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A GENERAL FIXED POINT THEOREM FOR IMPLICIT CYCLIC MULTI-VALUED CONTRACTION MAPPINGS

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Abstract. In this paper, a general fixed point theorem for cyclic multi-valued mappings satisfying an implicit relation from [19] different from implicit relations used in [13] and [23], generalizing some results from [22], [15], [13], [14], [16], [10] and from other papers, is proved.

1. Introduction

In 2003, Kirk et al. [11] extended Banach contraction principle to a case of cyclic contractive mappings. In [16], Petric extended most of the fundamental metrical fixed point theorems in literature (Chatterjee, Reich, Hardy–Rogers, Ćirić) to a cyclic contractive mappings. Other new results are obtained in [22], [13], [14], [15], [16], [5], [9], [10]. Several extensions of these results have appeared in literature.

On the other hand, Banach's contraction principle is extended to multi-valued mappings by Nadler in [12]. Afterward, an interesting and rich fixed point theory for set-valued mapping was developed in many directions. The theory of multi-valued mappings has applications in optimization problem, control theory, differential and integral theory, economics, informatics and

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many branches of analysis. Quite recently, Sintunavarat and Kumam [23] initiated the study of common fixed points for cyclic generalized multi-valued contraction mappings.

The study of fixed points for mappings satisfying implicit relations is initiated in [17], [18], [5], [6], [3]. Quite recently, the method is used in the study of fixed points for mappings satisfying a contractive condition of integral type, in fuzzy metric spaces and intuitionistic metric spaces. With this method, the proofs of some fixed point theorems are more simple. Also, the method allows the study of local and global properties of fixed point structures.

Quite recently, Nashin et al. [13] used a method from [1] and introduced an implicit relation-type-cyclic contractive for mappings in metric spaces for such mappings.

The study of fixed points for multi-valued mappings satisfying an implicit relation is initiated in [19], [20] and other papers.

In this paper, a general fixed point theorem for cyclic multi-valued mappings satisfying an implicit relation from [19] different from implicit relations used in [13] and [23], generalizing some results from [22], [11], [13], [14], [16] and from other papers, is proved.

2. Preliminaries

Definition 2.1 ([11], [15]). Let (X, d) be a metric space. Let p be a positive integer, $A_1, A_2, \dots, A_p$ be nonempty closed subsets of X, $Y = \bigcup_{i=1}^{p} A_i$ and $T\colon Y \to Y$. Then Y is said to be a *cyclical representation of* Y *with respect to* T if

1) A_i, $i = 1, 2, \dots, p$, are nonempty closed sets,
2) $T(A_1) \subset A_2$, $T(A_2) \subset A_3, \dots, T(A_{p-1}) \subset A_p$, $T(A_p) \subset A_1$.

In [2] is introduced a new class of implicit relations.

Definition 2.2 ([2]). Let $\mathcal{T}$ be the set of all real continuous function $T\colon \mathbb{R}_+^6 \to \mathbb{R}$ where $\mathbb{R}$ is the set of all real numbers and $\mathbb{R}_+ = [0, \infty)$, satisfying the following conditions:

(T_1) $T(t_1, \dots, t_6)$ is non-increasing in variables $t_2, \dots, t_6$,

(T_2) there exists a right continuous function $f\colon \mathbb{R}_+ \to \mathbb{R}$, $f(0) = 0$, $f(t) < t$ for $t > 0$, such that for $u, v > 0$, the condition $T(u, v, u, v, 0, u + v) \le 0$ or $T(u, v, 0, 0, v, v) \le 0$ implies $u \le f(v)$,

(T_3) $T(u, 0, u, 0, 0, u) > 0$, $T(u, u, 0, 0, u, u) > 0$, $\forall u > 0$.

Definition 2.3. Let (X,d) be a metric space, p a positive integer, A_1, $A_2,\dots,A_p$ nonempty closed subsets of X and $Y=\bigcup_{i=1}^{p} A_i$. An operator $f\colon Y\to Y$ is called an *implicit relation type cyclic contractive mappings* if

1) $Y=\bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to f,
2) for every $(x,y)\in A_i\times A_{i+1}$, $i=1,2,...,p$ and $A_{p+1}=A_1$,

$$T(d(fx,fy),d(x,y),d(x,fx),d(y,fy),d(x,fy),d(y,fx))\le 0$$

for some $T\in\mathcal{T}$.

The following theorem is proved in [13].

Theorem 2.1. *Let (X,d) be a complete metric space, $p\in\mathbb{N}$, $A_1,\dots,A_p$, nonempty closed subsets of X and $Y=\bigcup_{i=1}^{p} A_i$. Suppose that $f\colon Y\to Y$ is an implicit relation type-cyclic contractive mapping for some $T\in\mathcal{T}$. Then f has an unique fixed point. Moreover, the fixed point of f belongs to $\bigcap_{i=1}^{p} A_i$.*

By this theorem, as particular cases we obtain some results by [11], [14], [15], [16].

Let (X,d) be a metric space. We denote by $CB(X)$ the set of all nonempty bounded closed subsets of (X,d) and by H the Hausdorff–Pompeiu metric on $CB(X)$, i.e.

$$H(A,B)=\max\{\sup_{x\in A}\{d(x,B)\},\sup_{x\in B}\{d(x,A)\}\},$$

where $A,B\in CB(X)$ and

$$d(x,A)=\inf_{y\in A}\{d(x,y)\}.$$

It is well known that $(CB(X),H)$ is a metric space and the completness of (X,d) implies the completness of $(CB(X),H)$.

Let $A,B\in CB(X)$ and $k>1$. In the sequence, the following well known fact will be used [12]: for $a\in A$, there is $b\in B$ such that

$$d(a,b)\le kH(A,B).$$

Definition 2.4. Let (X,d) be a metric space and $F\colon X\to CB(X)$, a multi-valued mappings. A point $x\in X$ is a *fixed point of* F if $x\in Fx$.

In the following we denote by $\mathrm{Fix}(F)$ the set of all fixed points of F.

Definition 2.5. Let (X,d) be a metric space, p be a positive integer, $A_1, A_2, \dots, A_p$ be nonempty closed subsets of X, $Y = \bigcup_{i=1}^{p} A_i$ and $F\colon Y \to CB(X)$. Then Y is called a *cyclic representation of* Y *with respect to* F if:

1) A_i, $i = 1, 2, \dots, p$, are nonempty closed sets,
2) $F(A_1) \subset A_2$, $F(A_2) \subset A_3, \dots, F(A_{p-1}) \subset A_p$, $F(A_p) \subset A_1$.

3. Implicit relations

Definition 3.1. Let $\mathcal{T}_6$ be the family of all real continuous functions $T(t_1, \dots, t_6)\colon \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(T_1) : T is decreasing in variable t_3, t_4, t_5;
(T_2) : there exist $h \in [0,1)$ and $k > 1$ such that for all $u, v, t \geq 0$, $u \leq kt$ and $T(t, v, v, u, u+v, 0) \leq 0$ implies $u \leq hv$.

Example 3.1. $T(t_1, \dots, t_6) = t_1 - a_1 t_2 - a_2 t_3 - a_3 t_4 - a_4 t_5 - a_5 t_6$, where $a_1, \dots, a_5 \geq 0$ and $a_1 + a_2 + a_2 + 2a_4 < 1$.

(T_1) : Obviously.

(T_2) : Let $u, v, t \geq 0$,

$$1 < k < \frac{1}{a_1 + a_2 + a_2 + 2a_4},$$

$u \leq kt$ and $T(t, v, v, u, u+v, 0) = t - a_1 v - a_2 v - a_3 v - a_4(u+v) \leq 0$. Then $u \leq kt \leq k(a_1 v + a_2 v + a_3 u - a_4\,(u+v))$ which implies $u \leq hv$, where

$$0 \leq h = \frac{k\,(a_1 + a_2 + a_4)}{1 - (ka_3 + ka_4)} < 1.$$

Example 3.2. $T(t_1, \dots, t_6) = t_1 - p \max\{t_2, t_3, \dots, t_6\}$, where $p \in \left[0, \frac{1}{2}\right)$.

(T_1) : Obviously.

(T_2) : Let $u, v, t \geq 0$, $1 < k < \frac{1}{2p}$, $u \leq kt$ and $T(t, v, v, u, u+v, 0) = t - p(u+v) \leq 0$. Then $u \leq kt \leq kp\,(u+v)$, which implies $u \leq hv$, where

$$0 \leq h = \frac{kp}{1 - kp} < 1.$$

Example 3.3. $T(t_1, \dots, t_6) = t_1 - p \max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\}$, where $p \in [0,1)$.

(T_1) : Obviously.

(T_2) : Let $u, v, t \geq 0$, $1 < k < \frac{1}{p}$, $u \leq kt$ and $T(t, v, v, u, u+v, 0) = t - p\max\left\{u, v, \frac{u+v}{2}, 0\right\} \leq 0$. If $u > v$, then $u \leq kpu < u$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $0 \leq h = kp < 1$.

Example 3.4. $T(t_1, \ldots, t_6) = t_1 - p\max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}$, where $p \in [0, 1)$. The proof is similar to the proof from Example 3.3.

Example 3.5. $T(t_1, \ldots, t_6) = t_1^2 - at_2^2 - bt_3t_4 - t_5t_6$, where $a, b \geq 0$ and $0 < a + b < 1$.

(T_1) : Obviously.

(T_2) : Let $u, v, t \geq 0$, $1 < k < \frac{1}{\sqrt{a+b}}$, $u \leq kt$ and $T(t, v, v, u, u+v, 0) = t^2 - av^2 - buv \leq 0$. Then $u^2 \leq k^2t^2 \leq k^2\left(av^2 + buv\right)$. If $u > v$, then $u^2 \leq k^2(a+b)u^2 < u^2$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $0 \leq h = k\sqrt{a+b} < 1$.

Example 3.6.

$$T(t_1, \ldots, t_6) = t_1^2 + \frac{t_1}{1 + t_5 + t_6} - \left(at_2^2 + bt_3^2 - ct_4^2\right),$$

where $a, b, c \geq 0$ and $0 < a + b + c < 1$.

(T_1) : Obviously.

(T_2) : Let $u, v, t \geq 0$, $1 < k < \frac{1}{\sqrt{a+b+c}}$, $u \leq kt$ and $T(t, v, v, u, u+v, 0) = t^2 + t - av^2 - bv^2 - cu^2 \leq 0$, which implies $t^2 \leq av^2 + bv^2 + cu^2$. Then $u^2 \leq k^2t^2 \leq k^2\left(av^2 + bv^2 + cu^2\right)$, which implies $u \leq hv$, where $0 \leq h = k\sqrt{a+b+c} < 1$.

Example 3.7. $T(t_1, \ldots, t_6) = t_1^2 - p\max\left\{t_2^2, t_3t_4, \frac{1}{2}(t_3t_5 + t_4t_6)\right\}$, where $p \in [0, 1)$.

(T_1) : Obviously.

(T_2) : Let $u, v, t \geq 0$, $1 < k < \frac{1}{\sqrt{p}}$, $u \leq kt$ and $T(t, v, v, u, u+v, 0) = t^2 - p\max\left\{v^2, uv, \frac{1}{2}\left[v\left(u+v\right)\right]\right\} \leq 0$. Then $u^2 \leq k^2t^2 \leq k^2p\max\{v^2, uv, \frac{1}{2}\left(uv + v^2\right)\}$. If $u > v$, then $u^2 \leq k^2t^2 \leq k^2pu^2 < u^2$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $0 \leq h = k\sqrt{p} < 1$.

Example 3.8. $T(t_1, \ldots, t_6) = t_1^2 - p\max\{t_2^2, t_3t_5, t_4t_6\}$, where $p \in \left[0, \frac{1}{2}\right)$.

(T_1) : Obviously.

(T_2) : Let $u, v, t \geq 0$, $1 < k < \frac{1}{\sqrt{2p}}$, $u \leq kt$ and $T(t, v, v, u, u+v, 0) = t^2 - p\max\{v^2, v\left(u+v\right)\} \leq 0$. Then $u^2 \leq k^2t^2 \leq k^2p\max\left\{v^2, v\left(u+v\right)\right\}$. If $u > v$, then $u^2 \leq 2pk^2 < u^2$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $0 \leq h = k\sqrt{2p} < 1$.

Definition 3.2. Let (X,d) be a metric space. Let p be a positive integer, $A_1, A_2, \dots, A_p$ be nonempty closed subsets of X, $Y = \bigcup_{i=1}^{p} A_i$. A multi-valued $F\colon Y \to CB(X)$ is called an *implicit cyclic contractive mapping* if:

1) $Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to F,
2) for any $(x,y) \in A_i \times A_{i+1}$, $i = 1, 2, \dots, p$ and $A_{p+1} = A_1$,

$$T\left(H\left(Fx, Fy\right), d\left(x, y\right), d\left(x, Fx\right), d\left(y, Fy\right), d\left(x, Fy\right), d\left(y, Fx\right)\right) \leq 0 \tag{3.1}$$

for some $T \in \mathcal{T}_6$.

In this paper we extend Theorem 2.1 for multi-valued mappings.

4. Main results

Theorem 4.1. *Let (X,d) be a complete metric space, $p \in \mathbb{N}$, $A_1, \dots, A_p$ be nonempty closed subsets of X and $Y = \bigcup_{i=1}^{p} A_i$. If $F\colon Y \to CB(X)$ is an implicit cyclic contractive mapping for some $T \in \mathcal{T}_6$. Then F has at least a fixed point in $\bigcap_{i=1}^{p} A_i$.*

Proof. Let $x_0 \in A_1$ and $x_1 \in Tx_0 \subset A_2$. Then, there exists $x_2 \in Tx_1 \subset A_3$ such that

$$d\left(x_1, x_2\right) \leq kH\left(Tx_0, Tx_1\right). \tag{4.1}$$

Similarly, there exists $x_3 \in Tx_2 \subset A_4$ such that

$$d\left(x_2, x_3\right) \leq kH\left(Tx_1, Tx_2\right), \tag{4.2}$$

and there exists $x_4 \in Tx_3 \subset A_5$ such that

$$d\left(x_3, x_4\right) \leq kH\left(Tx_2, Tx_3\right), \tag{4.3}$$

$$\vdots$$

and there exists $x_{p-2} \in Tx_{p-3} \subset A_{p-1}$ and $x_{p-1} \in Tx_{p-2} \subset A_p$ such that

$$d\left(x_{p-2}, x_{p-1}\right) \leq kH\left(Tx_{p-3}, Tx_{p-2}\right). \tag{4.4}$$

Similarly, there exists $x_p \in Tx_{p-1} \subset A_{p+1} = A_1$ such that

$$d\left(x_{p-1}, x_p\right) \leq kH\left(Tx_{p-2}, Tx_{p-1}\right). \tag{4.5}$$

Hence, using this method, we defined a sequence (x_n) in X which contain the following subsequences

$$\{x_{np}, n = 0, 1, 2, \ldots\} \subset A_1,$$
$$\{x_{np+1}, n = 0, 1, 2, \ldots\} \subset A_2,$$
$$\vdots$$
$$\{x_{(n+1)p-1}, n = 0, 1, 2, \ldots\} \subset A_p.$$

If $x = x_0 \in A_1$ and $y = x_1 \in A_2$ we obtain by (3.1) that

$$T\left(H(Tx_0, Tx_1), d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), d(x_0, Tx_1), d(x_1, Tx_0)\right) \leq 0.$$

This by (T_1) leads to

$$T\left(H\left(Tx_0, Tx_1\right), d\left(x_0, x_1\right), d(x_0, x_1), d\left(x_1, x_2\right), d\left(x_0, x_2\right), 0\right) \leq 0.$$

By (T_1) and triangle inequality we have

$$T\left(H(Tx_0, Tx_1), d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_1) + d(x_1, x_2), 0\right) \leq 0.$$

Then by (4.1) and (T_2) we obtain

$$d(x_1, x_2) \leq hd\left(x_0, x_1\right).$$

Similarly, by (4.2) and (T_2) we obtain

$$d(x_2, x_3) \leq hd\left(x_1, x_2\right).$$
$$\vdots$$

Similarly, by (4.5) and (T_2) we obtain

$$d(x_{p-1}, x_p) \leq hd\left(x_{p-2}, x_{p-1}\right).$$

By induction we obtain

$$d(x_n, x_{n-1}) \leq hd\left(x_{n-1}, x_{n-2}\right) \leq \ldots \leq h^n d\left(x_0, x_1\right), n = 1, 2, \ldots.$$

By a routine calculation it follows that (x_n) is a Cauchy sequence in X. Since (X, d) is complete it follows that (x_n) is convergent in X to x^*. Then, also the subsequences (x_{np+1}), (x_{np+2}), $\ldots$, $(x_{(n+1)p-1})$ converge to x^*. Since

$A_1, A_2, \ldots, A_p$ are closed subsets in X, then $x^* \in A_i, i = 1, 2, \ldots, p$ which implies that $x^* \in \bigcap_{i=1}^{p} A_i$. □

Definition 4.1. Let (X,d) be a metric space, $p \in \mathbb{N}$, $A_1, \ldots, A_p$ be nonempty closed subsets of X, $Y = \bigcup_{i=1}^{p} A_i$. A multi-valued mapping $F\colon Y \to CB(X)$ is called *Kannan-type-cyclic contractive mapping* if

1) $Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to T,
2) for any $(x,y) \in A_i \times A_{i+1}$, $i = 1, 2, \ldots, p$ and $A_{p+1} = A_1$,

$$H(Fx, Fy) \leq a\left[d(x, Tx) + d(y, Ty)\right],$$

where $0 \leq a < \frac{1}{2}$.

Corollary 4.1. *Let (X,d) be a complete metric space, $p \in \mathbb{N}$, $A_1, \ldots, A_p$ nonempty closed subsets of X and $Y = \bigcup_{i=1}^{p} A_i$.*

If $F\colon Y \to CB(X)$ is a Kannan-type-cyclic contractive mapping then F have at least a fixed point in $\bigcap_{i=1}^{p} A_i$.

Proof. The proof it follows by Theorem 4.1 and Example 3.1 where $a_1 = a_5 = a_6 = 0$ and $a_2 = a_3 = a$. □

Remark 4.1. This corollary extends the result by [17] for multi-valued mappings.

Definition 4.2. Let (X,d) be a metric space, $p \in \mathbb{N}$, $A_1, \ldots, A_p$ nonempty closed subsets of X and $Y = \bigcup_{i=1}^{p} A_i$. A multi-valued mapping $F\colon Y \to CB(X)$ is called a *Chatterjee-type-cyclic contractive mapping* if

1) $Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to T,
2) for any $(x,y) \in A_i \times A_{i+1}$, $i = 1, 2, \ldots, p$ and $A_{p+1} = A_1$,

$$H(Fx, Fy) \leq a\left[d(x, Ty) + d(y, Tx)\right],$$

where $0 \leq a < \frac{1}{2}$.

Corollary 4.2. *Let (X,d) be a complete metric space, $p \in \mathbb{N}$, $A_1, \ldots, A_p$ nonempty closed subsets of X and $Y = \bigcup_{i=1}^{p} A_i$.*

If $F\colon Y \to CB(X)$ is a Chatterjee-type-cyclic contractive mapping then F have at least a fixed point in $\bigcap_{i=1}^{p} A_i$.

Proof. The proof it follows by Theorem 4.1 and Example 3.1 where $a_1 = a_2 = a_3 = 0$ and $a_4 = a_5 = a$. □

REMARK 4.2. This corollary extends [16, Theorem 3] for multi-valued mappings.

DEFINITION 4.3. Let (X,d) be a metric space, $p \in \mathbb{N}$, $A_1,\dots,A_p$ nonempty closed subsets of X and $Y = \bigcup_{i=1}^{p} A_i$. A multi-valued mapping $F\colon Y \to CB(X)$ is called a *Reich-type-cyclic contraction* if

1) $Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to T,
2) for all $(x,y) \in A_i \times A_{i+1}$, $i = 1,2,\dots,p$ and $A_{p+1} = A_1$,

$$H(Fx,Fy) \le ad(x,y) + bd(x,Tx) + cd(y,Ty),$$

where $a,b,c \ge 0$ and $a+b+c < 1$.

COROLLARY 4.3. *Let (X,d) be a complete metric space, $p \in \mathbb{N}$, $A_1,\dots,A_p$ nonempty closed subsets of X and $Y = \bigcup_{i=1}^{p} A_i$.*

If $F\colon Y \to CB(X)$ is a Reich-type-cyclic contractive mapping then F have at least a fixed point in $\bigcap_{i=1}^{p} A_i$.

PROOF. The proof it follows by Theorem 4.1 and Example 3.1 where $a_1 = a, a_2 = b, a_2 = c$ and $a_4 = a_5 = 0$. □

REMARK 4.3. This corollary extends [16, Theorem 7] for multi-valued mappings.

DEFINITION 4.4. Let (X,d) be a metric space, $p \in \mathbb{N}$, $A_1,\dots,A_p$ nonempty closed subsets of X and $Y = \bigcup_{i=1}^{p} A_i$. A multi-valued mapping $F\colon Y \to CB(X)$ is called a *Hardy–Roger-type-cyclic contractive mapping* if

1) $Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to T,
2) for all $(x,y) \in A_i \times A_{i+1}$, $i = 1,2,\dots,p$ and $A_{p+1} = A_1$,

$$H(Fx,Fy) \le a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty) + a_4 d(x,Ty) + a_5 d(y,Tx),$$

where $a_1,\dots,a_5 \ge 0$ and $a_1 + a_2 + a_3 + 2a_4 < 1$.

COROLLARY 4.4. *Let (X,d) be a complete metric space, $p \in \mathbb{N}$, $A_1,\dots,A_p$ nonempty closed subsets of X and $Y = \bigcup_{i=1}^{p} A_i$.*

If $F\colon Y \to CB(X)$ is a Hardy–Rogers-type-cyclic contractive mapping, then F have at least a fixed point in $\bigcap_{i=1}^{p} A_i$.

REMARK 4.4. This corollary extends [16, Theorem 9] for multi-valued mappings.

Remark 4.5.

a) Also, by Theorem 4.1 and Example 3.1 for $a_2 = a_3 = a_4 = 0$ we obtain a result which extend the results from [14, Theorem 3.1 (1)].

b) Also, by Theorem 4.1 and Example 3.3 we obtain an result which extend [13, Corollary 3.3] for multi-valued mappings.

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