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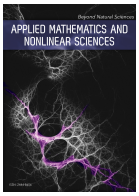

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Vortex Theory for Two Dimensional Boussinesq Equations

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Abstract

In this paper, the single center vortex method (SCVM) is extended to find some vortex solutions of finite core size for dissipative 2D Boussinesq equations. Solutions are expanded in to series of Hermite eigenfunctions. After confirmation the convergence of series of the solution, we show that, by considering the effect of temperature on the evolution of the vortex for the same initial condition as in [19] the symmetry of the vortex destroyed rapidly.

Keywords: Boussinesq equation, Vortex, Hermite function, Single center vortex, Multi convective equation.
AMS 2010 codes: 26A33, 37L05, 35D35, 76D17.

1 Introduction

The present study considers two dimensional Boussinesq equations in all of the plane, to find some exact solution of vortex type. On the best knowledge of authors, these exact Solutions are the first solutions of vortex type for Boussinesq equations. These equations are derived from a low degree approximation to the affiliate between the Navier-Stokes equations and the temperature [3, 21] and perform an main pattern in the perusal of Rayleigh-Bernard convection [4, 5]. The respective equations are as below:

$$\begin{aligned} \partial_t u + u\cdot\nabla u &= -\nabla p + \nu\triangle u + (g\alpha T)e_2 \\ \partial_t T + u\cdot\nabla T &= k_T\triangle T \\ \nabla\cdot u &= 0, \end{aligned} \tag{1}$$

where u is the fluid speed, T stands for temperature, g is gravitational acceleration constant, e_2 is monad vector in the x_2-direction, α is thermal expansion coefficient, K_T is diffusion coefficient of temperature and ν represents the kinematic viscosity.

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Thermally driven convections such as Boussinesq equations, are an active area of research, at present, with various applications from geophysics [22], ocean circulation [13] clued dynamics, inner core of the planets to astrophysics [4, 5]. These equations are one of the most commonly used fluid models in the atmospheric sciences to model Jet streams as a narrow fast flowing air currents, cold front (as a transition zone replacing cold and warm air) [15], thermohaline circulation and the El Nino Southern Oscillationas [13].

For the purpose of displaying the way in wich the presence of temperature and density influence the invisible point vortex dynamics, we concentrate on some numeric that investigate the viscous evolution of N point vortices in the Boussinesq equations.

The vorticity, in mathematics, are studied as the curl of the flow velocity. For this purpose, suppose that the field of vorticity $\omega = \nabla \times u$ is enough localized, then the Boussinesq equations for vorticity on the whole plane are include:

$$\begin{aligned} &\partial_t \omega + u \cdot \nabla \omega = \nu \triangle \omega + g\alpha \partial_{x_1} T \\ &\partial_t T + u \cdot \nabla T = k_T \triangle T \\ &\nabla \cdot \omega = 0, \qquad \omega = \nabla \times u. \end{aligned} \tag{2}$$

We are able to restore the speed of the fluid through Biot-Savart legislation:

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^2} \omega(y) dy, \tag{3}$$

which $z = (z_1, z_2)$, $z^{\perp} = (-z_2, z_1)$. For the sake of simplification, we focus on (2), but the overall results are applicable to the Thermohaline equations too.

In dimension 2, the vorticity equation is reducing to a scaler. Employing the traditional method, Ting and Tung in 1965 studied the movement of a vortex in a two dimensional incompressible flow while including the viscous influence in the internal kernel of the vortex [14]. In 1994, F. Lingevitch and A. J. Bernoff obtained the motion of vortex as integral of the background irrational current [2]. In 2002, Gallay and Wayne showed that the solutions of vorticity equation tend to Oseen vortex rapidly [7]. Afterwards, Nagem and coauthors employed the method and results of [7] to find an approximate solution for vorticity equation [18]. In the next step, they generalized the theory of single point vortex for viscose flow in two dimensions. Finally, their theory captures multi vortex problem for viscous two-dimensional flows [19]. Jing, Kanso and Newton, in 2010, described the viscous progress of a collinear three-vortex structure that at first corresponds to an inviscid point vortex fixed balance [11]. In 2011, Gallay proved that the replay of the Navier-Stockes equations converges, as $\nu \to 0$, to a superposition of Lamb-Oseen vortices which the centers evolve at a viscous regularization of the point vortex system [6]. After one year, Uminsky and Wayne introduced simplified and precise formulas that resulted in the effective performance and expansion of a new multi-moment vortex method (MMVM) using Hermite extension to resemble 2D vorticity [25]. In continue, by the use of MMVM Smith and Nagem studied vortex pairs and dipoles [23].

The content of the paper is as follows, utilizing the method presented in [19] and [25], we offer an expansion of solutions for the Boussinesq equations in the vorticity form. In section 2, the foundation of the theory of single center vortex method is reviewed. In section 3, the theory is extended for Boussinesq equations and it is shown that the series of the solution is converged. The numerical simulation of the solution of the Boussinesq equation is presented in section 4 with the same initial condition arose in [25] Then, we compare our results with [25].

2 Mathematical foundations of SCVM

In this section, we summarize the expansion of vorticity and temperature including the Hermite functions as described in [19]. Let

$$\phi_{00}(x,t;\lambda)=\frac{1}{\pi\lambda^2}e^{-|x|^2/\lambda^2},\qquad T_{00}(x,t;\sigma)=\frac{1}{\pi\sigma^2}e^{-|x|^2/\sigma^2}$$

where $\lambda^2=\lambda_0^2+4\nu t$ and $\sigma^2=\sigma_0^2+4k_Tt$. The Hermite functions of degree (k_1,k_2) is defined as follows:

$$\phi_{k_1,k_2}(x,t;\lambda)=D_{x_1}^{k_1}D_{x_2}^{k_2}\phi_{00}(x,t;\lambda),\qquad \psi_{k_1,k_2}(x,t;\sigma)=D_{x_1}^{k_1}D_{x_2}^{k_2}T_{00}(x,t;\sigma).$$

The moment expansion of functions is defined as follows:

$$\begin{aligned}\omega(x,t)&=\sum_{k_1,k_2=1}^{\infty}M[k_1,k_2;t]\phi_{k_1,k_2}(x,t;\lambda),\\ T(x,t)&=\sum_{k_1,k_2=1}^{\infty}I[k_1,k_2;t]\psi_{k_1,k_2}(x,t;\sigma).\end{aligned}\tag{4}$$

Let $(\omega,T)(x,t)$ be the resolvent of the equation (2), then Biot-Savrat law implies that the speed field is as below:

$$V(x,t)=\sum_{k_1,k_2=1}^{\infty}M[k_1,k_2;t]V_{k_1,k_2}(x,t;\lambda),\tag{5}$$

where $V_{k_1,k_2}(x,t;\lambda)=D_{x_1}^{k_1}D_{x_2}^{k_2}V_{00}(x,t;\lambda)$ and $V_{00}(x,t;\lambda)$ is the induced speed from $\phi_{00}(x,t;\lambda)$ which is determined as follows:

$$V_{00}(x,t;\lambda)=\frac{1}{2\pi}\frac{(-x_2,x_1)}{|x|^2}(1-e^{-|x|^2/\lambda^2}).\tag{6}$$

Hermite polynomials are defined by their generator functions:

$$H_{n_1,n_2}(z,\lambda)=(D_{t_1}^{n_1}D_{t_2}^{n_2}e^{(\frac{2t\cdot z-t^2}{\lambda^2})})|_{t=0},\ F_{n_1,n_2}(z,\sigma)=(D_{t_1}^{n_1}D_{t_2}^{n_2}e^{(\frac{2t\cdot z-t^2}{\sigma^2})})|_{t=0}.\tag{7}$$

Notice that the standard Hermite multinomial occur when $\lambda=1$ and $k=1$. In this case, they constitute the orthogonal sets:

$$\int_{\mathbb{R}^2}H_{n_1,n_2}(z,\lambda=1)H_{m_1,m_2}(z,\lambda=1)e^{-z^2}dz=\pi 2^{n_1+n_2}(n_1!)(n_2!)\delta_{n_1,m_1}\delta_{n_2,m_2},\tag{8}$$

$$\int_{\mathbb{R}^2}F_{n_1,n_2}(z,\sigma=1)F_{m_1,m_2}(z,\sigma=1)e^{-z^2}dz=\pi 2^{n_1+n_2}(n_1!)(n_2!)\delta_{n_1,m_1}\delta_{n_2,m_2}.\tag{9}$$

Consequently, the following projection operators determine the coefficients in the expansion (4):

$$M[k_1,k_2;t]=(P_{k_1,k_2}\omega)(t)=\rho(k_1,k_2,\lambda)\int_{\mathbb{R}^2}H_{k_1,k_2}(z,\lambda)\omega(z,t)dz,\tag{10}$$

$$I[k_1,k_2;t]=(Q_{k_1,k_2}T)(t)=\rho(k_1,k_2,\sigma)\int_{\mathbb{R}^2}F_{k_1,k_2}(z,\sigma)T(z,t)dz,\tag{11}$$

where

$$\rho(k_1,k_2,\tau)=\frac{(-1)^{(k_1+k_2)}\tau^{2(k_1+k_2)}}{2^{k_1+k_2}(k_1!)(k_2!)}. \tag{12}$$

Let

$$L^{\lambda}\phi=\frac{1}{4}\lambda^2\triangle\phi+\frac{1}{2}\nabla\cdot(x\phi), \tag{13}$$

and

$$\Phi_{\lambda}(x,t)=\phi_{00}(x,t;\lambda),\ \Psi_{\sigma}(x,t)=T_{00}(x,t;\sigma). \tag{14}$$

In the [19] Nagem and coauthors proved the convergence of the expansions (4), when:

$$\int_{\mathbb{R}^2}\Phi_{\lambda}^{-1}(x)(\omega(x,t))^2dx<\infty,\int_{\mathbb{R}^2}\Psi_{\sigma}^{-1}(x)(T(x,t))^2dx<\infty. \tag{15}$$

3 Main Result

In this section, we prove the criteria (15) and obtain the ODE for $M[k_1,k_2,t]$ and $I[k_1,k_2,t]$. In order the proof of theorem 2 we say the following fundamental lemma:

Lemma 1. *Suppose that (ω,T) satisfies the equations (2), $\omega(x,0)=\omega_0(x)$ and $T(x,0)=T_0(x)$ then the following assertions are true:*

i) For all $1\leq p\leq\infty$ and $t\geq 0$, $||T(x,t)||_p\leq||T_0(x)||_p$

ii) There exist constant $c=c(\omega_0,T_0,t)$ such that for all $2\leq q<\infty$ and $t\geq 0$, $||\omega(x,t)||_q\leq c(\omega_0,T_0)$

iii) For all $t\geq 0$, $||\nabla u||_{\infty}\leq c(\omega_0,T_0,t)$

Proof. For (i) see [1] and for (ii) see [10] and for (iii) see [26].

Now we are ready to prove criteria (15).

Theorem 2. *Define*

$$\varepsilon(t)=\int_{\mathbb{R}^2}\Phi_{\lambda}^{-1}(\omega(x,t))^2dx,\gamma(t)=\int_{\mathbb{R}^2}\Psi_{\sigma}^{-1}(T(x,t))^2dx. \tag{16}$$

If $k_T<2\nu$ and the primary vorticity and temperature, i.e. ω_0 and T_0, guarantee that $\varepsilon(0)<\infty$ and $\gamma(0)<\infty$ for some λ_0 and σ_0 respectively and ω_0 and T_0 are in the L^3, then $\varepsilon(t)$ and $\gamma(t)$ will be finite for all times of $t>0$.

Proof. According to lemma 2.1 in [7] we have: $||u||_{\infty}\leq c||\omega||_p^{\alpha}||\omega||_q^{1-\alpha}$ where $1\leq p<2<q\leq\infty$ and $\frac{\alpha}{p}+\frac{1-\alpha}{q}=\frac{1}{2}$, as a result according to lemma (1) we obtain: $||u||_{\infty}\leq c(\omega_0,T_0)$. Therefore by assumption it is concluded that $||u||_{\infty}\leq c(\omega_0,T_0)$. Now similar to the proof of theorem 3.4 in [19] it could be proved that:

$$\frac{d\gamma(t)}{dt}\leq(\frac{4c(\omega_0,T_0)}{K_T}+\frac{4K_T}{\sigma^2})\gamma(t),$$

and this means that $\gamma(t)$ is limited for each $t>0$ if $\gamma(0)$ is finite. Now to prove that $\varepsilon(t)<\infty$, differentiate $\varepsilon(t)$, we have:

$$\begin{aligned}\frac{d\varepsilon(t)}{dt}&=\frac{4\nu}{\lambda^2}\varepsilon(t)-\frac{4\nu}{\lambda^4}\int_{\mathbb{R}^2}|x|^2\Phi_{\lambda}^{-1}(\omega(x,t))^2dx+2\int_{\mathbb{R}^2}|x|^2\Phi_{\lambda}^{-1}\omega(x,t)\partial_t\omega(x,t)dx\\&=\frac{4\nu}{\lambda^2}\varepsilon(t)-\frac{4\nu}{\lambda^4}\int_{\mathbb{R}^2}|x|^2\Phi_{\lambda}^{-1}(\omega(x,t))^2dx\\&+2\int_{\mathbb{R}^2}\Phi_{\lambda}^{-1}\omega(\nu\triangle\omega-u\cdot\nabla\omega+g\alpha\partial_{x_1}T)dx.\end{aligned} \tag{17}$$

Integrating by parts in the last term in (17) implies that:

$$2\int_{\mathbb{R}^2}\Phi_\lambda^{-1}\omega(\nu\triangle\omega)dx=-2\nu\int_{\mathbb{R}^2}\Phi_\lambda^{-1}(x)(|\nabla\omega|^2+\frac{2}{\lambda^2}\omega x\cdot\nabla\omega)dx, \tag{18}$$

and the second item in the right side of (18) satisfies the following relation:

$$2\nu\int_{\mathbb{R}^2}\Phi_\lambda^{-1}(x)(\frac{2}{\lambda^2}\omega x\cdot\nabla\omega)dx\le\nu\int_{\mathbb{R}^2}\Phi_\lambda^{-1}(x)|\nabla\omega|^2dx+\frac{4\nu}{\lambda^4}\int_{\mathbb{R}^2}\Phi_\lambda^{-1}(x)(x^2\omega^2)dx. \tag{19}$$

Now using $||u||_\infty\le c(\omega_0,T_0)$ and Cauchy's inequality we have :

$$\begin{aligned}&2\int_{\mathbb{R}^2}\Phi_\lambda^{-1}\omega(u\cdot\nabla\omega)dx\le 2c(\omega_0,T_0)\int_{\mathbb{R}^2}\Phi_\lambda^{-1}|\omega(x,t)||\nabla\omega|dx\\&\le\frac{c^2(\omega_0,T_0)}{\nu}\int_{\mathbb{R}^2}\Phi_\lambda^{-1}(\omega(x,t))^2dx+\nu\int_{\mathbb{R}^2}\Phi_\lambda^{-1}(x)|\nabla\omega|^2dx\end{aligned} \tag{20}$$

also

$$\begin{aligned}&2g\alpha\int_{\mathbb{R}^2}\Phi_\lambda^{-1}\omega(\partial_{x_1}T)dx\le 2g\alpha\int_{\mathbb{R}^2}\Phi_\lambda^{-1}(\omega^2+(\partial_{x_1}T)^2)dx\\&=2g\alpha\int_{\mathbb{R}^2}\Phi_\lambda^{-1}\omega^2(x,t)dx+2g\alpha\int_{\mathbb{R}^2}\Phi_\lambda^{-1}(\partial_{x_1}T)^2dx\\&\le 2g\alpha\varepsilon(t)+2g\alpha||\nabla T||_\lambda^2.\end{aligned} \tag{21}$$

Now we bound the term $||\nabla T||_\lambda^2$. Let $f(x,t)=\nabla T(x,t)$ and define:

$$\delta(t)=\int_{\mathbb{R}^2}\Phi_\lambda^{-1}(\nabla T(x,t))^2dx.$$

Differentiate $\delta(t)$ obtain the following equation:

$$\begin{aligned}\frac{d\delta(t)}{dt}&=\frac{4\nu}{\lambda^2}\delta(t)-\frac{4\nu}{\lambda^4}\int_{\mathbb{R}^2}|x|^2\Phi_\lambda^{-1}f^2(x,t)dx+2\int_{\mathbb{R}^2}\Phi_\lambda^{-1}f(x,t)\partial_t f(x,t)dx\\&=\frac{4\nu}{\lambda^2}\delta(t)-\frac{4\nu}{\lambda^4}\int_{\mathbb{R}^2}|x|^2\Phi_\lambda^{-1}(f(x,t))^2dx\\&+2\int_{\mathbb{R}^2}\Phi_\lambda^{-1}f\nabla(K_T\triangle T-u\cdot\nabla T)dx.\end{aligned} \tag{22}$$

Now by considering that the last term in (22) we have:

$$\begin{aligned}2\int_{\mathbb{R}^2}\Phi_\lambda^{-1}f\nabla(K_T\triangle T)dx&=2K_T\int_{\mathbb{R}^2}\Phi_\lambda^{-1}f(\triangle f)dx\\&=-2K_T\int_{\mathbb{R}^2}\Phi_\lambda^{-1}(x)(|\nabla f|^2+\frac{2}{\lambda^2}f\cdot x\cdot\nabla f)dx.\end{aligned} \tag{23}$$

The second term in the last part of the equation (23) satisfy the following inequality:

$$2K_T\int_{\mathbb{R}^2}\Phi_\lambda^{-1}(x)(\frac{2}{\lambda^2}f\cdot x\cdot\nabla f)dx\le\frac{4\nu}{\lambda^4}\int_{\mathbb{R}^2}\Phi_\lambda^{-1}(x^2f^2)dx+\frac{K_T^2}{\nu}\int_{\mathbb{R}^2}\Phi_\lambda^{-1}|\nabla f|^2dx. \tag{24}$$

On the other hand inequality $||u||_\infty \leq c(\omega_0,T_0,t)$ and $||\nabla u||_\infty \leq c(\omega_0,T_0,t)$ in [10] implies that:

$$\begin{aligned}
&-2\int_{\mathbb{R}^2}\Phi_\lambda^{-1} f\nabla(u\cdot\nabla T)dx = \\
&-2\int_{\mathbb{R}^2}(\Phi_\lambda^{-1} f)\nabla u\cdot\nabla T dx - 2\int_{\mathbb{R}^2}(\Phi_\lambda^{-1} f)u\cdot\nabla(\nabla T)dx \\
&\leq 2c(\omega_0,T_0,t)\int_{\mathbb{R}^2}\Phi_\lambda^{-1} f^2 dx + 2c(\omega_0,T_0,t)\int_{\mathbb{R}^2}\Phi_\lambda^{-1}|f||\nabla f|dx. \\
&\leq 2c(\omega_0,T_0,t)\delta(t) + 2c(\omega_0,T_0,t)\int_{\mathbb{R}^2}\Phi_\lambda^{-1}|f||\nabla f|dx.
\end{aligned} \tag{25}$$

We now assume that $K_T < 2\nu$, then:

$$\begin{aligned}
&2c(\omega_0,T_0,t)\int_{\mathbb{R}^2}\Phi_\lambda^{-1}|f||\nabla f|dx \\
&\leq \frac{\nu c^2(\omega_0,T_0,t)}{2\nu K_T - K_T^2}\int_{\mathbb{R}^2}\Phi_\lambda^{-1}(f^2(x,t))dx + \frac{2K_T\nu - K_T^2}{\nu}\int_{\mathbb{R}^2}\Phi_\lambda^{-1}|\nabla f|^2 dx.
\end{aligned} \tag{26}$$

As a consequence of (23)-(26) we obtain:

$$\frac{d\delta(t)}{dt} \leq (2c(\omega_0,T_0,t) + \frac{\nu c^2(\omega_0,T_0,t)}{2\nu K_T - K_T^2} + \frac{4\nu}{\lambda^2})\delta(t), \tag{27}$$

and this means that if $\delta(0)$ is limited then $\delta(t)$ will be limited for all $t > 0$. So according to (17)-(21) we can write:

$$\frac{d\varepsilon(t)}{dt} \leq (\frac{4\nu}{\lambda^2} + \frac{4c(\omega_0,T_0)}{\nu} + 2g\alpha)\varepsilon(t) + 2g\alpha c_1(\omega_0,t_0,t), \tag{28}$$

where $||\nabla T||_\lambda^2 \leq c_1(\omega_0,t_0,t)$. Using the Gronwall lemma if $\varepsilon(0)$ is limited then $\varepsilon(t)$ remains limited for all $t > 0$.

In the following we look for differential equations generating the coefficient $M[k_1,k_2;t]$ and $I[k_1,k_2;t]$. Assuming that the $(\omega,T)(x,t)$ is a solution of (2) and Define

$$\sum_{k_1,k_2}^{m} f(k_1,k_2) := \sum_{i=0}^{m}\sum_{\substack{k_1+k_2=i \\ k_1\geq 0,k_2\geq 0}}^{n} f(k_1,k_2), \tag{29}$$

and also

$$\omega^m(x,t) = \sum_{k_1,k_2}^{m} M[k_1,k_2;t]\phi_{k_1,k_2}(x,t;\lambda) \tag{30}$$

$$u^m(x,t) = \sum_{k_1,k_2}^{m} M[k_1,k_2;t]V_{k_1,k_2}(x,t;\lambda) \tag{31}$$

$$T^m(x,t) = \sum_{k_1,k_2}^{m} I[k_1,k_2;t]\psi_{k_1,k_2}(x,t;\sigma) \tag{32}$$

where ω^m, u^m, and T^m are Hermit approximations of order m (Glerkin approximation by Hermit functions) then

by the use of Glerkin standard approximation for equation (2) we have:

$$\begin{aligned}\partial_t \omega^m &= \sum_{k_1,k_2}^{m} \frac{dM[k_1,k_2;t]}{dt}\phi_{k_1,k_2}(x,t;\lambda) + \sum_{k_1,k_2}^{m} M[k_1,k_2;t]\partial_t\phi_{k_1,k_2} \\ &= \sum_{k_1,k_2}^{m} M[k_1,k_2;t](\nu\triangle\phi_{k_1,k_2}(x,t;\lambda)) \\ &- P^m\left[(\sum_{l_1,l_2}^{m} M[l_1,l_2;t]V_{l_1,l_2}(x,t;\lambda))\cdot\nabla(\sum_{k_1,k_2}^{m} M[k_1,k_2;t]\phi_{k_1,k_2}(x,t;\lambda))\right] \\ &+ g\alpha\partial_{x_1}(\sum_{k_1,k_2}^{m} I[k_1,k_2;t]\psi_{k_1,k_2}(x,t;\sigma))\end{aligned} \tag{33}$$

$$\begin{aligned}\partial_t T^m &= \sum_{k_1,k_2}^{m} \frac{dI[k_1,k_2;t]}{dt}\psi_{k_1,k_2}(x,t;\sigma) + \sum_{k_1,k_2}^{m} I[k_1,k_2;t]\partial_t\psi_{k_1,k_2} \\ &= \sum_{k_1,k_2}^{m} I[k_1,k_2;t](K_T\triangle\psi_{k_1,k_2}(x,t;\sigma)) \\ &- P^m\left[(\sum_{l_1,l_2}^{m} M[l_1,l_2;t]V_{l_1,l_2}(x,t;\lambda))\cdot\nabla(\sum_{k_1,k_2}^{m} I[k_1,k_2;t]\psi_{k_1,k_2}(x,t;\sigma))\right].\end{aligned} \tag{34}$$

where $P^m[\cdot]$ is a projector on the subspace produced by Hermit functions of degree m or less. Noting that:

$$\partial_t\phi_{k_1,k_2} = \nu\triangle\phi_{k_1,k_2},\ \partial_t\psi_{k_1,k_2} = K_T\triangle\psi_{k_1,k_2},$$

and applying the projection operators P_{k_1,k_2} and Q_{k_1,k_2}, defined in (10) on the equation (33) and (34) we have:

$$\begin{aligned}\frac{dM[k_1,k_2;t]}{dt} = \\ -P_{k_1,k_2}\left[(\sum_{l_1,l_2}^{m} M[l_1,l_2;t]V_{l_1,l_2}(x,t;\lambda))\cdot\nabla(\sum_{m_1,m_2}^{m} M[m_1,m_2;t]\phi_{m_1,m_2}(x,t;\lambda))\right] \\ -P_{k_1,k_2}\left[g\alpha\partial_{x_1}(\sum_{m_1,m_2}^{m} I[m_1,m_2;t]\psi_{m_1,m_2}(x,t;\sigma))\right]\end{aligned} \tag{35}$$

$$\begin{aligned}\frac{dI[k_1,k_2;t]}{dt} = \\ -Q_{k_1,k_2}\left[(\sum_{l_1,l_2}^{m} M[l_1,l_2;t]V_{l_1,l_2}(x,t;\lambda))\cdot\nabla(\sum_{m_1,m_2}^{m} I[m_1,m_2;t]\psi_{m_1,m_2}(x,t;\sigma))\right].\end{aligned} \tag{36}$$

Note that $k_1+k_2 \le m$ then

$$\phi_{m_1,m_2}(x,t;\lambda) = (D_{a_1}^{m_1}D_{a_2}^{m_2}\phi_{00}(x+a,\lambda))|_{a=0} \tag{37}$$

$$V_{l_1,l_2}(x,t;\lambda) = (D_{b_1}^{l_1}D_{b_2}^{l_2}V_{00}(x+b,\lambda))|_{b=0} \tag{38}$$

$$\psi_{m_1,m_2}(x,t;\sigma) = (D_{c_1}^{m_1}D_{c_2}^{m_2}\psi_{00}(x+c,\sigma))|_{c=0}, \tag{39}$$

then the system of ordinary differential equations (35) and (36) become as follows:

$$\begin{aligned}\frac{dM[k_1,k_2;t]}{dt} &= -\rho(k_1,k_2,\lambda)\sum_{l_1,l_2=1}^{m}\sum_{m_1,m_2=1}^{m} M[l_1,l_2;t]M[m_1,m_2;t]\\ &\times\int_{\mathbb{R}^2} H_{k_1,k_2}(x)(D_{x_1}^{m_1}D_{x_2}^{m_2}V_{00}(x,\lambda))\cdot\nabla_x(D_{x_1}^{l_1}D_{x_2}^{l_2}\phi_{00}(x,\lambda))dx\\ &-\rho(k_1,k_2,\lambda)\sum_{m_1,m_2=1}^{m} I[m_1,m_2;t]\\ &\times\int_{\mathbb{R}^2} H_{k_1,k_2}(x)(D_{x_1}^{m_1}D_{x_2}^{m_2}T_{00}(x,\sigma))dx\end{aligned} \tag{40}$$

$$\begin{aligned}\frac{dI[k_1,k_2;t]}{dt} &= -\rho(k_1,k_2,\sigma)\sum_{l_1,l_2=1}^{m}\sum_{m_1,m_2=1}^{m} M[l_1,l_2;t]I[m_1,m_2;t]\\ &\times\int_{\mathbb{R}^2} F_{k_1,k_2}(x)(D_{x_1}^{m_1}D_{x_2}^{m_2}V_{00}(x,\lambda))\cdot\nabla_x(D_{x_1}^{l_1}D_{x_2}^{l_2}T_{00}(x,\sigma))dx.\end{aligned} \tag{41}$$

where $\rho(k_1,k_2,\tau)$ is defined in (12). The first integral in (40) is calculated in [25] and the two remaining integrals are calculated in appendix. Finally using (A.26)-(A.27) in appendix and (40)-(41) we have corrected the differential equations for $M[k_1,k_2,t]$ and $I[k_1,k_2,t]$ to:

$$\begin{aligned}\frac{dM[k_1,k_2;t]}{dt} &= \rho(k_1,k_2,\lambda)\sum_{l_1,l_2=1}^{m}\sum_{m_1,m_2=1}^{m} M[l_1,l_2;t]M[m_1,m_2;t]\\ &\times\tilde{\Gamma}[k_1,k_2,l_1,l_2,m_1,m_2;\lambda]+\sum_{m_1,m_2}^{m} I[m_1,m_2;t]B[k_1,k_2,m_1,m_2;\lambda,\sigma]\end{aligned} \tag{42}$$

$$\begin{aligned}\frac{dI[k_1,k_2;t]}{dt} &= \rho(k_1,k_2,\sigma)\sum_{l_1,l_2=1}^{m}\sum_{m_1,m_2=1}^{m} M[l_1,l_2;t]I[m_1,m_2;t]\\ &\times\tilde{\theta}[k_1,k_2,l_1,l_2,m_1,m_2;\lambda,\sigma],\end{aligned} \tag{43}$$

where B and $\tilde{\theta}$ is introduced in appendix, $\tilde{\Gamma}$ is introduced in [19] and

$$\tilde{\theta}[k_1,k_2,l_1,l_2,m_1,m_2;\lambda,\sigma]=\theta^1[k_1,k_2,l_1,l_2,m_1,m_2;\lambda,\sigma]+\theta^2[k_1,k_2,l_1,l_2,m_1,m_2;\lambda,\sigma].$$

4 Numerical Simulation

In this section, some numerical examples of the equation (2) are presented. Moreover, the effect of α (thermal expansion coefficient) and K_T (diffusion coefficient of temperature) on these solutions are investigated.

First, we present an example with zero temporal expansion, i.e. $\alpha=0$. Wayne and Uminsky, in [25] have shown that if we start with an initial vorticity of the following equation, where $\delta=0.1$ and core size $\lambda_0=2$,

$$\omega(x,0)=\phi_{00}(x,0)+4\delta(\phi_{20}+\phi_{02}), \tag{44}$$

then it will become quickly axisymmetric in the absence of temperature (see Figure.2 in [25]). In this section, the initial vorticity would be considered as (44) which leads to elliptical deformations of the Lamb-Oseen vortex as shown in Figure.1, and the initial temperature with $k_0 = 1$ is as follows:

$$T(x,0) = \psi_{00}(x,0) + 4\delta(\psi_{20} + \psi_{02}). \tag{45}$$

Now we present some examples with different values of α.

4.1 Zero thermal expansion coefficient $\alpha = 0$

In the differential equations (42) and (43) put $\alpha = 0$, $m = 4$, $\nu = 1/500$, and $K_T = 1/500$. As you can see in Figure 1.b, at time $t = 400$, the axisymmetric is increased. In this case, this result is similar to the result obtained by Nagem and coauthors in [25]. The enstrophy E of the vortex which is a criterion for axisymmetry of the vortex is defined as follows:

$$E = \int_{\mathbb{R}^2} (\omega(x) - <\omega(|x|)>)^2 dx, \qquad <\omega(|x|)> = \frac{1}{2\pi}\int_0^{2\pi} \omega(x) d\theta. \tag{46}$$

The values of E shows the nonaxisymmetric portion in L^2 norm. As shown in Figure 2 the values of E are decreased in time and the solution goes rapid axisymmetrization. In continue, we present two examples for

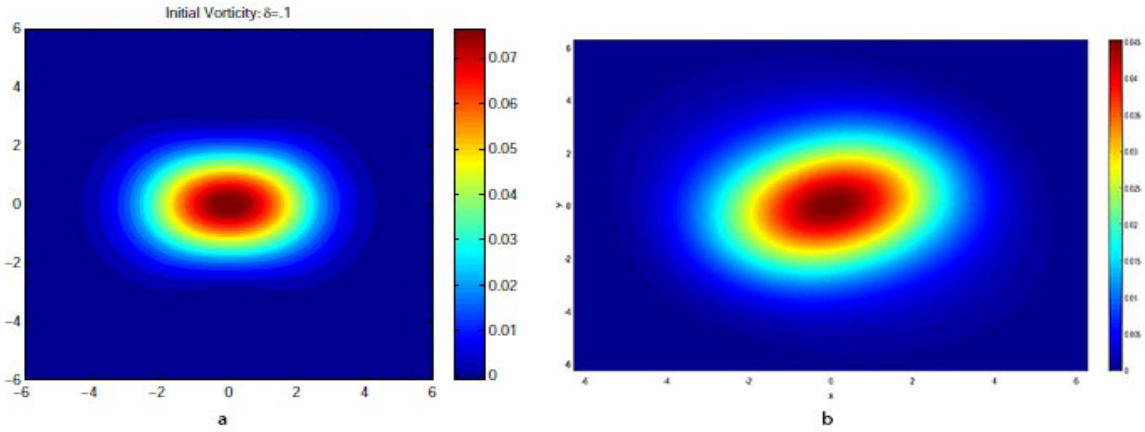

Fig. 1 a) Initial perturbation with $\delta = 0.1$ and b) Vorticity distribution at $t = 400$ with $m = 4$ and values of $\alpha = 0$, $\nu = 1/500$, and $K_T = 1/500$.

high and low values of α and the effect of α on the vorticity is investigated.

4.2 Nonzero thermal expansion coefficient (small values of α).

In this subsection, we assume that $\alpha = 69 \times 10^{-6}\,(k^{-1})$ (thermal expansion coefficient of water in 20 degrees centigrade) and other parameters are considered as follows:

$$m = 4, \quad \nu = \frac{1}{500}\,(m^2/s), \quad K_T = \frac{1}{500}, \frac{1}{800}, \frac{1}{1500}\,(m^2/s).$$

As it is displayed in Figure 3, at time $t = 8$, the portion begins to increase. For the large K_T nonaxisymmetric is increased rapidly. These results reveal two important feature of the equation (1). First, unlike the case of zero thermal expansion coefficient ($\alpha = 0$) the solution tends to be nonaxisymmetric in time and the monopole state of the vorticity breaks down. Second, as K_T decrease, the symmetry of the solution breaks faster in time. This is due to the fact that the effect of temperature on the vorticity decreases when K_T increased.(see Figure 4).

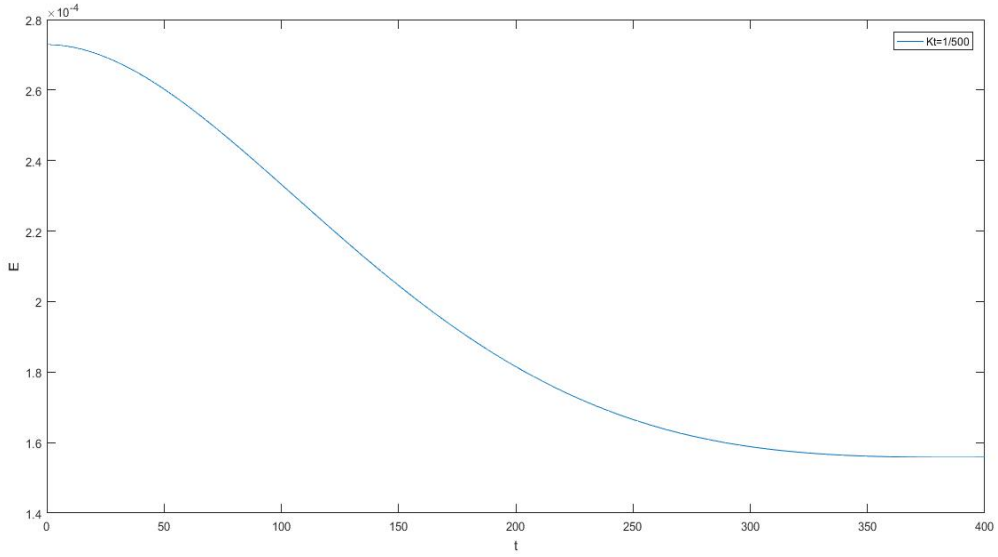

Fig. 2 the graph of the nonaxisymmetric portion (E) with $m = 4$ and $\alpha = 0$.

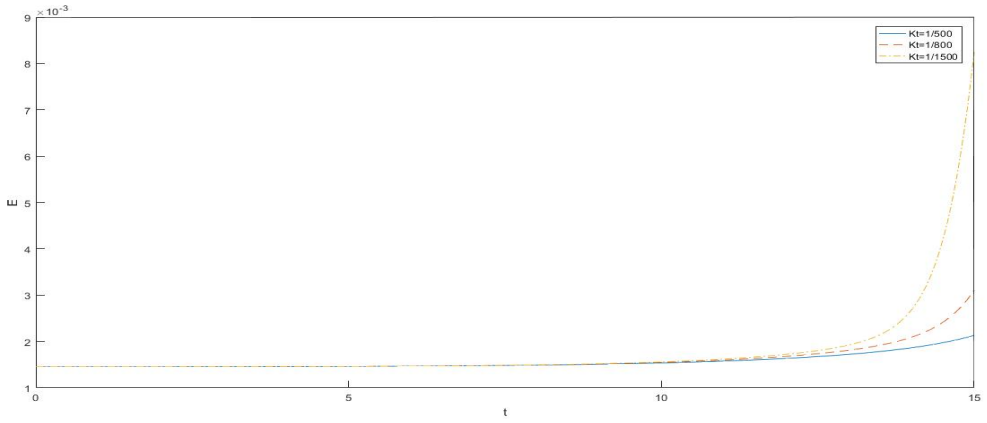

Fig. 3 The graph of the nonaxisymmetric portion (E) with $m = 4$ and $\alpha = 69 \times 10^{-6}$.

4.3 Nonzero thermal expansion coefficient (great values of α).

Now let $\alpha = 69 \times 10^{-4}\,(k^{-1})$ (suitable thermal expansion coefficient for gases), and other parameters are given as below

$$m = 4, \quad \nu = \frac{1}{500}\,(m^2/s), \quad K_T = \frac{1}{500}, \frac{1}{800}, \frac{1}{1500}\,(m^2/s).$$

Then, as can be seen in Figure 5, the results are as same as the results of the previous subsection with this difference that the nonsymmetrization process occurs faster in time.(You may see Figure 6)

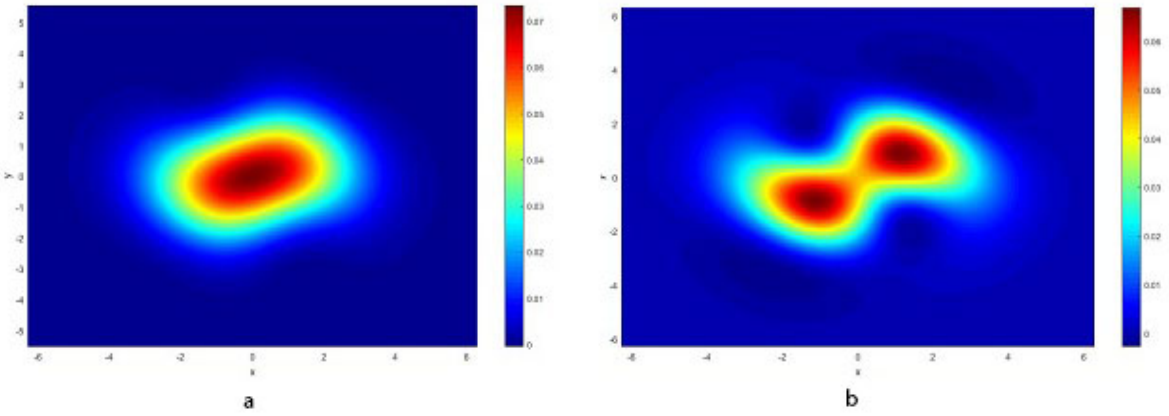

Fig. 4 Vorticity distribution for small values of α where a) at time $t = 15$, $K_T = 1/500$, $\nu = 1/500$, and $m = 4$, b) at time $t = 15$, $K_T = 1/1500$, $\nu = 1/500$, and $m = 4$.

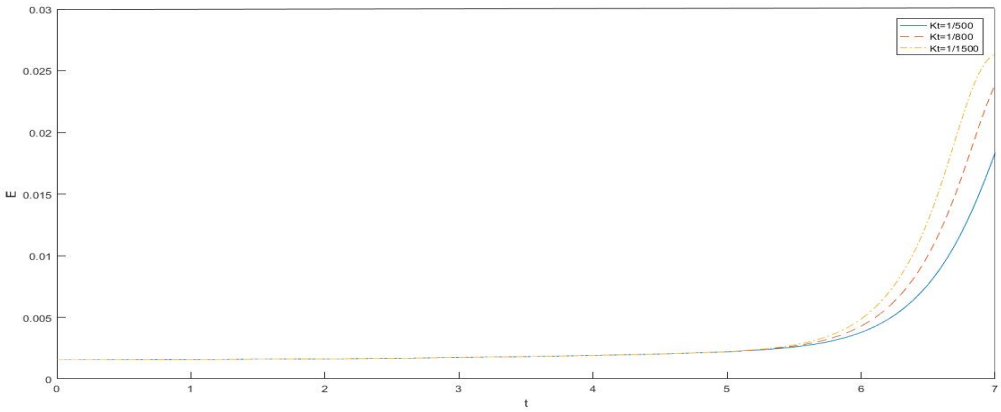

Fig. 5 The graph of the nonaxisymmetric portion (E) with $m = 4$ and $\alpha = 69 \times 10^{-4}$.

APPENDIX

The two remaining integrals in (40) and (41) are

$$1) \int_{\mathbb{R}^2} H_{k_1,k_2}(x)(D_{x_1}^{m_1} D_{x_2}^{m_2} T_{00}(x,\sigma))dx \tag{A1}$$

$$2) \int_{\mathbb{R}^2} F_{k_1,k_2}(x)(D_{x_1}^{m_1} D_{x_2}^{m_2} V_{00}(x,\lambda)) \cdot \nabla_x (D_{x_1}^{l_1} D_{x_2}^{l_2} T_{00}(x,\sigma))dx. \tag{A2}$$

Using the fact that:

$$\partial_{x_1} H_{n,m} = (\frac{2n}{\lambda^2}) H_{n-1,m}(x),\ \partial_{x_2} H_{n,m} = (\frac{2m}{\lambda^2}) H_{n,m-1}(x) \tag{A3}$$

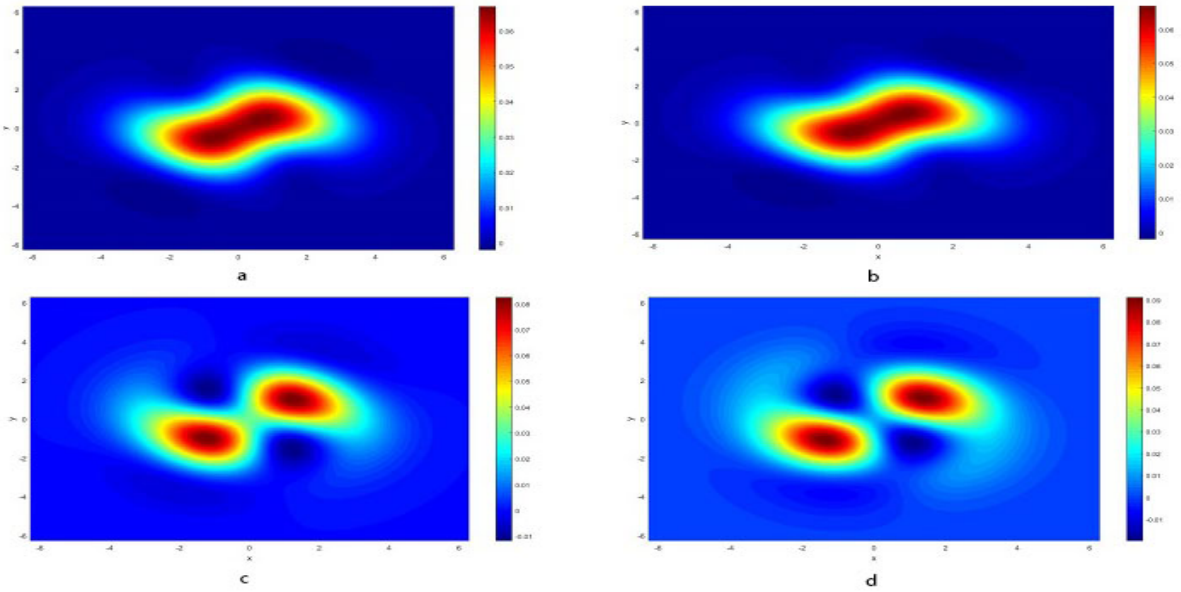

Fig. 6 Vorticity distribution for great values of α where a) at time $t = 6$, $K_T = 1/500$, $\nu = 1/500$, and $m = 4$, b) at time $t = 6$, $K_T = 1/1500$, $\nu = 1/500$, and $m = 4$, c) at time $t = 7$, $K_T = 1/500$, $\nu = 1/500$, and $m = 4$, d) at time $t = 7$, $K_T = 1/1500$, $\nu = 1/500$, and $m = 4$.

we have:

$$\int_{\mathbb{R}^2} H_{k_1,k_2}(x)(D_{x_1}^{m_1} D_{x_2}^{m_2} T_{00}(x,\sigma))dx = \tag{A4}$$
$$(-1)^{k_1+k_2} \int_{\mathbb{R}^2} \phi_{00}^{-1}(x) D_{x_1}^{k_1} D_{x_2}^{k_2} \phi_{00} D_{x_1}^{m_1} D_{x_2}^{m_2} T_{00} dx =$$
$$(-1)^{k_1+k_2} D_{b_1}^{k_1} D_{b_2}^{k_2} D_{a_1}^{m_1} D_{a_2}^{m_2} \int_{\mathbb{R}^2} \phi_{00}^{-1}(x)\phi_{00}(x+b;t)T_{00}(x+a;t)dx$$

where the first equality comes from the below equality:

$$H_{n,m} = (-1)^{n+m} \phi_{00}^{-1} D_{x_1}^n D_{x_2}^m \phi_{00},$$

and the secondary equality is an outcome of applying integration by parts. But to calculate the last integral of (A4) note that:

$$\int_{\mathbb{R}^2} \phi_{00}^{-1}(x)\phi_{00}(x+b;t)T_{00}(x+a;t)dx =$$
$$\int_{\mathbb{R}^2} \pi\lambda^2 e^{\frac{x_1^2+x_2^2}{\lambda^2}} \cdot \frac{1}{\pi\lambda^2} e^{\frac{-x_1^2-2b_1x_1-b_1^2-x_2^2-2b_2x_2-b_2^2}{\lambda^2}}$$
$$\cdot \frac{1}{\pi\sigma^2} e^{\frac{-x_1^2-2a_1x_1-a_1^2-x_2^2-2a_2x_2-a_2^2}{\sigma^2}} dx|_{a=0,b=0} =$$
$$\frac{1}{\pi\sigma^2} \cdot e^{\frac{-b_1^2-b_2^2}{\lambda^2}} \cdot e^{\frac{-a_1^2-a_2^2}{\sigma^2}} \times \int_{\mathbb{R}^2} e^{\frac{-2b_1x_1-2b_2x_2}{\lambda^2}} \cdot e^{\frac{-x_1^2-2a_1x_1-x_2^2-2a_2x_2}{\sigma^2}} dx$$
$$= \beta_1\beta_2 \int_{\mathbb{R}^2} e^{\frac{-(x_1+\sigma^2b_1+\lambda^2a_1)^2}{\lambda^2\sigma^2}} dx_1 \cdot \int_{\mathbb{R}^2} e^{\frac{-(x_2+\sigma^2b_2+\lambda^2a_2)^2}{\lambda^2\sigma^2}} dx_2 = \beta_1\beta_2 \cdot \pi\sigma^4,$$

where

$$\beta_1 = \frac{1}{\pi\sigma^2} e^{\frac{-b_1^2-b_2^2}{\lambda^2}} \cdot e^{\frac{-a_1^2-a_2^2}{\sigma^2}}, \beta_2 = e^{\frac{(\sigma^2b_1+\lambda^2a_1)^2}{\lambda^4\sigma^2}} \cdot e^{\frac{(\sigma^2b_2+\lambda^2a_2)^2}{\lambda^4\sigma^2}}$$

and this implies that:

$$\int_{\mathbb{R}^2} H_{k_1,k_2}(x)(D_{x_1}^{m_1}D_{x_2}^{m_2}T_{00}(x,\sigma))dx = \tag{A5}$$
$$(-1)^{k_1+k_2}D_{b_1}^{k_1}D_{b_2}^{k_2}D_{a_1}^{m_1}D_{a_2}^{m_2}[\beta_1\beta_2.\pi\sigma^4].$$

To calculate the integral in (A2) by use of the integration by parts we have:

$$\int_{\mathbb{R}^2} F_{k_1,k_2}(x)(D_{x_1}^{m_1}D_{x_2}^{m_2}V_{00}(x,\lambda))\cdot\nabla_x(D_{x_1}^{l_1}D_{x_2}^{l_2}T_{00}(x,\sigma))dx = \tag{A6}$$
$$-\int_{\mathbb{R}^2}\nabla_x F_{k_1,k_2}(x)(D_{x_1}^{m_1}D_{x_2}^{m_2}V_{00}(x,\lambda))\cdot(D_{x_1}^{l_1}D_{x_2}^{l_2}T_{00}(x,\sigma))dx =$$
$$-(\theta^1[k_1,k_2,l_1,l_2,m_1,m_2;\lambda,\sigma]+\theta^2[k_1,k_2,l_1,l_2,m_1,m_2;\lambda,\sigma]),$$

where

$$\theta^1 = \int_{\mathbb{R}^2} D_{x_1}^{m_1}D_{x_2}^{m_2}V_{00}^1\cdot D_{x_1}F_{k_1,k_2}(x)\cdot D_{x_1}^{l_1}D_{x_2}^{l_2}T_{00}(x,\sigma)dx \tag{A7}$$
$$\theta^2 = \int_{\mathbb{R}^2} D_{x_1}^{m_1}D_{x_2}^{m_2}V_{00}^2\cdot D_{x_2}F_{k_1,k_2}(x)\cdot D_{x_1}^{l_1}D_{x_2}^{l_2}T_{00}(x,\sigma)dx. \tag{A8}$$

Using repeated integration by parts from (A7) l_1 times toward x_1 and l_2 times toward x_2 conclude that:

$$\theta^1 = \sum_{i=0}^{l_1}\sum_{j=0}^{l_2}\binom{i}{l_1}\binom{j}{l_2}(-1)^{l_1+l_2}\int_{\mathbb{R}^2} T_{00}D_{x_1}^{i+1}D_{x_2}^{j}F_{k_1,k_2}(x)D_{x_1}^{l_1+m_1-i}D_{x_2}^{m_2+l_2-j}V_{00}^1dx$$
$$= \sum_{i=0}^{\min(l_1,k_1-1)}\sum_{j=0}^{\min(l_2,k_2)}\binom{i}{l_1}\binom{j}{l_2}(-1)^{l_1+l_2}\left(\frac{2^{i+1}k_1!}{\sigma^{2(i+1)}(k_1-i-1)!}\right)\left(\frac{2^j k_2!}{\sigma^{2(j)}(k_2-j)!}\right)$$
$$\times\int_{\mathbb{R}^2} T_{00}F_{k_1-i-1,k_2-j}(x)D_{x_1}^{l_1+m_1-i}D_{x_2}^{m_2+l_2-j}V_{00}^1dx, \tag{A9}$$

where in the secondary equality we used the following equation:

$$\partial_{x_1}F_{n,m} = \left(\frac{2n}{\sigma^2}\right)F_{n-1,m}(x),\ \partial_{x_2}F_{n,m} = \left(\frac{2m}{\sigma^2}\right)F_{n,m-1}(x).$$

In the last integral of (A9) using the relation $F_{n,m} = (-1)^{n+m}T_{00}^{-1}D_{x_1}^nD_{x_2}^mT_{00}$ arrive at the following formula:

$$\int_{\mathbb{R}^2} T_{00}F_{k_1-i-1,k_2-j}(x)D_{x_1}^{l_1+m_1-i}D_{x_2}^{m_2+l_2-j}V_{00}^1dx = \tag{A10}$$
$$(-1)^{k_1-i-1+k_2-j}\int_{\mathbb{R}^2} D_{x_1}^{k_1-i-1}D_{x_2}^{k_2-j}T_{00}D_{x_1}^{l_1+m_1-i}D_{x_2}^{m_2+l_2-j}V_{00}^1dx =$$
$$(-1)^{k_1-i-1+k_2-j}(-1)^{k_1-i-1+k_2-j}\int_{\mathbb{R}^2} T_{00}D_{x_1}^{m_1+k_1-i-1+l_1-i}D_{x_2}^{m_2+k_2-j+l_2-j}V_{00}^1dx$$
$$= \int_{\mathbb{R}^2} T_{00}D_{x_1}^{m_1+k_1-i-1+l_1-i}D_{x_2}^{m_2+k_2-j+l_2-j}V_{00}^1dx,$$

where the secondary equality is captured from an integration by parts. On the other hand:

$$\int_{\mathbb{R}^2} T_{00}D_{x_1}^{\alpha_1}D_{x_2}^{\alpha_2}V_{00}^1dx = \int_{\mathbb{R}^2} T_{00}D_{d_1}^{\alpha_1}D_{d_2}^{\alpha_2}V_{00}^1(x+d)dx|_{d=0}, \tag{A11}$$

and by the use of Biot-Savart law and by re-write the speed V_{00} in item of the vorticity ϕ_{00} :

$$V_{00}(x+d;\lambda) = -\nabla_d^*(\triangle_d)^{-1}\phi_{00}(x+d), \tag{A12}$$

where $\nabla_d^* f = (\partial_{x_2} f, -\partial_{x_1} f)$. Thus we can obtain:

$$\int_{\mathbb{R}^2} T_{00} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} V_{00} dx = D_{d_1}^{\alpha_1} D_{d_2}^{\alpha_2} \int_{\mathbb{R}^2} T_{00} V_{00}(x+d) dx|_{d=0} = \tag{A13}$$
$$-D_{d_1}^{\alpha_1} D_{d_2}^{\alpha_2} \nabla_d^* (\triangle_d)^{-1} \int_{\mathbb{R}^2} T_{00}\phi_{00}(x+d) dx|_{d=0},$$

and finally for the last integral in (A13) we have:

$$\int_{\mathbb{R}^2} T_{00}\phi_{00}(x+d)dx|_{d=0} = \int_{\mathbb{R}^2} \frac{1}{\pi\sigma^2} e^{\frac{-x_1^2-x_2^2}{\sigma^2}} \cdot \frac{1}{\pi\lambda^2} e^{\frac{-x_1^2-2d_1x_1-d_1^2-x_2^2-2d_2x_2-d_2^2}{\lambda^2}} dx|_{d=0} =$$
$$(\frac{1}{\pi\sigma\lambda})^2 \cdot e^{\frac{-d_1^2-d_2^2}{\lambda^2}} \int_{\mathbb{R}} e^{\frac{-(\lambda^2+\sigma^2)x_1^2-2\sigma^2 d_1 x_1}{\lambda^2\sigma^2}} dx_1|_{d=0}$$
$$\times \int_{\mathbb{R}} e^{\frac{-(\lambda^2+\sigma^2)x_2^2-2\sigma^2 d_2 x_2}{\lambda^2\sigma^2}} dx_2|_{d=0} = (\frac{1}{\pi\sigma\lambda})^2 e^{\frac{-d_1^2-d_2^2}{\lambda^2}} \cdot e^{\xi_1} \cdot e^{\xi_2}$$
$$\times \int_{\mathbb{R}} e^{\frac{-(\lambda^2+\sigma^2)(x_1+\xi_1)^2}{\lambda^2\sigma^2}} dx_1|_{d=0} \times \int_{\mathbb{R}} e^{\frac{-(\lambda^2+\sigma^2)(x_2+\xi_2)^2}{\lambda^2\sigma^2}} dx_1|_{d=0}$$
$$= \frac{2\lambda^2\sigma^2}{\pi(\lambda^2+\sigma^2)^2} (e^{\frac{-d_1^2-d_2^2}{\lambda^2}} \cdot e^{\xi_1} \cdot e^{\xi_2})|_{d=0}, \tag{A14}$$

where

$$\xi_1 = \frac{\sigma^4}{(\lambda^2+\sigma^2)^2} d_1^2,\ \xi_2 = \frac{\sigma^4}{(\lambda^2+\sigma^2)^2} d_2^2.$$

Replacing ξ_1 and ξ_2 in (A14), implies that:

$$\int_{\mathbb{R}^2} T_{00}\phi_{00}(x+d)dx|_{d=0} = \frac{2\lambda^2\sigma^2}{\pi(\lambda^2+\sigma^2)} e^{-(\frac{1}{\lambda^2} - \frac{\sigma^4}{(\lambda^2+\sigma^2)^2})(d_1^2+d_2^2)}|_{d=0}. \tag{A15}$$

As a result

$$\int_{\mathbb{R}^2} T_{00} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} V_{00} dx = -\frac{2\lambda^2\sigma^2}{\pi(\lambda^2+\sigma^2)} D_{d_1}^{\alpha_1} D_{d_2}^{\alpha_2} \nabla_d^* (\triangle_d)^{-1} (e^{\frac{-(d_1^2+d_2^2)}{\varepsilon}})|_{d=0}, \tag{A16}$$

where

$$\varepsilon = \frac{1}{\frac{1}{\lambda^2} - \frac{\sigma^4}{(\lambda^2+\sigma^2)^2}} = \frac{\lambda^2(\lambda^2+\sigma^2)^2}{(\lambda^2+\sigma^2)^2 - \lambda^2\sigma^4}.$$

And so

$$\int_{\mathbb{R}^2} T_{00} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} V_{00} dx = \varepsilon \frac{2\lambda^2\sigma^2}{(\lambda^2+\sigma^2)} D_{d_1}^{\alpha_1} D_{d_2}^{\alpha_2} V_{00}(d,\varepsilon)|_{d=0}$$
$$= \zeta D_{d_1}^{\alpha_1} D_{d_2}^{\alpha_2} V_{00}(d,\varepsilon)|_{d=0}, \tag{A17}$$

where

$$\zeta = \varepsilon \frac{2\lambda^2\sigma^2}{(\lambda^2+\sigma^2)} = \frac{2\lambda^4\sigma^2}{(\lambda^2+\sigma^2)^2 - \lambda^2\sigma^4}. \tag{A18}$$

According to the above computing for (A7) and similar computation for (A8) the following equalities are obtained:

$$\begin{aligned} &1)\ \int_{\mathbb{R}^2} T_{00} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} V_{00}^1 dx = \zeta D_{d_1}^{\alpha_1} D_{d_2}^{\alpha_2} V_{00}^1(d,\varepsilon)|_{d=0} \\ &2)\ \int_{\mathbb{R}^2} T_{00} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} V_{00}^2 dx = \zeta D_{d_1}^{\alpha_1} D_{d_2}^{\alpha_2} V_{00}^2(d,\varepsilon)|_{d=0}. \end{aligned} \tag{A19}$$

To simplify equation (A5), note that:

$$\begin{aligned} \beta_1\beta_2 &= \frac{1}{\pi\sigma^2}\cdot e^{-\frac{|b|^2}{\lambda^2}}\cdot e^{-\frac{|a|^2}{\sigma^2}}\cdot e^{-\frac{\sigma^2|b|^2}{\lambda^4}}\cdot e^{-\frac{|a|^2}{\sigma^2}}\cdot e^{-\frac{2\lambda^2\sigma^2 a_1 b_1}{\sigma^2\lambda^4}}.e^{-\frac{2\lambda^2\sigma^2 a_2 b_2}{\sigma^2\lambda^4}} \\ &= \frac{1}{\pi\sigma^2} e^{\frac{1}{\lambda^2}[(-1+\frac{\sigma^2}{\lambda^2})|b|^2+\frac{2(a_1b_1+a_2b_2)}{\lambda^2}]}, \end{aligned} \tag{A20}$$

so

$$\begin{aligned} &\int_{\mathbb{R}^2} H_{k_1,k_2}(x)(D_{x_1}^{m_1} D_{x_2}^{m_2} T_{00}(x,\sigma))dx = \\ &\quad (-1)^{k_1+k_2}(\frac{1}{\pi\sigma})^2\pi\sigma^4 D_{b_1}^{k_1} D_{b_2}^{k_2} D_{a_1}^{m_1} D_{a_2}^{m_2}[e^{\frac{1}{\lambda^2}[(-1+\frac{\sigma^2}{\lambda^2})|b|^2+\frac{2(a_1b_1+a_2b_2)}{\lambda^2}]}]|_{a=0,b=0} \\ &\quad = (-1)^{k_1+k_2}\frac{\sigma^2}{\pi} D_{b_1}^{k_1} D_{b_2}^{k_2} D_{a_1}^{m_1} D_{a_2}^{m_2}[e^{\frac{1}{\lambda^2}[(-1+\frac{\sigma^2}{\lambda^2})|b|^2+\frac{2(a_1b_1+a_2b_2)}{\lambda^2}]}]|_{a=0,b=0}, \end{aligned} \tag{A21}$$

but

$$\begin{aligned} &e^{\frac{1}{\lambda^2}[(-1+\frac{\sigma^2}{\lambda^2})|b|^2+\frac{2(a_1b_1+a_2b_2)}{\lambda^2}]} = \\ &\sum_{n=0}^{\infty}\frac{1}{\lambda^{2n}}\cdot\frac{1}{n!}\sum_{r=0}^{n}\binom{n}{r}2^r\cdot(-1+\frac{\sigma^2}{\lambda^2})^{n-r}(a_1b_1+a_2b_2)^r|b|^{2(n-r)} \\ &\sum_{n=0}^{\infty}\frac{1}{\lambda^{2n}}\cdot\frac{1}{n!}\sum_{r=0}^{n}\binom{n}{r}2^r\cdot(-1+\frac{\sigma^2}{\lambda^2})^{n-r}\left(\sum_{h_1=0}^{r}\binom{r}{h_1}(a_1b_1)^{h_1}(a_2b_2)^{r-h_1}\right) \\ &\times\left(\sum_{h_2=0}^{n-r}\binom{n-r}{h_2}(b_1)^{2h_2}(b_2)^{2(n-r-h_2)}\right) = \sum_{n=0}^{\infty}\frac{1}{\lambda^{2n}}\cdot\frac{1}{n!}\sum_{r=0}^{n}\binom{n}{r}2^r\cdot(-1+\frac{\sigma^2}{\lambda^2})^{n-r} \\ &\sum_{h_1=0}^{r}\sum_{h_2=0}^{n-r}\binom{r}{h_1}\binom{n-r}{h_2}(a_1)^{h_1}(a_2)^{r-h_1}(b_1)^{h_1+2h_2}(b_2)^{2(n-r-h_2)+r-h_1}, \end{aligned}$$

so

$$\begin{aligned} &D_{b_1}^{k_1} D_{b_2}^{k_2} D_{a_1}^{m_1} D_{a_2}^{m_2}(e^{\frac{1}{\lambda^2}[(-1+\frac{\sigma^2}{\lambda^2})|b|^2+\frac{2(a_1b_1+a_2b_2)}{\lambda^2}]})|_{a=0,b=0} = \\ &\left[\sum_{n=0}^{\infty}\frac{1}{\lambda^{2n}}.\frac{1}{n!}\sum_{r=0}^{n}\binom{n}{r}2^r.(-1+\frac{\sigma^2}{\lambda^2})^{n-r}\sum_{h_1=0}^{r}\sum_{h_2=0}^{n-r}\binom{r}{h_1}\binom{n-r}{h_2}\right. \\ &\times\frac{h_1!}{(h_1-m_1)!}(a_1)^{h_1-m_1}.\frac{(r-h_1)!}{(r-h_1-m_2)!}.(a_1)^{(r-h_1-m_2)}.\frac{(h_1+2h_2)!}{(h_1+2h_2-k_1)!}.(b_1)^{h_1+2h_2-k_1} \\ &\left.\times.\frac{(2(n-r-h_2)+r-h_1)!}{(2(n-r-h_2)+r-h_1-k_2)!}.(b_2)^{2(n-r-h_2)+r-h_1-k_2}\right]\Bigg|_{a=0,b=0}. \end{aligned} \tag{A22}$$

Assume $h_1 = m_1$, $r = m_1 + m_2$, $h_2 = \frac{k_1 - m_1}{2}$, $n = \frac{m_1 + k_1 + m_1 + k_2}{2}$, and define:

$$B[k_1,k_2,m_1,m_2;\lambda,\sigma] = \begin{cases} \frac{\sigma^2 g\alpha}{\pi} \cdot \frac{2^{m_1+m_2-k_1-k_2}}{\lambda^{(m_1+m_2-k_1-k_2+2)}} & \text{if } k_1 - m_1 \\ \times(-1+\frac{\sigma^2}{\lambda^2})^{\frac{k_1+k_2-m_1-m_2}{2}} & \text{and } k_2 - m_2 \\ \times \frac{1}{(\frac{k_1-m_1}{2})!(\frac{k_2-m_2}{2})!} & \text{is positive and even} \\ 0 & \text{otherwise} \end{cases}$$

the last term in (40) takes the form:

$$\sum_{m_1,m_2}^{m} I[m_1,m_2;t]B[k_1,k_2,m_1,m_2;\lambda,\sigma]. \tag{A23}$$

According to equations (A.18) to (A.21) of appendix (A) in the [25], we have:

$$\begin{aligned} &\theta^1[k_1,k_2,l_1,l_2,m_1,m_2;\lambda,\sigma] = \\ &\sum_{i=0}^{\min(l_1,k_1-1)} \sum_{j=0}^{\min(l_2,k_2)} \frac{\sigma^2}{\pi\lambda^2}\varepsilon \binom{i}{l_1}\binom{j}{l_2}(-1)^{l_1+l_2}\left(\frac{2^{i+1}k_1!}{\sigma^{2(i+1)}(k_1-i-1)!}\right) \\ &\times\left(\frac{2^j k_2!}{\sigma^{2(j)}(k_2-j)!}\right)\int_{\mathbb{R}^2} \phi_{00} D_{x_1}^{m_1+k_1-i-1+l_1-i} D_{x_2}^{m_2+k_2-j+l_2-j} V_{00}^1 dx \\ &= \sum_{i=0}^{\min(l_1,k_1-1)} \sum_{j=0}^{\min(l_2,k_2)} \frac{\sigma^2}{\pi\lambda^2}\varepsilon \binom{i}{l_1}\binom{j}{l_2}(-1)^{l_1+l_2}\left(\frac{2^{i+1}k_1!}{\sigma^{2(i+1)}(k_1-i-1)!}\right) \\ &\times\left(\frac{2^j k_2!}{\sigma^{2(j)}(k_2-j)!}\right) R_1(m_1+k_1-i-1+l_1-i, m_2+k_2-j+l_2-j) \end{aligned} \tag{A24}$$

$$\begin{aligned} &\theta^2[k_1,k_2,l_1,l_2,m_1,m_2;\lambda,\sigma] = \\ &\sum_{i=0}^{\min(l_1,k_1)} \sum_{j=0}^{\min(l_2,k_2-1)} \frac{\sigma^2}{\pi\lambda^2}\varepsilon \binom{i}{l_1}\binom{j}{l_2}(-1)^{l_1+l_2}\left(\frac{2^{i}k_1!}{\sigma^{2i}(k_1-i)!}\right) \\ &\times\left(\frac{2^{j+1} k_2!}{\sigma^{2(j+1)}(k_2-j-1)!}\right)\int_{\mathbb{R}^2} \phi_{00} D_{x_1}^{m_1+k_1-i-1+l_1-i} D_{x_2}^{m_2+k_2-j+l_2-j} V_{00}^1 dx \\ &= \sum_{i=0}^{\min(l_1,k_1-1)} \sum_{j=0}^{\min(l_2,k_2)} \frac{\sigma^2}{\pi\lambda^2}\varepsilon \binom{i}{l_1}\binom{j}{l_2}(-1)^{l_1+l_2}\left(\frac{2^{i}k_1!}{\sigma^{2i}(k_1-i)!}\right) \\ &\times\left(\frac{2^{j+1} k_2!}{\sigma^{2(j+1)}(k_2-j-1)!}\right) R_2(m_1+k_1-i+l_1-i, m_2+k_2-j-1+l_2-j). \end{aligned} \tag{A25}$$

And R_1 and R_2 the similar computation in appendix (A) in the [25] give rise to the following :

$$R_1(\alpha_1,\alpha_2) = \begin{cases} \eta(\alpha_1,\alpha_2,\varepsilon)\binom{\frac{\alpha_1+\alpha_2-1}{2}}{\frac{\alpha_1}{2}} & \text{if } \alpha_1 \text{ even and } \alpha_2 \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

$$R_2(\alpha_1,\alpha_2)=\begin{cases}\eta(\alpha_1,\alpha_2,\varepsilon)\binom{\frac{\alpha_1+\alpha_2-1}{2}}{\frac{\alpha_1-1}{2}} & \text{if } \alpha_1 \text{ odd and } \alpha_2 \text{ is even}\\ 0 & \text{otherwise}\end{cases}$$

where $\eta(\alpha_1,\alpha_2,\delta)=-\frac{1}{2\pi}(\frac{1}{2\delta})^{\frac{\alpha_1+\alpha_2+1}{2}}\frac{(-1)^{(\alpha_1+\alpha_2-1)/2}}{((\alpha_1+\alpha_2+1)/2)!}(\alpha_1)!(\alpha_2)!$.

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