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The Marichev-Saigo-Maeda Fractional Calculus Operators Pertaining to the Generalized K-Struve Function

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Abstract

In the present paper, we establish some compositions formulas for Marichev-Saigo-Maeda (MSM) fractional calculus operators with k-Struve function $S^k_{\nu,c}$ as of the kernel. The results are presented in terms of generalized k-Wright function ${}_p\Psi^k_q$.

Keywords: Generalized k-Struve function, Marichev-Saigo-Maeda fractional calculus operators, Generalized k-Wright function, k-Pochhammer symbol, k-Gamma function.
AMS 2010 codes: 26A33, 33C65, 33C20.

1 Introduction and Preliminaries

The Wright function play an important role in the partial differential equation of fractional order which is familiar and extensively treated in papers by a number of authors including Gorenflo et al. [6].

For ς_i, $\tau_j \in \mathbb{R}\backslash\{0\}$ and $a_i, b_j \in \mathbb{C}, i = (\bar{1,p}); j = (\bar{1,q})$ the generalized form of Wright function is defined by

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Wright ([13–17]) as following:

$$
{}_p\Psi_q(z) = {}_p\Psi_q \left[\begin{matrix} (a_i, \varsigma_i)_{1,p} \\ (b_j, \tau_j)_{1,q} \end{matrix} \middle| z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + n\varsigma_i)}{\prod_{j=1}^{q} \Gamma(b_j + n\tau_j)} \frac{z^n}{n!}, z \in \mathbb{C}, \tag{1.1}
$$

where $\Gamma(z)$ is the well-known Euler gamma function [4].The condition for existence of (1.1) with its depiction in terms of Mellin-Barnes integral and the H-function were obtained by Kilbas et al. [10].

The generalized form of the above Wright function (1.1) was given by Gehlot and Pra- japati [5], named as generalized K-Wright function which is defined as

$$
{}_p\Psi_q^k(z) = {}_p\Psi_q^k \left[\begin{matrix} (a_i, \varsigma_i)_{1,p} \\ (b_j, \tau_j)_{1,q} \end{matrix} \middle| z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma_k(a_i + n\varsigma_i)}{\prod_{j=1}^{q} \Gamma_k(b_j + n\tau_j)} \frac{z^n}{n!}, z \in \mathbb{C}, \tag{1.2}
$$

where $k \in \mathbb{R}^+$ and $(a_i + n\varsigma_i)$, $(b_j + b\tau_j) \in \mathbb{C}\backslash k\mathbb{Z}^-$ for all $n \in \mathbb{N}_0$. The generalized k-gamma function [3] is defined as

$$
\Gamma_k(z) = \int_0^{\infty} e^{-\frac{t^k}{k}} t^{z-1} dt; \ \ (\Re(z) > 0;\, k \in \mathbb{R}^+) \tag{1.3}
$$

and

$$
\Gamma_k(z) = \lim_{n \to \infty} \frac{n! k^n (nk)^{\frac{z}{k}-1}}{(z)_{n,k}}, \ \ k \in \mathbb{R}^+,\, z \in \mathbb{C}\backslash k\mathbb{Z}^- \tag{1.4}
$$

Also

$$
\Gamma_k(z) = k^{\frac{z}{k}-1} \Gamma\left(\frac{z}{k}\right), \tag{1.5}
$$

where $(z)_{n,k}$ is the k-Pochammer symbol introduced by Diaz and Pariguan [3] defined for complex $z \in \mathbb{C}$ and $k \in \mathbb{R}$ as

$$
(z)_{n,k} = \begin{Bmatrix} 1 & if \ \ n = 0, \\ z(z+k)(z+2k)\dots(z+(n-1)k) & if \ \ n \in \mathbb{N}. \end{Bmatrix} \tag{1.6}
$$

On taking $k = 1$, then the generalized K-Wright function (1.2) diminishes to the generalized Wright function (1.1).

1.1 Saigo fractional calculus operators

Saigo [18] defined the fractional integral and differential operators with the Gauss hyergeometric function as kernel, which are remarkable generalizations of the Riemann-Liouville (R-L) and Erdélyi-Kober fractional calculus operators (see; [11]).

For $\varsigma,\ \tau,\ \gamma \in \mathbb{C}$ and $x \in \mathbb{R}^+$ with $\Re(\varsigma) > 0$, the left-hand and the right-hand sided generalized fractional integral operators connected with Gauss hypergeometric function are defined as below:

$$
(I_{0+}^{\varsigma,\tau,\gamma} f)(x) = \frac{x^{-\varsigma-\tau}}{\Gamma(\varsigma)} \int_0^x (x-t)^{\varsigma-1} {}_2F_1(\varsigma+\tau, -\gamma; \varsigma; 1 - \frac{t}{x}) f(t) dt \tag{1.7}
$$

and

$$
(I_{-}^{\varsigma,\tau,\gamma} f)(x) = \frac{1}{\Gamma(\varsigma)} \int_x^{\infty} \frac{(t-x)^{\varsigma-1}}{t^{\varsigma+\tau}} {}_2F_1(\varsigma+\tau, -\gamma; \varsigma; 1 - \frac{x}{t}) f(t) dt \tag{1.8}
$$

respectively. Here, ${}_2F_1(\varsigma,\tau;\gamma;z)$ is the Gauss hypergeometric function [11] defined for $z\in\mathbb{C}, |z|<1$ and $\varsigma,\tau\in\mathbb{C}, \gamma\in\mathbb{C}\backslash\mathbb{Z}_0^-$ by

$${}_2F_1(\varsigma,\tau;\gamma;z)=\sum_{n=0}^{\infty}\frac{(\varsigma)_n(\tau)_n}{(\gamma)_n}\frac{z^n}{n!},$$

where $(z)_n=(z)_{n,1}$. The corresponding fractional differential operators are

$$(D_{0+}^{\varsigma,\tau,\gamma}f)(x)=\left(\frac{d}{dx}\right)^l(I_{0+}^{-\varsigma+l,-\tau-l,\varsigma+\gamma-l}f)(x) \tag{1.9}$$

and

$$(D_{-}^{\varsigma,\tau,\gamma}f)(x)=\left(-\frac{d}{dx}\right)^l(I_{-}^{-\varsigma+l,-\tau-l,\varsigma+\gamma}f)(x) \tag{1.10}$$

where $l=[\Re(\varsigma)]+1$ and $[\Re(\varsigma)]$ is the integer part of $\Re(\varsigma)$. Substituting $\tau=-\varsigma$ and $\tau=0$ in equation (1.7) – (1.10), we get the corresponding R-L and Erdélyi-Kober fractional operators, respectively.

1.2 Marichev-Saigo-Maeda fractional operators

Marichev [13] was introduced and studied fractional calculus operators which are the generalization of the Saigo operators, later generalized by Saigo and Maeda [19]. For ς, ς', τ, τ', $\gamma\in\mathbb{C}$ and $x\in\mathbb{R}^+$ with $\Re(\gamma)>0$, the left-hand and right-hand sided MSM fractional integral and derivative operators associated with third Appell function F_3 are defined as

$$(I_{0+}^{\varsigma,\varsigma',\tau,\tau',\gamma}f)(x)=\frac{x^{-\varsigma}}{\Gamma(\gamma)}\int_0^x\frac{(x-t)^{\gamma-1}}{t^{\varsigma'}}F_3(\varsigma,\varsigma',\tau,\tau,\gamma,1-\frac{t}{x},1-\frac{x}{t})f(t)dt \tag{1.11}$$

and

$$(I_{-}^{\varsigma,\varsigma',\tau,\tau',\gamma}f)(x)=\frac{x^{-\varsigma'}}{\Gamma(\gamma)}\int_x^\infty\frac{(t-x)^{\gamma-1}}{t^{\varsigma}}F_3(\varsigma,\varsigma',\tau,\tau,\gamma,1-\frac{x}{t},1-\frac{t}{x})f(t)dt \tag{1.12}$$

$$(D_{0+}^{\varsigma,\varsigma',\tau,\tau',\gamma}f)(x)=\left(\frac{d}{dx}\right)^m(I_{0+}^{-\varsigma',-\varsigma,-\tau'+m,-\tau,-\gamma+m}f)(x) \tag{1.13}$$

and

$$(D_{-}^{\varsigma,\varsigma',\tau,\tau',\gamma}f)(x)=\left(-\frac{d}{dx}\right)^m(I_{-}^{-\varsigma',-\varsigma,-\tau',-\tau+m,-\gamma+m}f)(x) \tag{1.14}$$

respectively, where $m=[\Re(\gamma)]+1$ and the third Appell function [17], is defined by

$$F_3(\varsigma,\varsigma',\tau,\tau',\gamma;x,y)=\sum_{m,n,=0}^{\infty}\frac{(\varsigma)_m(\varsigma')_n(\tau)_m(\tau')_n}{(\gamma)_{m+n}}\frac{x^my^n}{m!n!},\quad \max\{|x|,|y|\}<1. \tag{1.15}$$

1.3 Generalized *k*-Struve function

The generalized k-Struve function was defined by Nisar et al. [14] as

$$S_{\nu,c}^k(t)=\sum_{n=0}^{\infty}\frac{(-c)^n}{\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma(n+\frac{3}{2})n!}\left(\frac{t}{2}\right)^{2n+\frac{\nu}{k}+1} \tag{1.16}$$
$$(k\in\mathbb{R}^+;c\in\mathbb{R};\nu>-1)$$

taking $k\to 1$ and $c=1$; (1.15) reduces to yield the well-known Struve function of order ν is defined by [1] as

$$H_\nu(t)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\nu+\frac{3}{2})\Gamma(n+\frac{3}{2})n!}\left(\frac{t}{2}\right)^{2n+\nu+1} \tag{1.17}$$

For more details about Struve functions, their generalizations and properties, the esteemed reader is invited to consider references [2,7,8,14,15,20–22].

The following MSM integral operators are required here [19, p. 394] to obtain the MSM fractional integration of generalized k-Struve function.

Lemma 1. Let $\varsigma,\varsigma',\tau,\tau',\gamma,\rho\in\mathbb{C}$ such that $\Re(\varsigma)>0$

(i). $\Re(\rho)>0\max\{0,\Re(\varsigma'-\tau'),\Re(\varsigma+\varsigma'+\tau-\gamma)\}$, then

$$(I_{0+}^{\varsigma,\varsigma',\tau,\tau',\gamma}t^{\rho-1})(x)=\frac{\Gamma(\rho)\Gamma(-\varsigma'+\tau'+\rho)\Gamma(-\varsigma-\varsigma'-\tau+\gamma+\rho)}{\Gamma(\tau'+\rho)\Gamma(-\varsigma-\varsigma'+\gamma+\rho)\Gamma(-\varsigma'-\tau+\gamma+\rho)}x^{-\varsigma-\varsigma'+\gamma+\rho-1} \tag{1.18}$$

(ii). If $\Re(\rho)>\max\{\Re(\tau),\Re(-\varsigma-\varsigma'+\gamma),\Re(-\varsigma-\tau'+\gamma\}$, then

$$(I_{-}^{\varsigma,\varsigma',\tau,\tau',\gamma}t^{-\rho})(x)=\frac{\Gamma(-\tau+\rho)\Gamma(\varsigma+\varsigma'-\gamma+\rho)\Gamma(\varsigma+\tau'-\gamma+\rho)}{\Gamma(\rho)\Gamma(\varsigma-\tau+\rho)\Gamma(\varsigma+\varsigma'+\tau'-\gamma+\rho)}x^{-\varsigma-\varsigma'+\gamma-\rho} \tag{1.19}$$

Further, to obtain the MSM fractional differentiation of the generalized k-Struve function, following results will be used from [9] as below:

Lemma 2. Let $\varsigma,\varsigma',\tau,\tau',\gamma,\rho\in\mathbb{C}$, such that $\Re(\varsigma)>0$;

(i). If $\Re(\rho)>\max\{0,\Re(-\varsigma+\tau),\Re(-\varsigma-\varsigma'-\tau'+\gamma)\}$

$$(D_{0+}^{\varsigma,\varsigma',\tau,\tau',\gamma}t^{\rho-1})(x)=\frac{\Gamma(\rho)\Gamma(-\tau+\varsigma+\rho)\Gamma(\varsigma+\varsigma'+\tau'-\gamma+\rho)}{\Gamma(-\tau+\rho)\Gamma(\varsigma+\varsigma'-\gamma+\rho)\Gamma(\varsigma+\tau'-\gamma+\rho)}x^{\varsigma+\varsigma'-\gamma+\rho-1} \tag{1.20}$$

(ii). If $\Re(\rho)>\max\{\Re(-\tau'),\Re(\varsigma'+\tau-\gamma),\Re(\varsigma+\varsigma'-\gamma)+[\Re(\gamma)]+1\}$, then

$$(D_{-}^{\varsigma,\varsigma',\tau,\tau',\gamma}t^{-\rho})(x)=\frac{\Gamma(\tau'+\rho)\Gamma(-\varsigma-\varsigma'+\gamma+\rho)\Gamma(-\varsigma'-\tau+\gamma+\rho)}{\Gamma(\rho)\Gamma(-\varsigma'+\tau'+\rho)\Gamma(-\varsigma-\varsigma'-\tau+\gamma+\rho)}x^{\varsigma+\varsigma'-\gamma-\rho} \tag{1.21}$$

2 Fractional Calculus Approach

In this section, the following six theorems for k-Struve function concerning to MSM fractional integral and differential operators are established here as main results.

Theorem 1. Let $\varsigma,\varsigma',\tau,\ tau',\gamma,\rho\in\mathbb{C}$ and $k\in\mathbb{R}^+$ be such that $\Re(\gamma)>0,\Re\left(\frac{\lambda}{k}\right)>\max\{0,\Re(\varsigma'-\tau'),\Re(\varsigma+\varsigma'+\tau-\gamma)\}$. Also let $c\in\mathbb{R};\nu>-1$, then for $t>0$

$$\begin{aligned}\left(I_{0+}^{\varsigma,\varsigma',\tau,\tau',\gamma}\left(t^{\frac{\lambda}{k}-1}S_{\nu,c}^{k}(t)\right)\right)(x)&=\frac{k^{\gamma+\frac{1}{2}}x^{-\varsigma-\varsigma'+\gamma+\frac{\lambda}{k}+\frac{\nu}{k}}}{2^{\frac{\nu}{k}+1}}\\ \times{}_3\Psi_5^k&\left[\begin{matrix}(\lambda+\nu+k,2k), & (-k\varsigma'+k\tau'+\lambda+\nu+k,2k),\\ (k\tau'+\lambda+\nu+k,2k), & (-k\varsigma-k\varsigma'+k\gamma+\lambda+\nu+k,2k),\end{matrix}\right.\\ &\left.\begin{matrix}(-k\varsigma-k\varsigma'-k\tau+k\gamma+\lambda+\nu+k,2k) & \\ (-k\varsigma'-k\tau+k\gamma+\lambda+\nu+k,2k), & (\nu+\frac{3k}{2},k),(\frac{3k}{2},k)\end{matrix}\middle|\frac{-cx^2k}{4}\right].\end{aligned} \tag{2.1}$$

Proof. On using (1.16) and taking the left-hand sided MSM fractional integral operator inside the summation, the left-hand side of (2.1) becomes

$$=\sum_{n=0}^{\infty}\frac{(-c)^n}{\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma(n+\frac{3}{2})n!2^{2n+\frac{\nu}{k}+1}}\left(I_{0+}^{\varsigma,\varsigma',\tau,\tau',\gamma}\{t^{\frac{\lambda}{k}+\frac{\nu}{k}+2n}\}\right)(x),$$

Making use of (1.18), we obtain

$$=\sum_{n=0}^{\infty}\frac{(-c)^n x^{-\varsigma-\varsigma'+\gamma\frac{\lambda}{k}+\frac{\nu}{k}+2n}}{\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma(n+\frac{3}{2})n!2^{2n+\frac{\nu}{k}+1}}\frac{\Gamma(\frac{\lambda}{k}+\frac{\nu}{k}+2n+1)}{\Gamma(\tau'+\frac{\lambda}{k}+\frac{\nu}{k}+2n+1)}$$
$$\times\frac{\Gamma(-\varsigma-\varsigma'-\tau+\gamma+\frac{\lambda}{k}+\frac{\nu}{k}+2n+1)\Gamma(-\varsigma'+\tau'+\frac{\lambda}{k}+\frac{\nu}{k}+2n+1)}{\Gamma(-\varsigma'-\tau+\gamma+\frac{\lambda}{k}+\frac{\nu}{k}+2n+1)\Gamma(-\varsigma-\varsigma'+\gamma+\frac{\lambda}{k}\frac{\nu}{k}+2n+1)},$$

Now, using equation (1.5) on above term, then we get

$$=\frac{x^{-\varsigma-\varsigma'+\gamma+\frac{\lambda}{k}+\frac{\nu}{k}}}{2^{\frac{\nu}{k}+1}k^{-\gamma-\frac{1}{2}}}\sum_{n=0}^{\infty}\frac{\Gamma_k(\lambda+\nu+k+2nk)\Gamma_k(-k\varsigma'+k\tau'+\lambda+\nu+k+2nk)}{\Gamma_k(k\tau'+\lambda+\nu+k+2nk)\Gamma_k(-k\varsigma-k\varsigma'+k\gamma+\lambda+\nu+k+2nk)}$$
$$\times\frac{\Gamma_k(-k\varsigma-k\varsigma'-k\tau+k\gamma+\lambda+\nu+k+2nk)}{\Gamma_k(-k\varsigma'-k\tau+k\gamma+\lambda+\nu+k+2nk)\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma_k(\frac{3k}{2}+nk)n!}\left(\frac{-cx^2k}{2^2}\right)^n.$$

Using the definition of (1.2) in the above term, we arrive at the result (2.1).

Next theorem gives the right-hand MSM fractional integration of $S^k_{\nu,c}(.)$.

Theorem 2. Let $\varsigma,\varsigma',\tau,\tau',\gamma,\rho\in\mathbb{C}$ and $k\in\mathbb{R}^+$ be such that $\Re(\gamma)>0,\Re\left(\frac{\lambda}{k}\right)>\max\{\Re(\tau),\Re(-\varsigma-\varsigma'+\gamma),\Re(-\varsigma-\tau'+\gamma)\}$. Also let $c\in\mathbb{R}$; $\nu>-1$, then for $t>0$

$$\left(I_{-}^{\varsigma,\varsigma',\tau,\tau',\gamma}\left(t^{\frac{\lambda}{k}-1}S^k_{\nu,c}(t)\right)\right)(x)=\frac{k^{\gamma+\frac{1}{2}}x^{-\varsigma-\varsigma'+\gamma+\frac{\lambda}{k}+\frac{\nu}{k}}}{2^{\frac{\nu}{k}+1}}$$
$$\times{}_3\Psi_5^k\left[\begin{matrix}(-k\tau-\lambda-\nu,-2k),\ (k\varsigma+k\varsigma'-k\gamma-nu,-2k),\\ (-\lambda-\nu,-2k),\qquad (k\varsigma-k\tau-\lambda-\nu,-2k),\end{matrix}\right.$$
$$\left.\begin{matrix}(k\varsigma+k\tau'-k\gamma-\lambda-\nu,-2k)\\ (k\varsigma+k\varsigma'+k\tau'-k\gamma-\lambda-\nu,-2k),\ (\nu+\frac{3k}{2},k),(\frac{3k}{2},k)\end{matrix}\ \middle|\ \frac{-cx^2k}{4}\right].\tag{2.2}$$

Proof. On using (1.16) and taking the right-hand sided MSM fractional integral operator inside the summation, the left hand side of (2.2) becomes

$$=\sum_{n=0}^{\infty}\frac{(-c)^n}{\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma(n+\frac{3}{2})n!2^{2n+\frac{\nu}{k}+1}}\left(I_{-}^{\varsigma,\varsigma',\tau,\tau',\gamma}\{t^{\frac{\lambda}{k}+\frac{\nu}{k}+2n}\}\right)(x)$$

On using (1.19), we get

$$=\sum_{n=0}^{\infty}\frac{(-c)^n x^{-\varsigma-\varsigma'+\gamma+\frac{\lambda}{k}+\frac{\nu}{k}+2n}}{\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma(n+\frac{3}{2})n!2^{2n+\frac{\nu}{k}}+1}\frac{\Gamma(-\tau-\frac{\lambda}{k}-\frac{\nu}{k}-2n)}{\Gamma(-\frac{\lambda}{k}-\frac{\nu}{k}-2n)}$$
$$\times\frac{\Gamma(\varsigma+\tau'-\gamma-\frac{\lambda}{k}-\frac{\nu}{k}-2n)\Gamma(\varsigma+\varsigma'-\gamma-\frac{\lambda}{k}-\frac{\nu}{k}-2n)}{\Gamma(\varsigma+\varsigma'+\tau'-\gamma-\frac{\lambda}{k}-\frac{\nu}{k}-2n)\Gamma(\varsigma-\tau-\frac{\lambda}{k}-\frac{\nu}{k}-2n)}$$
$$=\sum_{n=0}^{\infty}\frac{(-cx^2)^n x^{-\varsigma-\varsigma'+\gamma+\frac{\lambda}{k}+\frac{\nu}{k}}}{\Gamma_k(nk+\nu+\frac{3k}{2})n!2^{2n+\frac{\nu}{k}+1}}\frac{\Gamma(-\tau-\frac{\lambda}{k}-\frac{\nu}{k}-2n)\Gamma(\varsigma+\varsigma'-\gamma-\frac{\lambda}{k}-\frac{\nu}{k}-2n)}{\Gamma(n+\frac{3}{2})\Gamma(-\frac{\lambda}{k}-\frac{\nu}{k}-2n)\Gamma(\varsigma-\tau-\frac{\lambda}{k}-\frac{\nu}{k}-2n)}$$
$$\times\frac{\Gamma(\varsigma+\tau'-\gamma-\frac{\lambda}{k}-\frac{\nu}{k}-2n)}{\Gamma(\varsigma+\varsigma'+\tau'-\gamma-\frac{\lambda}{k}-\frac{\nu}{k}-2n)}$$
$$=\frac{x^{-\varsigma-\varsigma'+\gamma+\frac{\lambda}{k}+\frac{\nu}{k}}}{2^{\frac{\nu}{k}+1}k^{-\gamma-\frac{1}{2}}}\sum_{n=0}^{\infty}\left(\frac{-ckx^2}{4}\right)^n\frac{1}{n!}\frac{\Gamma_k(-k\tau-\lambda-\nu-2nk)}{\Gamma_k(-\lambda-\nu-2nk)\Gamma_k(k\varsigma-k\tau-\lambda-\nu-2nk)}$$
$$\times\frac{\Gamma_k(k\varsigma+k\varsigma'-k\gamma-\lambda-\nu-2nk)\Gamma_k(k\varsigma+k\tau'-k\gamma-\lambda-\nu-2nk)}{\Gamma_k(k\varsigma+k\varsigma'+k\tau'-k\gamma-\lambda-\nu-2nk)\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma_k(\frac{3k}{2}+nk)}$$

and the result follows on making use of (1.5) and definition of generalized k-Wright function.

Theorem 3. Let $\varsigma,\varsigma',\tau,\tau',\gamma,\rho\in\mathbb{C}$ and $k\in\mathbb{R}^+$ be such that $\Re(\gamma)>0,\Re(\frac{\lambda}{k})>\max\{\Re(\tau),\Re(-\varsigma-\varsigma'+\gamma),\Re(-\varsigma-\tau'+\gamma)\}$. Also let $c\in\mathbb{R}$; $\nu>-1$, then for $t>0$

$$\left(I_{-}^{\varsigma,\varsigma',\tau,\tau',\gamma}\left(t^{\frac{-\lambda}{k}}S_{\nu,c}^{k}(t)\right)\right)(x)=\frac{k^{\gamma-\frac{1}{2}}x^{-\varsigma-\varsigma'+\gamma+\frac{\nu}{k}-\frac{\lambda}{k}+1}}{2^{\frac{\nu}{k}+1}}$$
$$\times{}_3\Psi_5^k\left[\begin{matrix}(-k\tau+\lambda-\nu,-2k),\ (k\varsigma+k\varsigma'-k\gamma+\lambda-nu-k,-2k),\\ (-\lambda-\nu-k,-2k),\quad (k\varsigma-k\tau+\lambda-\nu-k,-2k),\end{matrix}\right.$$
$$\left.\begin{matrix}(k\varsigma+k\tau'-k\gamma+\lambda-\nu-k,-2k)\\ (k\varsigma+k\varsigma'+k\tau'-k\gamma+\lambda-\nu-k,-2k),\ (\nu+\frac{3k}{2},k),(\frac{3k}{2},k)\end{matrix}\ \middle|\ \frac{-cx^2k}{4}\right]. \tag{2.3}$$

Proof. On using (1.16) and taking the right-hand sided MSM fractional integral operator inside the summation, the left hand side of (2.3) becomes

$$=\sum_{n=0}^{\infty}\frac{(-c)^n}{\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma(n+\frac{3}{2})n!2^{2n+\frac{\nu}{k}+1}}\left(I_{-}^{\varsigma,\varsigma',\tau,\tau',\gamma}\{t^{\frac{\nu-\lambda}{k}+2n+1}\}\right)$$

On using (1.19), we obtain

$$=\frac{(-c)^n t^{-\varsigma-\varsigma'+\gamma+\frac{\nu}{k}-\frac{\lambda}{k}+2n+1}}{\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma(n+\frac{3}{2})n!2^{2n+\frac{\nu}{k}+1}}\frac{\Gamma(-\tau+\frac{\lambda-\nu}{k}-2n-1)}{\Gamma(\frac{\lambda-nu}{k}-2n-1)}$$
$$\times\frac{\Gamma(\varsigma+\varsigma'-\gamma+\frac{\lambda-\nu}{k}-2n-1)\Gamma(\varsigma+\tau'-\gamma+\frac{\lambda-\nu}{k}-2n-1)}{\gamma(\varsigma-\tau+\frac{\lambda-\nu}{k}-2n-1)\Gamma(\varsigma+\varsigma'+\tau'-\gamma+\frac{\lambda-\nu}{k}-2n-1)}$$

Making use of (1.5), we get

$$=\frac{x^{-\varsigma-\varsigma'+\gamma+\frac{\nu}{k}-\frac{\lambda}{k}+1}}{k^{-\gamma+\frac{1}{2}}2^{\frac{\nu}{k}+1}}\sum_{n=0}^{\infty}\frac{(-ckx^2)^n}{4^n n!}\frac{\Gamma_k(-k\tau+\lambda-\nu-k-2nk)}{\Gamma_k(\lambda-\nu-k-2nk)\Gamma_k(k\varsigma-k\tau+\lambda-\nu-k-2nk)}$$
$$\times\frac{\Gamma_k(k\varsigma+k\varsigma'-k\gamma+\lambda-\nu-k-2nk)\Gamma_k(k\varsigma+k\tau'-k\gamma+\lambda-\nu-k-2nk)}{\Gamma_k(k\varsigma+k\varsigma'+k\tau'-k\gamma+\lambda-\nu-k-2nk)\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma_k(nk+\frac{3k}{2})}$$

This on expressing in terms of k-Wright function ${}_p\Psi_q^k$ using (1.2) leads to the right-hand side of (2.3). This completes the proof of theorem.

The next theorem obtains the left-hand sided MSM fractional differentiation of k-Struve function.

Theorem 4. Let $\varsigma,\varsigma',\tau,\tau',\gamma,\rho\in\mathbb{C}$ and $k\in\mathbb{R}^+$ be such that $\Re(\frac{\lambda}{k})>\max\{0,\Re(-\varsigma+\tau),\Re(-\varsigma-\varsigma'-\tau'+\gamma)\}$. Also let $c\in\mathbb{R}$; $\nu>-1$, then for $t>0$

$$\left(D_{0+}^{\varsigma,\varsigma',\tau,\tau',\gamma}\left(t^{\frac{\lambda}{k}-1}S_{\nu,c}^{k}(t)\right)\right)(x)=\frac{k^{-\gamma+\frac{1}{2}}x^{\varsigma+\varsigma'-\gamma+\frac{\lambda}{k}+\frac{\nu}{k}}}{2^{\frac{\nu}{k}+1}}$$
$$\times{}_3\Psi_5^k\left[\begin{matrix}(\lambda+\nu+k,2k),\quad (-k\tau+k\varsigma+\lambda+\nu+k,2k),\\ (-k\tau+\lambda+\nu+k,2k),(k\varsigma+k\varsigma'-k\gamma+\lambda+\nu+k,2k),\end{matrix}\right.$$
$$\left.\begin{matrix}(k\varsigma+k\varsigma'+k\tau-k\gamma+\lambda+\nu+k,2k)\\ (k\varsigma+k\tau'-k\gamma+\lambda+\nu+k,2k),\quad (\nu+\frac{3k}{2},k),(\frac{3k}{2},k)\end{matrix}\ \middle|\ \frac{-cx^2k}{4}\right]. \tag{2.4}$$

Proof. On using (1.16) and taking the left-hand sided MSM fractional derivative inside the summation, the left-hand side of (2.4) becomes

$$=\sum_{n=0}^{\infty}\frac{(-c)^n}{\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma(n+\frac{3}{2})n!2^{\frac{\nu}{k}+2n+1}}\left(D_{0+}^{\varsigma,\varsigma',\tau,\tau',\gamma}\left(t^{\frac{\lambda}{k}+\frac{\nu}{k}+2n}\right)\right)$$

Using (1.20) in above term, we obtain

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-c)^n \Gamma(\frac{\lambda}{k}+\frac{\nu}{k}+2n+1)\Gamma(-\tau+\varsigma+\frac{\lambda}{k}+\frac{\nu}{k}+2n+1)}{\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma(n+\frac{3}{2})n!2^{\frac{\nu}{k}+2n+1}\Gamma(-\tau+\frac{\lambda}{k}+\frac{\nu}{k}+2n+1)} \\
&\times \frac{\Gamma(\varsigma+\varsigma'+\tau'-\gamma+\frac{\lambda}{k}+\frac{\nu}{k}+2n+1)}{\Gamma(\varsigma+\varsigma'-\gamma+\frac{\lambda}{k}+\frac{\nu}{k}+2n+1)\Gamma(\varsigma+\tau'-\gamma+\frac{\lambda}{k}+\frac{\nu}{k}+2n+1)} x^{\varsigma+\varsigma'-\gamma+\frac{\lambda}{k}+\frac{\nu}{k}+2n} \\
&= \frac{x^{\varsigma+\varsigma'-\gamma\frac{\lambda}{k}+\frac{\nu}{k}}}{2^{\frac{\nu}{k}+1}} \sum_{n=0}^{\infty} \frac{(-cx^2)^n}{n!4^n\Gamma_k(nk+\nu+\frac{3k}{2})} \frac{\Gamma(\frac{\lambda}{k}+\frac{\nu}{k}+2n+1)}{\Gamma(n+\frac{3}{2})\Gamma(-\tau+\frac{\lambda}{k}+\frac{\nu}{k}+2n+1)} \\
&\times \frac{\Gamma(-\tau+\varsigma+\frac{\lambda}{k}+\frac{\nu}{k}+2n+1)\Gamma(\varsigma+\varsigma'+\tau'-\gamma+\frac{\lambda}{k}+\frac{\nu}{k}+2n+1)}{\Gamma(\varsigma+\varsigma'-\gamma+\frac{\lambda}{k}+\frac{\nu}{k}+2n+1)\Gamma(\varsigma+\tau'-\gamma+\frac{\lambda}{k}+\frac{\nu}{k}+2n+1)} \\
&= \frac{k^{-\gamma+\frac{1}{2}} x^{\varsigma+\varsigma'-\gamma+\frac{\lambda}{k}+\frac{\nu}{k}}}{2^{\frac{\nu}{k}+1}} \sum_{n=0}^{\infty} \frac{(-ckx^2)^n}{n!4^n} \\
&\times \frac{\Gamma_k(\lambda+\nu+k+2nk)\Gamma_k(-k\tau+k\varsigma+\lambda+\nu+k+2nk)}{\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma_k(nk+\frac{3k}{2})\Gamma_k(-k\tau+\lambda+\nu+k+2nk)} \\
&\times \frac{\Gamma_k(k\varsigma+k\varsigma'+k\tau'-k\gamma+\lambda+\nu+k+2nk)}{\Gamma_k(k\varsigma+k\varsigma'-k\gamma+\lambda+\nu+k+2nk)\Gamma_k(k\varsigma+k\tau'-k\gamma+\lambda+\nu+k+2nk)}
\end{aligned}
$$

In above term, we use equation (1.5), and the result follows by using (1.2), then we arrive at (2.4).

The next theorem gives the right-hand sided MSM fractional derivative of k-Struve function.

Theorem 5. Let $\varsigma,\varsigma',\tau,\tau',\gamma,\rho \in \mathbb{C}$ and $k \in \mathbb{R}^+$ be such that $\Re(\frac{\lambda}{k}) > \max\{\Re(-\tau'), \Re(\varsigma'+\tau-\gamma), \Re(\varsigma+\varsigma'-\gamma)+[\Re(\gamma)]+1\}$. Also let $c \in \mathbb{R}$; $\nu > -1$, then for $t > 0$

$$
\begin{aligned}
&\left(D_{-}^{\varsigma,\varsigma',\tau,\tau',\gamma}\left(t^{\frac{\lambda}{k}-1}S^k_{\nu,c}(t)\right)\right)(x) = \frac{k^{-\gamma+\frac{1}{2}} x^{\varsigma+\varsigma'-\gamma+\frac{\lambda}{k}+\frac{\nu}{k}}}{2^{\frac{\nu}{k}+1}} \\
&\times {}_3\Psi_5^k\left[\begin{matrix}(k\tau'-\lambda-\nu,-2k),\ (-k\varsigma-k\varsigma'+k\gamma-\lambda-\nu,-2k), \\ (-\lambda-\nu,-2k),\qquad (-k\varsigma'+k\tau'-\lambda-\nu,-2k), \end{matrix}\right. \\
&\left.\begin{matrix}(-k\varsigma'+k\tau+k\gamma-\lambda-\nu,-2k) \\ (-k\varsigma-k\varsigma'-k\tau+k\gamma-\lambda-\nu,-2k),\ (\nu+\frac{3k}{2},k),(\frac{3k}{2},k)\end{matrix}\ \middle|\ \frac{-cx^2k}{4}\right].
\end{aligned}
\tag{2.5}
$$

Proof. On using (1.16) and taking the left-hand sided MSM fractional derivative inside the summation, the left-hand side of (2.5) becomes

$$
= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma(n+\frac{3}{2})n!2^{\frac{\nu}{k}+2n+1}} \left(D_{-}^{\varsigma,\varsigma',\tau,\tau',\gamma}\left(t^{\frac{\lambda}{k}+\frac{\nu}{k}+2n}\right)\right)
$$

Using (1.21) in above term, we obtain

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-c)^n \Gamma(\tau' - \frac{\lambda}{k} - \frac{\nu}{k} - 2n)}{\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma(n+\frac{3}{2})n! 2^{\frac{\nu}{k}+2n+1}\Gamma(-\frac{\lambda}{k}-\frac{\nu}{k}-2n)} \\
&\times \frac{\Gamma(-\varsigma-\varsigma'+\gamma-\frac{\lambda}{k}-\frac{\nu}{k}-2n)\Gamma(-\varsigma'-\tau+\gamma-\frac{\lambda}{k}-\frac{\nu}{k}-2n)}{\Gamma(-\varsigma-\varsigma'-\tau+\gamma-\frac{\lambda}{k}-\frac{\nu}{k}-2n)\Gamma(-\varsigma'+\tau'-\frac{\lambda}{k}-\frac{\nu}{k}-2n)} x^{\varsigma+\varsigma'-\gamma+\frac{\lambda}{k}+\frac{\nu}{k}+2n} \\
&= \frac{x^{\varsigma+\varsigma'-\gamma\frac{\lambda}{k}+\frac{\nu}{k}}}{2^{\frac{\nu}{k}+1}} \sum_{n=0}^{\infty} \frac{(-cx^2)^n}{n! 4^n \Gamma_k(nk+\nu+\frac{3k}{2})} \frac{\Gamma(\tau'-\frac{\lambda}{k}-\frac{\nu}{k}-2n)}{\Gamma(-\frac{\lambda}{k}-\frac{\nu}{k}-2n)} \\
&\times \frac{\Gamma(-\varsigma-\varsigma'+\gamma-\frac{\lambda}{k}-\frac{\nu}{k}-2n)\Gamma(-\varsigma'-\tau+\gamma-\frac{\lambda}{k}-\frac{\nu}{k}-2n)}{\Gamma(-\varsigma'+\tau'-\frac{\lambda}{k}-\frac{\nu}{k}-2n)\Gamma(-\varsigma-\varsigma'-\tau+\gamma-\frac{\lambda}{k}-\frac{\nu}{k}-2n)} \\
&= \frac{k^{-\gamma+\frac{1}{2}} x^{\varsigma+\varsigma'-\gamma+\frac{\lambda}{k}+\frac{\nu}{k}}}{2^{\frac{\nu}{k}+1}} \sum_{n=0}^{\infty} \frac{(-ckx^2)^n}{n! 4^n} \\
&\times \frac{\Gamma_k(k\tau'-\lambda-\nu-2nk)\Gamma_k(-k\varsigma-k\varsigma'+k\gamma-\lambda-\nu-2nk)}{\Gamma_k(-\lambda-\nu-2nk)\Gamma_k(-k\varsigma'+k\tau'-\lambda-\nu-2nk)} \\
&\times \frac{\Gamma_k(-k\varsigma'-k\tau+k\gamma-\lambda-\nu-2nk)}{\Gamma_k(-k\varsigma-k\varsigma'-k\tau+k\gamma-\lambda-\nu-2nk) t \Gamma_k(\nu+\frac{3k}{2}+nk)\Gamma_k(\frac{3k}{2}+nk)}
\end{aligned}
$$

Thus, in accordance with (1.2), we get the required result (2.5).

Theorem 6. Let $\varsigma,\varsigma',\tau,\tau',\gamma,\rho \in \mathbb{C}$ and $k \in \mathbb{R}^+$ be such that $\Re(\frac{\lambda}{k}) > \max\{\Re(-\tau'), \Re(\varsigma'+\tau-\gamma), \Re(\varsigma+\varsigma'-\gamma)+[\Re(\gamma)]+1\}$. Also let $c \in \mathbb{R}$; $\nu > -1$, then for $t > 0$

$$
\begin{aligned}
&\left(D_{-}^{\varsigma,\varsigma',\tau,\tau',\gamma}\left(t^{-\frac{\lambda}{k}} S_{\nu,c}^{k}(t)\right)\right)(x) = \frac{k^{-\gamma+\frac{1}{2}} x^{\varsigma+\varsigma'-\gamma+\frac{\lambda}{k}+\frac{\nu}{k}+1}}{2^{\frac{\nu}{k}+1}} \\
&\times {}_3\Psi_5^k \left[\begin{matrix} (k\tau'+\lambda-\nu-k,-2k), (-k\varsigma-k\varsigma'+k\gamma-\lambda-\nu-k,-2k), \\ (-\lambda-\nu-k,-2k), \quad (-k\varsigma'+k\tau'+\lambda-\nu-k,-2k), \end{matrix} \right. \\
&\left. \begin{matrix} (-k\varsigma'-k\tau+k\gamma+\lambda-\nu-k,-2k) \\ (-k\varsigma-k\varsigma'-k\tau+k\gamma+\lambda-\nu-k,-2k), (\nu+\frac{3k}{2},k), (\frac{3k}{2},k) \end{matrix} \middle| \frac{-cx^2k}{4} \right].
\end{aligned} \tag{2.6}
$$

Proof. On using (1.16) and taking the right-hand sided MSM fractional derivative inside the summation, the left-hand side of (2.6) becomes

$$
= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma(n+\frac{3}{2})n! 2^{\frac{\nu}{k}+2n+1}} \left(D_{-}^{\varsigma,\varsigma',\tau,\tau',\gamma}\left(t^{\frac{\nu}{k}-\frac{\lambda}{k}+2n+1}\right)\right)
$$

Using (1.21), we have

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma(n+\frac{3}{2})n! 2^{\frac{\nu}{k}+2n+1}} \frac{\Gamma(\tau'+\frac{\lambda-\nu}{k}-2n-1)}{\Gamma(\frac{\lambda-\nu}{k}-2n-1)} \\
&\times \frac{\Gamma(-\varsigma-\varsigma'+\gamma+\frac{\lambda-\nu}{k}-2n-1)\Gamma(-\varsigma'-\tau+\gamma+\frac{\lambda-\nu}{k}-2n-1)}{\Gamma(-\varsigma'+\tau'+\frac{\lambda-\nu}{k}-2n-1)\Gamma(-\varsigma-\varsigma'-\tau+\gamma+\frac{\lambda-\nu}{k}-2n-1)} x^{\varsigma+\varsigma'-\gamma+\frac{\nu}{k}-\frac{\lambda}{k}+2n+1}
\end{aligned}
$$

Making use of (1.5), we obtain

$$= \frac{t^{\varsigma+\varsigma'-\gamma+\frac{\nu}{k}-\frac{\lambda}{k}+1}}{2^{\frac{\nu}{k}+1}} \sum_{n=0}^{\infty} \frac{(-ckt^2)^n}{n!4^n k^{\gamma-\frac{1}{2}}}$$
$$\times \frac{\Gamma_k(k\tau'+\lambda-\nu-k-2nk)}{\Gamma_k(\lambda-\nu-k-2nk)\Gamma_k(-k\varsigma'+k\tau'+\lambda-\nu-k-2nk)}$$
$$\times \frac{\Gamma_k(-k\varsigma-k\varsigma'+k\gamma+\lambda-\nu-k-2nk)\Gamma_k(-k\varsigma'-k\tau+k\gamma+\lambda-\nu-k-2nk)}{\Gamma_k(-k\varsigma-k\varsigma'-k\tau+k\gamma+\lambda-\nu-k-2nk)\Gamma_k(nk+\nu+\frac{3k}{2})\Gamma_k(nk+\frac{3k}{2})}$$

This on expressing in terms of k-Wright function ${}_p\Psi_q^k$ using (1.2) leads to the right-hand side of (2.6). This completes the proof.

3 Concluding Remark

MSM fractional calculus operators have more advantage due to the generalize of Riemann-Liouville, Weyl, Erdélyi-Kober, and Saigo's fractional calculus operators; there- fore, many authors are called as general operator. Now we are going to conclude of this paper by emphasizing that our leading results (Theorems 1 – 6) can be derived as the specific cases involving familiar fractional calculus operators as above said. On other hand , the k Struve function defined in (1.16) possesses the lead that a number of special functions occur to be the particular cases. Some of special cases respect to the integrals relating with kStruve function have been discovered in the earlier research works by various authors with not the same arguments.

References

[1] Baricz A., Generalized Bessel functions of the first kind, Lecture Notes in Mathematics, Springer, Berlin, 1994.
[2] Bhowmick K.N., A generalized Struve's function and its recurrence formula, Vijnana Parishad Anu- sandhan Patrika, 6, (1963), 1-11.
[3] Diaz R. and Pariguan E., On hypergeometric functions and Pochammer k-symbol, Divulgaciones mathematics, 15 (2), (2007), 179-192.
[4] Erdélyi A., Magnus W., Oberhettinger and Tricomi F.G., Higher Transcendental Functions, Vol.I, McGraw-Hill, New York-Toronto-London, 1953.
[5] Gehlot K.S. and Prajapati J.C., On generalization of K-Wright functions and its properties, Pac. J. Appl. Math., 5(2), (2013), 81-88.
[6] Gorenflo R., Luchko Y. and Mainardi F., Analytical properties and applications of the Wright function, Fract. Calc. Appl. Anal., 2(4), (1999), 383-414.
[7] Habenom H., Suthar D.L. and Gebeyehu M., Application of Laplace transform on fractional kinetic equation pertaining to the generalized Galué type Struve function, Adv. Math. Phys., volume (2019) Article ID 5074039, pp. 8.
[8] Kanth B.N., Integrals involving generalized Struve's function, Nepali Math. Sci. Rep., 6, (1981), 61-64.
[9] Kataria K.K. and Vellaisamy P., Some fractional calculus results associated with the *I*-function, Mathematiche (Catania), 70(2), (2015), 173-190.
[10] Kilbas A.A., Saigo M., and Trujillo J.J., On the generalized Wright function, Fract. Calc. Appl. Anal., 5(4), (2002), 437-460.
[11] Kilbas A.A., Srivastava H.M. and Trujillo J.J., Theory and Applications of Fractional Differential Equations, Elsevier, North Holland, 2006.
[12] Kiryakova V.S., Generalized Fractional Calculus and Applications, Pitman Res. Notes Math. Ser. , 301, Wiley, New York, 1994.
[13] Marichev O.I., Volterra equation of Mellin convolution type with a Horn function in the kernel, Izvestiya Akademii Nauk , SSSR, 1, (1974), 128-129.
[14] Nisar K.S., Mondal S.R. and Choi J., Certain inequalities involving the k-Struve function, J. Inequal. Appl., 71, (2017), 1-8.
[15] Nisar K.S., Purohit S.D. and Mondal S.R., Generalized fractional Kinetic equations involving generalized Struve function of the first Kind, Journal of King Saud University-Science (2015), http://dx.doi.org/10.1016/j.jksus.2015.08.005.

[16] Nisar K.S., Suthar D.L. , Purohit S.D. and Aldhaifallah M., Some unified integral associated with the generalized Struve function. Proc. Jangjeon Math. Soc., 20 (2), (2017), 261-267.
[17] Prudnikov A.P., Brychkov Y. A. and Marichev O.I., Integrals and Series, More Special Functions, Gordon and Breach, New York, 1990.
[18] Saigo M., A remark on integral operators involving the Gauss hypergeometric functions, Math. Rep. Kyushu Uni.,11(2), (1978), 135-143.
[19] Saigo M. and Maeda N., More generalization of fractional calculus, In: Rusev P., Dimovski I. and Kiryakova V. (Eds.), Transform Methods and Special Functions. 386-400, IMI-BAS, Sofia, Bulgaria, 1998.
[20] Singh R.P. , Some integral representation of generalized Struve's function, Math. Ed. (Siwan), 22(3), (1988), 91-94.
[21] Suthar D.L., Purohit S.D. and Nisar K.S., Integral transforms of the Galue type Struve function. TWMS J. Appl. Eng. Math., 8(1), (2018), 114-121.
[22] Yagmur N. and Orhan H. , Starlikeness and convexity of generalized Struve functions, Abstr. Appl. Anal., 2013, Art. ID 954513, pp. 6.