neq X$.

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In the paper [2] the authors introduced an open-open game. We say that G is *an open-open game* of length ω if two players take turns playing; a round consists of player I choosing a non-empty open set $U \subseteq X$ and player II choosing a non-empty open $V \subseteq U$; I wins if the union of II's open sets is dense in X, otherwise II wins. Suppose that there exists a function

$$s_{op}\colon \bigcup\{\mathcal{T}^n : n \geq 0\} \to \mathcal{T}$$

such that for each sequence $V_0, V_1, \ldots$ consisting of non-empty elements of $\mathcal{T}$ with $s_{op}(V_0) \subseteq V_0$ and $s_{op}(V_0, V_1, \ldots, V_n) \subseteq V_n$, for all $n \in \omega$, there holds $\mathrm{cl} \bigcup_{n\in\omega} V_n \neq X$. Then the function s_{op} is called a *winning strategy for II player* in the open-open game and we say that the space X is *II-favorable.*

It is known [2] that the open-open game G is equivalent to G_2. We consider only games with the length equal to ω. In [2] the authors introduced a game G_7 which is played as follows: In the n-th inning II chooses $\mathcal{O}_n$, a family of open sets with $\bigcup \mathcal{O}_n$ dense in X. I responds with $\mathcal{T}_n$, a finite subfamily of $\mathcal{O}_n$; I wins if $\bigcup_{n\in\omega} \mathcal{T}_n$ is dense subset of X; otherwise, II wins.

According to A. Szymański [13] a sequence $\{\mathcal{P}_n : n \in \omega\}$ of open families in X is a *tiny sequence* if

(1) $\bigcup \mathcal{P}_n$ is dense in X for all $n \in \omega$
(2) if $\mathcal{F}_n$ is a finite subfamily of $\mathcal{P}_n$ for each $n \in \omega$ then $\bigcup\{\bigcup \mathcal{F}_n : n \in \omega\}$ is not dense in X.

We call a sequence $\{\mathcal{P}_n : n \in \omega\}$ of open families in X a *1-tiny sequence* if

(1) $\bigcup \mathcal{P}_n$ is dense in X for all $n \in \omega$
(2) if F_n is a member of $\mathcal{P}_n$ for each $n \in \omega$ then $\bigcup\{F_n : n \in \omega\}$ is not dense in X.

M. Scheepers used in the paper [12] negation of the existence of tiny sequence, and 1-tiny sequence - called these properties $S_{fin}(\mathcal{D}, \mathcal{D})$ and $S_1(\mathcal{D}, \mathcal{D})$ respectively. In this paper we refer to notions tiny sequence and 1-tiny sequence, because in some situations (Theorem 1.1 and 1.2) we can define them.

Recall another game G_4 introduced in [2]. In the n-th inning player I chooses finite open family $\mathcal{A}_n$. Player II responds with a finite, open family $\mathcal{B}_n$ with $|\mathcal{B}_n| = |\mathcal{A}_n|$ and for each $V \in \mathcal{A}_n$ there exists $W \in \mathcal{B}_n$ such that $W \subseteq V$. I wins if $\bigcup_{n\in\omega} \bigcup \mathcal{B}_n$ is dense subset of X; otherwise, II wins. One can prove that the game G_7 is equivalent to the game G_4 in a way similar to the proof of the equivalence between games G and G_2.

From now on we consider only c.c.c. spaces.

Theorem 1.1 (M. Scheepers [12, Theorem 2]). *II has a winning strategy in the game G_7 if and only if there exists a tiny sequence.*

THEOREM 1.2 (M. Scheepers [12, Theorem 14]). *Player II has a winning strategy in the game G_2 if and only if there exists a 1-tiny sequence.*

2. The main results

Recall that X is called a *II-favorable* space if player II a has winning strategy in the game G. If player I has a winning strategy in the game G then we say that the space is *I-favorable.*

The following theorem was proven by K. Kuratowski and S. Ulam, see [9]. In order to formulate it, let us recall that: a π-base is a family of open, nonempty sets such that any open set contains a set from this family, and the π-weight of a space is the smallest cardinality of a π-base in this space.

Let X and Y be topological spaces such that Y has countable π-weight. If $E \subseteq X \times Y$ is a nowhere dense set, then there is a meager set $P \subseteq X$ such that the section $E_x = \{y : (x, y) \in E\}$ is nowhere dense in Y for each point $x \in X \setminus P$.

A space Y is *universally Kuratowski–Ulam* (for short, *uK-U space*), whenever for a topological space X and a nowhere dense set $E \subseteq X \times Y$ the set

$$\{x \in X : \{y \in Y : (x, y) \in E\} \text{ is not nowhere dense in } Y\}$$

is meager in X, see D. Fremlin [6] (compare [3]). In the paper [7] authors have shown that a compact I-favorable space is universally Kuratowski–Ulam and posed a question: *Does there exist a compact universally Kuratowski–Ulam space which is not I-favorable*? We will partially answer to this question, namely we will prove that a II-favorable space is not universally Kuratowski–Ulam space.

THEOREM 2.1. *Let X be a dense in itself space with a π-base $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$, where $\mathcal{B}_n$ is a maximal family of pairwise disjoint open sets for $n \in \omega$ and let Y be II-favorable c.c.c. space. Then the Kuratowski–Ulam theorem does not hold in $X \times Y$.*

PROOF. By Theorem 1.2 there is a 1-tiny sequence $\{\mathcal{P}_n : n \in \omega\}$. Since the space Y satisfies c.c.c., we can assume that each $\mathcal{P}_{n+1}$ is a countable, open, pairwise disjoint family. We can also assume that every $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$, i.e. each member of $\mathcal{P}_{n+1}$ is a subset of a member of $\mathcal{P}_n$. Let $\{V^n_\sigma : \sigma \in {}^n\mathbb{N}\}$ be an enumeration of the family $\mathcal{P}_n$ such that for each $\tau \in {}^{n-1}\mathbb{N}$, $\{V^n_{\tau\frown k} : k \in \mathbb{N}\} = \mathcal{P}_n$.

We can assume that $\mathcal{B}_{n+1}$ is a refinement of $\mathcal{B}_n$ and $|\{V \in \mathcal{B}_{n+1} : V \subseteq U\}| \geq \omega$ for each $U \in \mathcal{B}_n$. For each $n \in \mathbb{N}$ fix a function $f_n \colon \mathcal{B}_n \to {}^n\mathbb{N}$ such that for a fixed $U \in \mathcal{B}_n$ we have

$$\{f_{n+1}(V) : V \in \mathcal{B}_{n+1} \text{ and } V \subseteq U\} = f_n(U)^\frown\mathbb{N}. \tag{2.1}$$

Therefore, there holds the condition:

$$\text{if } V \subset U \text{ then } f_{n+1}(V) \supset f_n(U) \text{ for every } V \in \mathcal{B}_{n+1} \text{ and } U \in \mathcal{B}_n. \tag{2.2}$$

Consider an open set

$$F = \bigcup \Big\{ \bigcup \{U \times V^n_{f_n(U)} : U \in \mathcal{B}_n\} : n \in \mathbb{N} \Big\}.$$

We shall show that F is dense and $F_x = \{y \in Y : (x,y) \in F\}$ is not dense for each $x \in X$. If $x \in X \setminus \bigcap\{\bigcup \mathcal{B}_n : n \in \mathbb{N}\}$ then it is easy to see that E_x is not dense. If $x \in \bigcap\{\bigcup \mathcal{B}_n : n \in \mathbb{N}\}$ then by condition (2.2) there is $\sigma \in {}^{\mathbb{N}}\mathbb{N}$ such that for each $n \in \mathbb{N}$ there exists $U_n \in \mathcal{B}_n$ with $f_n(U_n) = \sigma|n$ and $x \in \bigcap\{U_n : n \in \mathbb{N}\}$, hence $F_x = \bigcup\{V^n_{\sigma|n} : n \in \mathbb{N}\}$. Since $V^n_{\sigma|n} \in \mathcal{P}_n$ for each $n \in \mathbb{N}$ and $\{\mathcal{P}_n : n \in \omega\}$ is a 1-tiny sequence the set $\bigcup\{V^n_{\sigma|n} : n \in \mathbb{N}\}$ is not dense.

Now we show that F is a dense set. Let $U \times W$ be any open set. Since $\mathcal{B}$ is a π-base there are $n \in \mathbb{N}$ and $U_0 \in \mathcal{B}_n$ such that $U_0 \subseteq U$. Let $\sigma = f_n(U_0)$, since $\{V^{n+1}_{\sigma^\frown k} \colon k \in \mathbb{N}\}$ is a dense family, we get that $W \cap V^{n+1}_{\sigma^\frown k} \neq \emptyset$ for some $k \in \mathbb{N}$. By (1), we may take $U_1 \subseteq U_0$ such that $U_1 \in \mathcal{B}_{n+1}$ and $f_{n+1}(U_1) = \sigma^\frown k$. Thus $U_1 \times V^{n+1}_{f_{n+1}(U_1)} \cap U \times W \neq \emptyset$. □

Since $\mathbb{R}$ with natural topology satisfies assumption of the above theorem and every universally Kuratowski–Ulam space is c.c.c. space, we get the following theorem.

Theorem 2.2. *A II-favorable space is not universally Kuratowski–Ulam space.*

Following [10, pp. 86–91] recall category measure space. If X is a topological space with finite measure μ defined on the σ-algebra S of sets having the Baire property, and if $\mu(E) = 0$ if and only if E is of a meager set, then (X, S, μ) is called a *category measure space*. An example of a regular Baire space which is a category measure space, is an open interval $(0,1)$ with Lebesgue measure μ_l and density topology $\mathcal{T}_d$, see [10]. For density topology and measurable set $A \subseteq (0,1)$ the following conditions are equivalent:

(1) $\mu_l(A) = 0$,
(2) A is closed and nowhere dense.

In the space $((0,1), \mathcal{T}_d)$ there is a 1-tiny sequence but there is no tiny sequence. Indeed, define a 1-tiny sequence in the following way: let $\mathcal{P}_n = \{U : U \in \mathcal{T}_d \text{ and } \mu_l(U) \leq \frac{1}{3^n}\}$. If $\{U_n : n \in \mathbb{N}\}$ is a family chosen by player I then $\mu_l(\bigcup\{U_n : n \in \mathbb{N}\}) \leq \frac{1}{2}$. Therefore $\{U_n : n \in \mathbb{N}\}$ is not a dense family. Now assume that there exists a tiny sequence $\{\mathcal{P}_n : n \in \mathbb{N}\}$. In each stage we choose a finite subfamily $\mathcal{R}_n \subset \mathcal{P}_n$ such that $\mu_l(\bigcup\{\bigcup \mathcal{R}_i : i \leq n\}) \geq 1 - \frac{1}{n}$, hence we get a dense family $\bigcup\{\mathcal{R}_n : n \in \mathbb{N}\}$.

The authors of the paper [2] posed a question (Question 4.3): *Does a player have a winning strategy in the game G if and only if the same player has a winning strategy in the game G_7.* The author of paper [12] showed that if $\text{cov}(\mathcal{M}) < \mathfrak{d}$ the answer is NO. We show that games G and G_7 are not equivalent.

Corollary 2.3. *The game G is not equivalent to the game G_7.*

Proof. By Theorem 1.2 a winning strategy of II player in the game G is equivalent to the existence of a 1-tiny sequence and by Theorem 1.1 the existence of a winning strategy of player II in the game G_7 is equivalent to the existence of a tiny sequence. Since in the space $((0,1), \mathcal{T}_d)$ there is a 1-tiny sequence but there is no tiny sequences we get that games G and G_7 are not equivalent. □

Since the game G_7 is equivalent to the game G_4, we get the following:

Corollary 2.4. *The game G is not equivalent to the game G_4.*

3. Some remarks

It is known that on the ω_1 with discrete topology II player has a winning strategy in the game G_7, but one can pose a question:

Is it possible to construct a tiny sequence $\{\mathcal{P}_n : n \in \omega\}$ on a discrete space of the size ω_1 with $|\mathcal{P}_n| = \omega$ for all $n \in \omega$?

The following Remark 3.1 gives us the answer - it is possible if and only if the dominating number is equal ω_1. This is reformulation of well know results about critical cardinal number, see W. Just, A.W. Miller, M. Scheepers and P.J. Szeptycki [5]; D. Fremlin, A.W. Miller [4] and B. Tsaban [14].

Recall that $f \leq^* g$ denotes that for almost all $n \in \omega$ holds $f(n) \leq g(n)$, where f, g are functions defined on natural numbers. A family $\mathcal{R} \subseteq {}^\omega\omega$ is a *dominating* family if for each $f \in {}^\omega\omega$ there is $g \in \mathcal{R}$ such that $f \leq^* g$. The *dominating number* $\mathfrak{d}$ is the smallest cardinality of a dominating family:

$$\mathfrak{d} = \min\{|\mathcal{R}| : \mathcal{R} \text{ is dominating }\}.$$

Remark 3.1. The smallest cardinality κ such that there exists a tiny sequence $\{\mathcal{P}_n \colon n \in \omega\}$ on the discrete space of the size κ with $|\mathcal{P}_n| = \omega$ for all $n \in \omega$ is equal to $\mathfrak{d}$.

Proof. Let X be any discrete space for which there exists a tiny sequence $\{\mathcal{P}_n \colon n \in \omega\}$. We can assume that every $\mathcal{P}_n$ is a partition of X into countably many blocks $\{X_0^n, X_1^n, \ldots\}$, so we may define for each $x \in X$ a function $f_x \colon \omega \to \omega$ in the following way: $f_x(n) = k$ whenever $x \in X_k^n$. Take an arbitrary function $f \colon \omega \to \omega$, and any $x \in X \setminus \bigcup\{\bigcup\{X_k^n \colon k \leq f(n)\} \colon n < \omega\}$, then f is dominated by the function f_x. It shows that $\{f_x \colon x \in X\}$ is a dominating family, hence $|X| \geq \mathfrak{d}$.

Now, let $\mathcal{F} \subset \omega^\omega$ be a dominating family of the cardinality $\mathfrak{d}$. Without loss of generality assume that for each function $f \colon \omega \to \omega$ there is $g \in \mathcal{F}$ such that $f(n) < g(n)$ for all $n < \omega$. We treat $\mathcal{F}$ as a discrete topological space. For $n, k \in \omega$ put $A_k^n = \{f \in \mathcal{F} \colon f(n) \leq k\}$ and set $\mathcal{P}_n = \{A_k^n \colon k < \omega\}$. Of course, each family $\mathcal{P}_n$ is increasing and has the union equal to $\mathcal{F}$. From each $\mathcal{P}_n$ take some single $A_{f(n)}^n$ where $f \colon \omega \to \omega$. If $\bigcup\{A_{f(n)}^n \colon n < \omega\}$ was equal to $\mathcal{F}$, then it would contain such a function g that $g(n) > f(n)$ for all $n \in \omega$, but it is not the case. Therefore $\{\mathcal{P}_n \colon n \in \omega\}$ is a tiny sequence. □

Recall a definition of a Baire number $\operatorname{cov}(\mathcal{M})$ for the ideal $\mathcal{M}$ of meager subsets of real line $\mathbb{R}$:

$$\operatorname{cov}(\mathcal{M}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{M} \text{ and } \bigcup \mathcal{A} = \mathbb{R}\}.$$

T. Bartoszyński [1] proved that $\operatorname{cov}(\mathcal{M})$ is the cardinality of the smallest family $\mathcal{F} \subseteq {}^\omega\omega$ such that

$$\forall (g \in {}^\omega\omega)\ \exists (f \in \mathcal{F})\ \forall (n \in \omega)\ f(n) \neq g(n).$$

We get another well known characterization of such families by a 1-tiny sequence.

Remark 3.2. The smallest cardinality κ such that there exists a 1-tiny sequence $\{\mathcal{P}_n \colon n \in \omega\}$ on the discrete space of the size κ with $|\mathcal{P}_n| = \omega$ for all $n \in \omega$ is equal to $\operatorname{cov}(\mathcal{M})$.

We give the proof for the sake of completeness. We shall prove that the smallest cardinality of a family $\mathcal{F} \subseteq {}^{\omega}\omega$ such that

$$(*) \qquad \forall (g \in {}^{\omega}\omega)\ \exists (f \in \mathcal{F})\ \forall (n \in \omega)\ f(n) \neq g(n)$$

is equal to the smallest cardinality κ such that there exists a 1-tiny sequence $\{\mathcal{P}_n : n \in \omega\}$ on the discrete space κ with $|\mathcal{P}_n| = \omega$ for all $n \in \omega$.

Proof. Let $\mathcal{F} = \{f_\alpha : \alpha < \kappa\} \subseteq {}^{\omega}\omega$ be a family with the property $(*)$. Define $A^i_n = \{f \in \mathcal{F} : f(i) = n\}$ for every $i, n \in \omega$. Let $\mathcal{P}_i = \{A^i_n : n \in \omega\}$ for $i \in \omega$. We will show that $\{\mathcal{P}_i : i \in \omega\}$ is a 1-tiny sequence. Assume that we have chosen $A^i_{n_i} \in \mathcal{P}_i$ for each $i \in \omega$. Define a function $g(i) = n_i$ for $i \in \omega$. Since $\mathcal{F}$ satisfies $(*)$ there is $f \in \mathcal{F}$ such that $f(i) \neq g(i)$ for each $i \in \omega$. Therefore we get $f \in \mathcal{F} \setminus \bigcup\{A^i_{n_i} : i \in \omega\}$.

Let $\{\mathcal{P}_n : n \in \omega\}$ be a 1-tiny sequence with $|\mathcal{P}_n| = \omega$ and $\bigcup \mathcal{P}_n = \kappa$ for each $n \in \omega$. We can assume that each $\mathcal{P}_n$ consists of pairwise disjoint subsets of κ. Let $\{A^n_k : k \in \omega\}$ be a enumeration of $\mathcal{P}_n$. We define a function $f_x \in {}^{\omega}\omega$ for each $x \in \kappa$ in the following way: $f_x(i) = n$, where $x \in A^i_n$ for each $i \in \omega$. The family $\{f_x : x \in \kappa\}$ satisfies $(*)$. Indeed, let $g \in {}^{\omega}\omega$ be any function. Since $\{\mathcal{P}_n : n \in \omega\}$ is a 1-tiny sequence, choose $x \in \kappa \setminus \bigcup\{A^i_{g(i)} : i \in \omega\}$. Finally, observe that $f_x(i) \neq g(i)$ for every $i \in \omega$. □

We shall recall definition of the bounding number

$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^{\omega}\omega \text{ and } \forall (g \in {}^{\omega}\omega)\ \exists (f \in \mathcal{F})\ \neg(f \leq^* g))\}.$$

We say that a sequence $\{\mathcal{P}_n : n \in \omega\}$ of open families in X is a $\mathfrak{b}$-*tiny sequence* if

(1) $\bigcup \mathcal{P}_n$ is dense in X for all $n \in \omega$;
(2) if $\mathcal{F}_n$ is a finite subfamily of $\mathcal{P}_n$ for each $n \in \omega$, then there exists strictly increasing sequence $\{n_i : i \in \omega\}$ such that

$$\bigcup\Big\{\bigcup \mathcal{F}_{n_i} : i \in \omega\Big\}$$

is not dense in X.

We get the next reformulation of the bounding number.

Remark 3.3. The smallest cardinality κ such that there exists a $\mathfrak{b}$-tiny sequence $\{\mathcal{P}_n : n \in \omega\}$ on the discrete space of the size κ with $|\mathcal{P}_n| = \omega$ for all $n \in \omega$ is equal to $\mathfrak{b}$.

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