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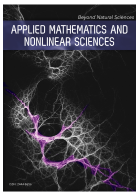

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Normal complex contact metric manifolds admitting a semi symmetric metric connection

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Abstract

In this paper, we study on normal complex contact metric manifold admitting a semi symmetric metric connection. We obtain curvature properties of a normal complex contact metric manifold admitting a semi symmetric metric connection. We also prove that this type of manifold is not conformal flat, concircular flat, and conharmonic flat. Finally, we examine complex Heisenberg group with the semi symmetric metric connection.

Keywords: normal complex contact metric manifold ; semi symmetric metric connection; curvature tensors; complex Heisenberg group.
AMS 2010 codes: 53C15, 53C25, 53D10.

1 Introduction

The Riemannian geometry of complex contact manifolds has been studied since 1970s. In the early 1980s some important developments were presented by Ishihara-Konishi. They obtained the normality conditions and curvature properties [9, 10]. Due to some important features that are different from real contact geometry, in 2000s some researchers have taken their attention to this notion. Blair, Korkmaz and Foreman gave results for the Riemannian geometry of complex contact manifolds [2,4,5,8,12] . Also two of presented authors examined curvature and symmetry notions [15, 16].

In Riemannian geometry the notion of connection gives information about transporting data along a curve or family of curves in a parallel and consistent manner. Affine connections and Levi-Civita connections are commonly used for to understand the geometry of manifolds. Levi-Civita connection is symmetric, i.e, has zero torsion, and also it is metric, i.e, the covariant derivation of metric vanish. In recent years some different

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connections were defined and worked on manifolds. One of them is semi symmetric metric connection. This type of connection were defined by Hayden and this was developed by Yano [13].

Blair and Molina [4], proved that a normal complex contact metric manifold could not be conformal flat. Also Turgut Vanlı and Unal, prove that, concircular, quasi-conformal, and conharmonic curvature tensors do not vanish on any normal complex contact metric manifold [16].

In this paper, we study on normal complex contact metric manifold with a semi symmetric metric connection. Firstly, we give some basic properties. Our starting point was the non-vanishing of special curvature tensors (conformal, concircular, quasi-conformal etc.) on normal complex contact manifolds with canonic connection. We research the flatness conditions of these special tensors on normal complex contact metric manifold with a semi symmetric metric connection. We proved that a normal complex contact metric manifold admitting the semi symmetric metric connection is not conformal flat, concircular flat and conharmonic flat. Finally we apply our results to complex-Heisenberg group as a well-known example of normal complex contact metric manifolds.

2 Preliminaries

In 1959 Kobyasahi [11] gave the definition of a complex contact manifold. A complex contact manifold is a $(2m+1)-$complex dimensional complex manifold with a holomorphic $1-$form ω such that $\omega\wedge(d\omega)^m\neq 0$. ω is not globally defined. For an open covering by coordinate neighborhoods $\mathscr{A}=\{\mathscr{O},\mathscr{O}',\dots\}$ of M, there is a non-vanishing $\lambda:\mathscr{O}\cap\mathscr{O}'\to\mathbb{C}\setminus\{0\}$ such that $\omega'=\lambda\omega$. We have a subbbundle $\mathscr{H}=ker\omega$ which is called the horizontal subbundle.

Complex almost contact structure on a complex contact manifold were given by Ishihara-Konishi [10]. For a Hermitian metric g and complex structure J, we have 1-forms u and $v=u\circ J$, with dual vector fields U and $V=-JU$, and $(1,1)$ tensor fields G and $H=GJ$ such that

$$H^2=G^2=-I+u\otimes U+v\otimes V$$

$$GJ=-JG,\quad GU=0,\quad g(X,GY)=-g(GX,Y),$$
$$g(U,X)=u(X),\quad g(U,U)=1.$$

Also there are functions a and b on $\mathscr{O}\cap\mathscr{O}'\neq\emptyset$ such that

$$u'=au-bv,\quad v'=bu+av,$$

$$a^2+b^2=1,$$

$$G'=aG-bH,\quad H'=bG+aH.$$

With these properties M is said to be a complex almost contact metric manifold.

On the other hand the verical subbundle of TM is spanned by U,V i.e $\mathscr{V}=sp\{U,V\}$. Thus we have $TM\cong\mathscr{H}\oplus\mathscr{V}$. Also $2-$forms du,dv are defined as follow;

$$du(Z,T)=g(Z,GT)+(\sigma\wedge v)(Z,T),$$
$$dv(Z,T)=g(Z,HT)-(\sigma\wedge u)(Z,T)$$

where $\sigma(Z)=g(\nabla_Z U,V)$ [10]. σ is called IK-connection [7]. Also if complex contact 1-form is globally defined then σ vanishes.

There are two normality notions for a complex almost contact metric manifold in literature. The fundamental difference between of these normality notions is to be a Kähler manifold. IK-tensors are given as below;

$$\mathscr{S}(Z,T)=[G,G](Z,T)+2g(Z,GT)U-2g(Z,HT)V$$

$$
\begin{aligned}
&+2(v(T)HZ - v(Z)HT) + \sigma(GT)HZ \\
&-\sigma(GZ)HT + \sigma(Z)GHT - \sigma(T)GHZ, \\
\mathscr{T}(Z,T) = {} & [H,H](Z,T) - 2g(Z,GT)U + 2g(Z,HT)V \\
&+2(u(T)GZ - u(Z)GT) + \sigma(HZ)GT \\
&-\sigma(HT)GZ + \sigma(Z)GHZ - \sigma(T)GHZ
\end{aligned}
$$

where $[G,G]$ and $[H,H]$ denote the Nijenhuis tensors of G and H, respectively [9, 10]. If these two tensors vanishes identically then M is called IK-normal and an IK-Normal complex contact metric manifold is Kähler. But this definition does not contain normality of complex Heisenberg group. So Korkmaz [12] presented a weak definition for normality and the complex Heisenberg group is normal in this sense.

Definition 1. *[12] A complex contact metric manifold M is called normal if it satisfied the following conditions:*

$$S\,|_{\mathscr{H}\wedge\mathscr{H}} = S\,|_{\mathscr{H}\wedge\mathscr{V}} = T\,|_{\mathscr{H}\wedge\mathscr{H}} = T\,|_{\mathscr{H}\wedge\mathscr{V}} = 0.$$

As similar to $\phi-$sectional curvature in real contact geometry, in complex contact geometry the definition of $\mathscr{GH}-$*sectional curvature* were given.

Definition 2. *[12] Let M be a normal complex contact metric manifold. Z be an unit horizontal vector field on M and $a^2+b^2=1$. A $\mathscr{GH}-$section is a plane which is spanned by Z and $T = aGZ + bHZ$ and the sectional curvature of this plane is called $\mathscr{GH}-$sectional curvature .*

$\mathscr{GH}-$*sectional curvature* is denoted by $\mathscr{GH}_{a,b}$ and we assume that it does not depend on the choice of a and b. So we will use $\mathscr{GH}(Z)$ notation. Also Korkmaz [12] proved that

$$K(Z,JZ) = \mathscr{GH}(Z) + 3.$$

Some basic curvature properties of a normal complex contact metric manifold can be seen in [7, 12, 15, 16]. On a normal complex contact metric manifold, for Z and T horizontal vector fields we have

$$
\begin{aligned}
R(U,V,V,U) = R(V,U,U,V) &= -2d\sigma(U,V) \\
R(Z,U)U = Z, \quad & R(Z,V)V = Z \\
R(Z,T)U &= 2(g(Z,JT) + d\sigma(Z,T))V \\
R(Z,T)V &= -2(g(Z,JT) + d\sigma(Z,T))U \\
R(Z,U)V &= \sigma(U)GZ + (\nabla_U H)Z - JZ \\
R(Z,V)U &= -\sigma(V)HZ + (\nabla_V G)Z + JZ \\
R(Z,U)T &= -g(Z,T)U - g(JZ,T)V + d\sigma(T,Z)V \\
R(Z,V)T &= -g(Z,T)V + g(JZ,T)U - d\sigma(T,Z)U \\
R(U,V)Z &= JZ\,.
\end{aligned}
$$

Details about complex contact geometry can be find in [3], page 233 . On the other hand for Z and T horizontal vector fields Ricci tensor of a $(2m+1)-$complex dimensional normal complex contact metric manifold has the following properties

$$Ric(GZ,GT) = Ric(HZ,HT) = Ric(Z,T)$$

$$Ric(U,U) = Ric(V,V) = 4m - 2d\sigma(U,V) \;\; and \;\; Ric(U,V) = 0$$

$$Ric(Z,U) = Ric(Z,V) = 0$$

[15].

3 Normal Complex Contact Metric Manifolds Admitting a Semi Symmetric Metric Connection

In this section the definition of a semi symmetric metric connection are given for normal complex contact metric manifolds. Some basic equalities are computed via this connection.
Let M be normal complex contact metric manifold and define $\overline{\nabla}:\chi(M)\times\chi(M)\to\chi(M)$ as follow

$$\overline{\nabla}_Z T=\nabla_Z T+u(T)Z-g(Z,T)U+v(T)Z-g(Z,T)V \tag{1}$$

where ∇ is Levi-Cevita connection on M, U,V are the structure vector fields and u,v are dual $1-$forms. It can be easily showed that $\overline{\nabla}$ is an linear connection. Also we could write the torsion tensor field of $\overline{\nabla}$ as follow;

$$\overline{\mathbf{T}}(Z,T)=u(T)Z+v(T)Z-u(Z)T-v(Z)T.$$

As we see $\overline{\nabla}$ is not torsion free and it is also a semi symmetric metric connection. In addition we have Lie bracket operator $\overline{[Z,T]}=[Z,T]$.
For brevity we use a abbreviation "NCCMM" for normal complex contact metric manifold , and $(M,\overline{\nabla})$ for a normal complex contact metric manifold M admitting a semi symmetric metric connection $\overline{\nabla}$.

Lemma 1. *On $(M,\overline{\nabla})$ we have*

$$(\overline{\nabla}_T U)=T-GT+\sigma(T)V-u(T)[U+V] \tag{2}$$
$$(\overline{\nabla}_T V)=T+HT-\sigma(T)U-v(T)[U+V] \tag{3}$$

for an arbitrary vector field T.

Proof. Let T be an arbitrary vector field on $(M,\overline{\nabla})$. From (1) we have

$$\begin{aligned}\overline{\nabla}_T U&=\nabla_T U+u(U)T-g(T,U)U+v(U)T-g(T,U)V\\&=-GT+\sigma(T)V+T-u(T)U-u(T)V\\&=-GT+T+\sigma(T)V-u(T)[U+V].\end{aligned}$$

Similarly we get

$$\begin{aligned}\overline{\nabla}_T V&=\nabla_T V+u(V)T-g(T,V)U+v(V)T-g(T,V)V\\&=HT-\sigma(T)U+T-v(T)U-v(T)V\\&=HT+T-\sigma(T)U-v(T)[U+V].\end{aligned}$$

□

Corollary 1. *On $(M,\overline{\nabla})$ we have*

$$\begin{aligned}&\overline{\nabla}_U U=(1+\sigma(U))V, &&\overline{\nabla}_V V=-(1+\sigma(V))U\\&\overline{\nabla}_U V=(1-\sigma(U))U, &&\overline{\nabla}_V U=(1+\sigma(V))V.\end{aligned}$$

Corollary 2. *On $(M,\overline{\nabla})$ we have*

$$\overline{\sigma}(Z)=\sigma(Z)+v(Z)-u(Z) \tag{4}$$

for arbitrary vector field Z on M.

4 Curvature Properties of Normal Complex Contact Metric Manifolds Admitting a Semi Symmetric Metric Connection

The Riemannian and Ricci curvature properties of $(M,\overline{\nabla})$ is given in this section.

Theorem 1. *On* $(M,\overline{\nabla})$ *we have*

$$\begin{aligned}\overline{R}(T,W)Z = R(T,W)Z &\\ +[u(Z)u(W)+u(Z)v(W)+v(Z)u(W)+v(Z)v(W)-2g(W,Z)&\\ +g(GW,Z)+g(HW,Z)+\sigma(W)[u(Z)-v(Z)]]T&\\ +[-u(Z)u(T)-u(Z)v(T)-v(Z)u(T)-v(Z)v(T)+2g(T,Z)&\\ -g(GT,Z)-g(HT,Z)+\sigma(T)[v(Z)-u(Z)]]W&\\ +[g(W,Z)[u(T)+v(T)]-g(T,Z)[u(W)+v(W)]+\sigma(T)g(W,Z)&\\ -\sigma(W)g(T,Z)]U&\\ +[g(W,Z)[u(T)+v(T)]-g(T,Z)[u(W)+v(W)]-\sigma(T)g(W,Z)&\\ +\sigma(W)g(T,Z)]V&\\ +g(W,Z)(GT+HT)-g(T,Z)(GW+HW)&\end{aligned} \tag{5}$$

where T,W,Z *are arbitrary vector fields on* M *and* $R,\overline{R}$ *are the Riemannian curvature tensor of* ∇ *and* $\overline{\nabla}$ *, respectively.*

Proof. It is known that for arbitrary vector fields X,Y,Z on M, the Riemannian curvature $\overline{R}$ is given by

$$\overline{R}(T,W,Z)=\overline{\nabla}_T\overline{\nabla}_W Z-\overline{\nabla}_W\overline{\nabla}_T Z-\overline{\nabla}_{[T,W]}Z. \tag{6}$$

From (1) we obtain $\overline{\nabla}_T\overline{\nabla}_W Z$, $\overline{\nabla}_W\overline{\nabla}_T Z$ and $\overline{\nabla}_{[W,W]}Z$ as below:

$$\begin{aligned}\overline{\nabla}_T\overline{\nabla}_W Z &= \nabla_T\nabla_W Z+u(\nabla_W Z)T-g(T,\nabla_W Z)U+v(\nabla_W Z)T-g(T,\nabla_W Z)V\\ &+g(\nabla_T U,Z)W+g(U,\nabla_T Z)W+u(Z)\nabla_T W+u(Z)u(W)T\\ &-u(Z)g(T,W)U+u(Z)v(W)T-u(Z)g(T,W)V-g(\nabla_T W,Z)U\\ &-g(W,\nabla_T Z)U-g(W,Z)\nabla_T U-g(W,Z)T+g(W,Z)u(T)U+g(W,Z)u(T)V\\ &+g(\nabla_T V,Z)W+g(V,\nabla_T Z)W+v(Z)\nabla_T W+v(Z)u(W)T\\ &-v(Z)g(T,W)U+v(Z)v(W)T-v(Z)g(T,W)V-g(\nabla_T W,Z)V\\ &-g(W,\nabla_T Z)V-g(W,Z)\nabla_T V+g(W,Z)v(T)U-g(W,Z)T+g(W,Z)v(T)V\end{aligned}$$

$$\begin{aligned}\overline{\nabla}_W\overline{\nabla}_T Z &= \nabla_W\nabla_T Z+u(\nabla_T Z)W-g(W,\nabla_T Z)U+v(\nabla_T Z)W-g(W,\nabla_T Z)V\\ &+g(\nabla_W U,Z)T+g(U,\nabla_W Z)T+u(Z)\nabla_W T+u(Z)u(T)W\\ &-u(Z)g(W,T)U+u(Z)v(T)W-u(Z)g(W,T)V-g(\nabla_W T,Z)U\\ &-g(T,\nabla_W Z)U-g(T,Z)\nabla_W U-g(T,Z)W+g(T,Z)u(W)U+g(T,Z)u(W)V\\ &+g(\nabla_W V,Z)T+g(V,\nabla_W Z)T+v(Z)\nabla_W T+v(Z)u(T)W-v(Z)g(W,T)U\\ &+v(Z)v(T)W-v(Z)g(W,T)V-g(\nabla_W T,Z)V\\ &-g(T,\nabla_W Z)V-g(T,Z)\nabla_W V+g(T,Z)v(W)U-g(T,Z)W+g(T,Z)v(W)V,\end{aligned}$$

$$\overline{\nabla}_{[T,W]}Z=\nabla_{[T,W]}Z+u(Z)[T,W]-g([T,W],Z)U+v(Z)[T,W]-g([T,W],Z)V$$

$$
\begin{aligned}
&= \nabla_{[T,W]}Z + u(Z)(\nabla_T W - \nabla_W T) - g(\nabla_T W - \nabla_W T, Z)U \\
&+ v(Z)(\nabla_T W - \nabla_W T) - g(\nabla_T W - \nabla_W T, Z)V \\
&= \nabla_{[T,W]}Z + u(Z)\nabla_T W - u(Z)\nabla_W T - g(\nabla_T W, Z)U + g(\nabla_W T, Z)U \\
&+ v(Z)\nabla_T W - v(Z)\nabla_W T - g(\nabla_T W, Z)V + g(\nabla_W T, Z)V.
\end{aligned}
$$

By consider all these equalities we get 6. □

Also we have

$$
\begin{aligned}
\overline{R}(T,W,Z,Y) = R(T,W,Z,Y)& \\
&+ [u(Z)u(W) + u(Z)v(W) + v(Z)u(W) + v(Z)v(W) - 2g(W,Z) \\
&+ g(GW,Z) + g(HW,Z) + \sigma(W)[u(Z) - v(Z)]]g(T,Y) \\
&+ [-u(Z)u(T) - u(Z)v(T) + v(Z)u(T) - v(Z)v(T) + 2g(T,Z) \\
&- g(GT,Z) - g(HT,Z) + \sigma(T)[v(Z) - u(Z)]]g(W,Y) \\
&+ [g(W,Z)[u(T) + v(T)] - g(T,Z)[u(W) + v(W)] + \sigma(T)g(W,Z) \\
&- \sigma(W)g(T,Z)]u(Y) \\
&+ [g(W,Z)[u(T) + v(T)] - g(T,Z)[u(W) + v(W)] - \sigma(T)g(W,Z) \\
&+ \sigma(W)g(T,Z)]v(Y) \\
&+ g(W,Z)[g(GT,Y) + g(HT,Y)] - g(T,Z)[g(GW,Y) + g(HW,Y)]
\end{aligned} \tag{7}
$$

As we know that Riemannian curvature tensor R has some symmetric properties. The Riemannian curvature tensor $\overline{R}$ of $(M,\overline{\nabla})$ has the following symmetry properties.

$$
\begin{aligned}
\overline{R}(T,W,Z,Y) &= -\overline{R}(W,T,Z,Y) \\
\overline{R}(T,W,Z,Y) &= -\overline{R}(T,W,Y,Z)
\end{aligned}
$$

$$
\begin{aligned}
\overline{R}(T,W,Z,Y) - \overline{R}(Z,Y,T,W) = &-2g(T,Y)g(GZ + HZ, W) + 2g(W,Y)g(GZ + HZ, T) \\
&+ 2g(T,Z)g(GY + HY, W) - 2g(W,Z)g(GY + HY, T) \\
&+ (u(T) - v(T))[\sigma(Z)g(W,Y) - \sigma(Y)g(W,Z)] \\
&+ (u(W) - v(W))[\sigma(Y)g(T,Z) - \sigma(Z)g(T,Y)] \\
&+ (u(Z) - v(Z))[\sigma(W)g(T,Z) - \sigma(T)g(W,Y)] \\
&+ (u(Y) - v(Y))[\sigma(T)g(W,Z) - \sigma(W)g(T,Z)].
\end{aligned}
$$

Also similar to Bianchi identity for R we have

$$
\begin{aligned}
\overline{R}(T,W)Z + \overline{R}(W,Z)T + \overline{R}(Z,T)W = &[2g(GW,Z) + 2g(HW,Z) \\
&+ \sigma(W)(u(Z) - v(Z)) + \sigma(Z)(v(W) - u(Z))]T \\
&+ [-2g(GT,Z) - 2g(HT,Z) \\
&+ \sigma(T)(v(Z) - u(Z)) + \sigma(Z)(u(T) - v(T))]W \\
&+ [2g(GT,W) + 2g(HT,W) \\
&+ \sigma(T)(u(W) - v(W)) + \sigma(W)(v(T) - u(T))]Z.
\end{aligned}
$$

As we know for a normal complex contact metric manifold [15], we have

$$
R(GT,GW,GZ,GY) = R(HT,HW,HZ,HY) = R(T,W,Y,Z).
$$

For T,W,Z,Y vector fields on M we get

$$\begin{aligned}\overline{R}(GT,GW,GZ,GY)-\overline{R}(HT,HW,HZ,HY) &= 2[g(GW,Z)+g(HW,Z)]g(T,Y)\\ &-2[g(GT,Z)+g(HT,Z)]g(W,Y)\\ &+2g(W,Z)[g(GT,Y)+g(T,HY)]\\ &+2g(T,Z)[g(W,GY)\\ &+g(HW,Y)],\\ \overline{R}(JT,JW,JZ,JY)-\overline{R}(T,W,Z,Y) &= -2[g(GW,Z)+g(HW,Z)]g(T,Y)\\ &+2[g(GT,Z)+g(HT,Z)]g(W,Y)\\ &-2g(W,Z)[g(GT,Y)+g(HT,Y)]\\ &+2g(T,Z)[g(GW,Y)+g(HW,Y)],\end{aligned}$$

$$\overline{R}(T,GT,HT,T)=R(T,GT,HT,T), \tag{8}$$

$$\overline{R}(T,HT,GT,T)=R(T,HT,GT,T), \tag{9}$$

$$\overline{R}(T,GT,HT,T)=\overline{R}(T,HT,GT,T). \tag{10}$$

These results let us to obtain curvature properties of $(M,\overline{\nabla})$.

Corollary 3. *For T,Y,U,V,W on $(M,\overline{\nabla})$ we have*

$$\begin{aligned}\overline{R}(T,Y)U &= R(T,Y)U+[v(Y)-u(Y)+\sigma(Y)]T\\ &+[u(T)-v(T)-\sigma(T)]Y\\ &+(u(Y)v(T)-u(T)v(Y)+\sigma(T)u(Y)-\sigma(Y)u(T))U\\ &+(u(Y)v(T)-u(T)v(Y)-\sigma(T)u(Y)+\sigma(Y)u(T))V\\ &+u(Y)(GT+HT)-u(T)(GY+HY),\\ \overline{R}(T,Y)V &= R(T,Y)V+[u(Y)-v(Y)-\sigma(Y)]T\\ &+[u(T)+v(T)+\sigma(T)]Y\\ &+(v(Y)u(T)-v(T)u(Y)+\sigma(T)v(Y)-\sigma(Y)v(T))U\\ &+(v(Y)u(T)-v(T)u(Y)-\sigma(T)v(Y)+\sigma(Y)v(T))V\\ &+v(Y)(GT+HT)-v(T)(GY+HY),\\ \overline{R}(T,U)U &= R(T,U)U+[\sigma(U)-1]T-u(T)[\sigma(U)-1]U\\ &+[v(T)-\sigma(T)+\sigma(U)u(T)]V+GT+HT,\\ \overline{R}(T,V)V &= R(T,V)V-[\sigma(V)+1]T+v(T)[\sigma(V)+1]V\\ &+[u(T)+\sigma(T)-\sigma(V)v(T)]U+GT+HT,\\ \overline{R}(T,V)U &= R(T,V)U+[\sigma(V)+1]T-u(T)[1+\sigma(V)]U\\ &+[-v(T)-\sigma(T)+\sigma(V)u(T)]V,\\ \overline{R}(T,U)V &= R(T,U)V+[-\sigma(U)+1]T-v(T)[1-\sigma(U)]V\\ &+[-u(T)+\sigma(T)-\sigma(U)v(T)]U,\\ \overline{R}(T,U)W &= R(T,U)W+[v(W)-u(W)+\sigma(U)[u(W)-v(W)]]T\\ &+[-v(W)u(T)-v(W)v(T)+g(T,W)-g(GT,W)-g(HT,W)\\ &+\sigma(T)v(W)-\sigma(U)g(T,W)]U\\ &+[u(W)u(T)+u(W)v(T)-g(T,W)-\sigma(T)u(W)+\sigma(U)g(T,W)]V\\ &+u(W)(GT+HT),\end{aligned}$$

$$
\begin{aligned}
\overline{R}(T,V)W &= R(T,V)W + [u(W) - v(W) + \sigma(V)[u(W) - v(W)]]T \\
&\quad + [-u(W)u(T) - u(W)v(T) + g(T,W) - g(GT,W) - g(HT,W) \\
&\quad - \sigma(T)u(W) + \sigma(V)g(T,Z)]V \\
&\quad + [u(T)v(Z) + v(Z)v(T) - g(T,Z) + \sigma(T)v(Z) - \sigma(V)g(T,Z)]U \\
&\quad + v(Z)(GT + HT), \\
\overline{R}(V,U)U &= R(V,U)U + [\sigma(U) - \sigma(V)]V, \\
\overline{R}(U,V)V &= R(V,V)V + [\sigma(U) - \sigma(V)]U.
\end{aligned}
$$

An other geometric important object in the complex contact geometry is $d\sigma$. In [15] an equality for $d\sigma$. By following Proposition we present a new version of $d\sigma$ on a normal complex contact metric manifold M was obtained.

Proposition 1. *We have on* $(M,\overline{\nabla})$

$$
d\overline{\sigma}(Z,T) = d\sigma(Z,T) + g(GZ,T) + g(HZ,T) + \frac{1}{2}[(u\wedge\sigma)(Z,T) + (v\wedge\sigma)(Z,T)]
$$

for all $Z,T \in \Gamma(TM)$.

Proof. For any $Z,T \in \Gamma(TM)$ we have

$$
\begin{aligned}
2d\overline{\sigma}(Z,T) &= Z(\overline{\sigma}(T)) - T(\overline{\sigma}(Z)) - \overline{\sigma}([Z,T]) \\
&= Zg(\overline{\nabla}_T U,V) - Tg(\overline{\nabla}_Z U,V) - g(\overline{\nabla}_{[Z,T]}U,V) \\
&= g(\overline{\nabla}_Z\overline{\nabla}_T U,V) + g(\overline{\nabla}_T U,\overline{\nabla}_Z V) \\
&\quad - g(\overline{\nabla}_T\overline{\nabla}_Z U,V) - g(\overline{\nabla}_Z U,\overline{\nabla}_T V) \\
&\quad - g(\overline{\nabla}_{[Z,T]}U,V).
\end{aligned}
$$

Also from (2) and (5) we get

$$
\begin{aligned}
2d\overline{\sigma}(Z,T) &= R(Z,T,U,V) + \sigma(T)[u(Z)+v(Z)] - \sigma(Z)[u(T)+v(T)] \\
&\quad + g(T - GT + \sigma(T)V - u(T)U - u(T)V, Z + HZ - \sigma(Z)U - v(Z)U - v(Z)V) \\
&\quad - g(Z - GZ + \sigma(Z)V - u(Z)U - u(Z)V, T + HT - \sigma(T)U - v(T)U - v(T)V) \\
&= R(Z,T,U,V) + \sigma(T)[u(Z)+v(Z)] - \sigma(Z)[u(T)+v(T)] \\
&\quad + g(T,Z) + g(T,HZ) - \sigma(Z)u(T) - v(Z)u(T) - v(Z)v(T) - g(GT,Z) \\
&\quad - g(GT,HZ) + \sigma(T)v(Z) - \sigma(T)v(Z) - u(T)u(Z) + u(T)\sigma(Z) \\
&\quad + u(T)v(Z) - u(T)v(Z) + u(T)v(Z) \\
&\quad - g(Z,T) - g(Z,HT) + \sigma(T)u(Z) + v(T)u(Z) + v(T)v(Z) + g(GZ,T) \\
&\quad - g(GZ,HT) - \sigma(Z)v(T) + \sigma(Z)v(T) + u(T)u(Z) - u(Z)\sigma(T) \\
&\quad - u(Z)v(T) + u(Z)v(T) - u(Z)v(T) \\
&= R(Z,T,U,V) + \sigma(T)[u(Z)+v(Z)] - \sigma(Z)[u(T)+v(T)] \\
&\quad + 2g(HZ,T) + 2g(GZ,T) - g(GHZ,T) + g(HGZ,T) \\
&= R(Z,T,U,V) + \sigma(T)[u(Z)+v(Z)] - \sigma(Z)[u(T)+v(T)] \\
&\quad - 2g(HZ,T) - 2g(GZ,T) - g(-JZ - u(Z)U + v(Z)V, T) \\
&\quad + g(JZ + u(Z)U - v(Z)V, T) \\
&= R(Z,T,U,V) + \sigma(T)[u(Z)+v(Z)] - \sigma(Z)[u(T)+v(T)]
\end{aligned}
$$

$$+2g(HZ,T)+2g(GZ,T)+g(JZ,T)+u(Z)u(T)-v(Z)v(T)$$
$$+g(JZ,T)+u(Z)u(T)-v(Z)v(T)$$
$$=R(Z,T,U,V)+\sigma(T)[u(Z)+v(Z)]-\sigma(Z)[u(T)+v(T)]$$
$$+2g(HZ,T)+2g(GZ,T)+2g(JZ,T)+2(u(Z)u(T)-v(Z)v(T))$$
$$=R(Z,T,U,V)+\sigma(T)[u(Z)+v(Z)]-\sigma(Z)[u(T)+v(T)]$$
$$+2g(HZ,T)+2g(GZ,T)+2g(JZ,T)+2u\wedge v(Z,T).$$

Here

$$2d\sigma(Z,T)=R(Z,T,U,V)+2g(JZ,T)+2u\wedge v(Z,T)$$

thus we obtain

$$\begin{aligned} d\overline{\sigma}(Z,T) &= d\sigma(Z,T)+g(GZ,T)+g(HZ,T) \\ &\quad +\frac{1}{2}[(u\wedge\sigma)(Z,T)+(v\wedge\sigma)(Z,T)]. \end{aligned}$$

□

From this Proposition we get following corollary.

Corollary 4. *On $(M,\overline{\nabla})$ we have*

$$d\overline{\sigma}(T,U)=d\sigma(T,U)-\frac{1}{2}\sigma(T) \text{ and } d\overline{\sigma}(U,V)=d\sigma(U,V)+\frac{1}{2}[\sigma(V)-\sigma(U)].$$

for $T\in\mathscr{H}$

Theorem 2. *The sectional curvature of $(M,\overline{\nabla})$ is*

$$\overline{k}(T,W)=k(T,W)-2 \tag{11}$$

where T,W are unit, mutually orthogonal and horizontal vector fields on M, and k is the sectional curvature of (M,∇).

Proof. Let W,T be unit, mutually orthogonal and horizontal vector fields on M. By setting $Z=W, Y=T$ in (7) we get

$$\begin{aligned} \overline{k}(T,W) &= \overline{R}(T,W,W,T) \\ &= R(T,W,W,T) \\ &\quad +[-2g(W,W)+g(GW,W)+g(HW,W)]g(T,T) \\ &\quad +[2g(T,W)-g(GT,W)-g(HT,W)]g(W,T) \\ &\quad +g(W,W)(g(GT,T)+g(HT,T)) \\ &\quad -g(T,W)(g(GW,T)+g(HW,T)). \end{aligned} \tag{12}$$

Also since $g(GW,W)=g(HT,T)=0$ and $k(T,W)=R(T,W,W,T)$ we have

$$\overline{k}(T,W)=k(T,W)-2.$$

□

Corollary 5. *For unit vector fields T on $(M,\overline{\nabla})$ we have*

$$\overline{k}(T,JT)=k(T,JT)-2 \tag{13}$$
$$\overline{k}(T,GT)=k(T,GT)-2 \tag{14}$$
$$\overline{k}(T,HT)=k(T,HT)-2. \tag{15}$$

Theorem 3. *The $\mathscr{GH}-$sectional curvature of $(M,\overline{\nabla})$ is given by*

$$\overline{\mathscr{GH}}_{a,b}(T) = \mathscr{GH}_{a,b}(T) - 2 \tag{16}$$

where T is unit horizontal vector field on M.

Proof. From the definition of $\mathscr{GH}-$sectional curvature, for unit horizontal vector field T on M we have

$$\begin{aligned}\overline{\mathscr{GH}}_{a,b}(T) &= \overline{k}(T, aGT + bHT) \\ &= \frac{\overline{R}(T, aGT + bHT, aGT + bHT, T)}{g(T,T)g(aGT + bHT, aGT + bHT) - g(T, aGT + bHT)^2}.\end{aligned}$$

Then the Riemannian curvature tensor is given by

$$\begin{aligned}\overline{R}(T, aGT + bHT, aGT + bHT, T) &= a\overline{R}(T, aGT + bHT, GT, T) \\ &\quad + b\overline{R}(T, aGT + bHT, HT, T) \\ &= a^2\overline{R}(T, GT, GT, T) + ab\overline{R}(T, HT, GT, T) \\ &\quad + ba\overline{R}(T, GT, HT, T) + b^2\overline{R}(T, HT, HT, T).\end{aligned}$$

From (14),(15), (8) and (9) we get

$$\begin{aligned}\overline{R}(T, aGT + bHT, aGT + bHT, T) &= a^2R(T, GT, GT, T) - 2a^2 + abR(T, HT, GT, T) \\ &\quad + baR(T, GT, HT, T) + b^2R(T, HT, HT, T) - 2b^2 \\ &= R(T, aGT + bHT, aGT + bHT, T) - 2(a^2 + b^2).\end{aligned}$$

Since $R(T, aGT + bHT, aGT + bHT, T) = \mathscr{GH}(T)$ and $a^2 + b^2 = 1$ we obtain (16). □

Proposition 2. *For unit and horizontal vector field T on $(M,\overline{\nabla})$ we get*

$$\overline{k}(T, JT) = \frac{1}{2}[\overline{\mathscr{GH}}(T + GT) + \overline{\mathscr{GH}}(T - GT)] + 3. \tag{17}$$

Proof. By using (16) and (13) in (17) we obtain

$$\overline{k}(T, JT) + 2 = \frac{1}{2}[\overline{\mathscr{GH}}(T + GT) + 2 + \overline{\mathscr{GH}}(T - GT) + 2] + 3.$$

So we get (17). □

Corollary 6. *If $\overline{\mathscr{GH}}-$sectional curvature of $(M,\overline{\nabla})$ is constant then we have*

$$\overline{k}(T, JT) = \overline{\mathscr{GH}}(T) + 3. \tag{18}$$

Theorem 4. *The Ricci curvature of $(M,\overline{\nabla})$ is given by*

$$\begin{aligned}\overline{Ric}(T,W) &= Ric(T,W) \\ &\quad + 4m[u(W)u(T) + u(W)v(T) + v(W)u(T) + v(W)v(T) \\ &\quad + g(GT,W) + g(HT,W) + \sigma(T)[u(W) - v(W)]] \\ &\quad + (-8m + \sigma(U) - \sigma(V))g(T,W)\end{aligned} \tag{19}$$

for all $T, W \in \Gamma(TM)$.

Proof. Let T and W be two arbitrary vector fields on $(M,\overline{\nabla})$ and $(E_1,E_2,E_3,...,E_{4m},U,V)$ be orthonormal basis of TM. Then from (5) we get

$$\begin{aligned}\overline{R}(E_i,T,W,E_i) = R(E_i,T,W,E_i) \\ &+[u(W)u(T)+u(W)v(T)+v(W)u(T)+v(W)v(T)-2g(T,W)\\ &+g(GT,W)+g(HT,W)+\sigma(T)[u(W)-v(W)]]g(E_i,E_i)\\ &+[-u(W)u(E_i)-u(W)v(E_i)-v(W)u(E_i)-v(W)v(E_i)\\ &+2g(E_i,W)-g(GE_i,W)-g(HE_i,W)\\ &-\sigma(E_i)[v(W)-u(W)]]g(T,E_i)+[g(T,W)[u(E_i)+v(E_i)]\\ &-g(E_i,W)[u(T)+v(T)]+\sigma(E_i)g(T,W)\\ &-\sigma(T)g(E_i,W)]g(U,E_i)+[g(T,W)[u(E_i)+v(E_i)]\\ &-g(E_i,W)[u(T)+v(T)]-\sigma(E_i)g(T,W)\\ &+\sigma(T)g(E_i,W)]g(V,E_i)+g(T,W)(g(GE_i,E_i)\\ &+g(HE_i,E_i))-g(E_i,W)(g(GT,E_i)+g(HT,E_i)).\end{aligned}$$

For brevity let state

$$\begin{aligned}A=&[u(W)u(T)+u(W)v(T)+v(W)u(T)+v(W)v(T)\\ &-2g(T,W)+g(GT,W)+g(HT,W)+\sigma(T)[u(W)-v(W)]]\end{aligned}$$

then we have

$$\begin{aligned}\sum_{i=1}^{4m+2}\overline{R}(E_i,T,W,E_i) = &\sum_{i=1}^{4m+2}R(E_i,T,W,E_i)+\sum_{i=1}^{4m+2}Ag(E_i,E_i)\\ &+\sum_{i=1}^{4m+2}[-u(W)u(E_i)-u(W)v(E_i)-v(W)u(E_i)\\ &-v(W)v(E_i)+2g(E_i,W)-g(GE_i,W)\\ &-g(HE_i,W)+\sigma(E_i)[v(W)-u(W)]]g(T,E_i)\\ &+\sum_{i=1}^{4m+2}[g(T,W)[u(E_i)+v(E_i)]-g(E_i,W)[u(T)+v(T)]\\ &+\sigma(E_i)g(T,W)-\sigma(T)g(E_i,W)]g(U,E_i)\\ &+\sum_{i=1}^{4m+2}[g(T,W)[u(E_i)+v(E_i)]-g(E_i,W)[u(T)+v(T)]\\ &-\sigma(E_i)g(T,W)+\sigma(T)g(E_i,W)]g(V,E_i)\\ &+\sum_{i=1}^{4m+2}(g(T,W)(g(GE_i,E_i)+g(HE_i,E_i))\\ &-g(E_i,W)(g(GT,E_i)+g(HT,E_i))).\end{aligned}$$

Thus by direct computations the proof is completed. □

Also from the above theorem we get following corollaries:

Corollary 7. *On a $(M,\overline{\nabla})$ we have*

$$\overline{Ric}(U,U)=-2d\sigma(U,V)+(4m+1)\sigma(U)-\sigma(V), \tag{20}$$

$$\overline{Ric}(V,V) = -2d\sigma(U,V) - (4m+1)\sigma(V) + \sigma(U), \tag{21}$$
$$\overline{Ric}(U,V) = 4m(1-\sigma(U)). \tag{22}$$

Corollary 8. *Let T, W and Z be horizontal vector fields on $(M, \overline{\nabla})$. Then we have*

$$\begin{aligned}
\overline{Ric}(GT,GW) - \overline{Ric}(HT,HW) &= +8m(g(GT,W) - g(HT,W)),\\
\overline{Ric}(GT,GW) - \overline{Ric}(T,W) &= -8mg(HT,W),\\
\overline{Ric}(HT,HW) - \overline{Ric}(T,W) &= -8mg(GT,W),\\
\overline{Ric}(JT,JW) - \overline{Ric}(T,W) &= -8m[v(W)u(T) + u(W)v(T) - g(GT,W)\\
&\quad - g(HT,W)] + 4m[\sigma(JT)(v(W) + u(W))\\
&\quad - \sigma(T)[u(W) - v(W)]]
\end{aligned}$$

Corollary 9. *The scalar curvature and Ricci operator of $(M, \overline{\nabla})$ is given by*

$$\begin{aligned}
\overline{Scal} &= Scal - 32m^2 + (\sigma(U) - \sigma(V))(8m+2) - 8m,\\
\overline{Q}T &= QT + 4m[(u(T) + v(T))(U+V) + GT + HT\\
&\quad + \sigma(T)[U-V]] + (-8m + \sigma(U) - \sigma(V))T.
\end{aligned}$$

5 Flatness conditions on Normal Complex Contact Metric Manifolds Admitting a Semi Symmetric Metric Connection

A Riemannian manifold is called flat if its Riemannian curvature tensor vanishes. That means manifold is locally Euclidean. Also the flatness of a Riemannian manifold can be provided by some special transformations like conformal, concircular etc. If the manifold is flat under these special transformations, is called conformally flat, concircularly flat etc. On the other hand there are several tensors which are invariant of these special transformations, and can give flatness of the manifold when vanishes. The three of them are conformal, concircular and quasi-conformal curvature tensor. The flatness of conformal curvature tensor on a NCCMM was studied by Blair and Molina [4]. Two of present authors studied the flatness of concircular and quasi-concircular curvature tensors. A NCCMM is not conformal, concircular and quasi-conformal flat. In this section we study on the flatness of these tensors on $(M, \overline{\nabla})$. Conformal curvature tensor $\overline{\mathscr{C}}$, concircular curvature tensor $\overline{\mathscr{Z}}$, quasi-conformal curvature tensor $\overline{\tilde{C}}$ and conharmonic curvature tensor $\overline{K}$ of a $(2m+1)$-complex dimensional normal complex contact metric manifold M is defined by

$$\begin{aligned}
\overline{\mathscr{C}}(T,W)Z &= \overline{R}(T,W)Z + \frac{\overline{Scal}}{(4m+1)4m}(g(W,Z)T - g(T,Z)W)\\
&\quad + \frac{1}{4m}(g(T,Z)\overline{Q}W - g(W,Z)\overline{Q}T + \overline{Ric}(T,Z)W - \overline{Ric}(W,Z)T),
\end{aligned}$$

$$\overline{\mathscr{Z}}(T,W)Z = \overline{R}(T,W)Z - \frac{\overline{Scal}}{(4m+2)(4m+1)}[g(W,Z)T - g(T,Z)W],$$

$$\begin{aligned}
\overline{\tilde{C}}(T,W)Z &= p\overline{R}(T,W)Z + q[\overline{Ric}(W,Z)T - \overline{Ric}(T,Z)W + g(W,Z)\overline{Q}T\\
&\quad - g(T,Z)\overline{Q}W] - \frac{\overline{Scal}}{(4m+2)}[\frac{p}{4m+1} + 2q][g(W,Z)T - g(T,Z)W],
\end{aligned}$$

$$\overline{K}(T,W)Z = \overline{R}(T,W)Z - \frac{1}{4m}[\overline{Ric}(W,Z)T - \overline{Ric}(T,Z)W + g(W,Z)\overline{Q}T - g(T,Z)\overline{Q}W].$$

Theorem 5.

$(M,\overline{\nabla})$ is not conformal, concircular, quasi-conformal and conharmonic flat.

Proof. Assume that $(M,\overline{\nabla})$ is quasi-conformal flat. Then for $\forall T,W,Z,Y \in \Gamma(TM)$ we have

$$\begin{aligned} g(\overline{R}(T,W)Z,Y) = &-\frac{q}{p}[\overline{Ric}(W,Z)g(T,Y) - \overline{Ric}(T,Z)g(W,Y) \\ &+ g(W,Z)\overline{Ric}(T,Y) - g(T,Z)\overline{Ric}(W,Y)] \\ &+ \frac{\overline{Scal}}{(4m+2)}[\frac{1}{4m+1} + \frac{2q}{p}][g(W,Z)g(T,Y) - g(T,Z)g(W,Y)] \end{aligned} \tag{23}$$

and setting $T = Y = U$ ve$W = Z = U$ we get

$$\begin{aligned} g(\overline{R}(U,V)V,U) = &-\frac{q}{p}[\overline{Ric}(V,V)g(U,U) - \overline{Ric}(U,V)g(V,U) \\ &+ g(V,V)\overline{Ric}(U,U) - g(U,V)\overline{Ric}(V,U)] \\ &+ \frac{\overline{Scal}}{(4m+2)}[\frac{1}{4m+1} + \frac{2q}{p}][g(V,V)g(U,U) - g(U,V)g(V,U)] \\ = &-\frac{q}{p}[\overline{Ric}(V,V) + \overline{Ric}(U,U)] \\ &+ \frac{\overline{Scal}}{(4m+2)}[\frac{1}{4m+1} + \frac{2q}{p}]. \end{aligned}$$

From (20), (21) and (22) we obtain

$$d\sigma(U,V) = \frac{q(4m+2)+p}{4q+2p}(\sigma(U) - \sigma(V)) - \frac{\overline{Scal}}{(4m+2)}\frac{p}{(4q+2p)}[\frac{1}{4m+1} + \frac{2q}{p}].$$

On the other hand for $W = Z = U$, unit and mutually orthogonal T,Y vector fields we have

$$\begin{aligned} g(\overline{R}(T,U)U,T) = &-\frac{q}{p}[\overline{Ric}(U,U)g(T,T) - \overline{Ric}(T,U)g(U,T) + g(U,U)\overline{Ric}(T,T) \\ &- g(T,U)\overline{Ric}(U,T)] \\ &+ \frac{\overline{Scal}}{(4m+2)}[\frac{1}{4m+1} + \frac{2q}{p}][g(U,U)g(T,T) - g(T,U)g(U,T)] \\ = &-\frac{q}{p}[\overline{Ric}(U,U) + \overline{Ric}(T,T)] + \frac{\overline{Scal}}{(4m+2)}[\frac{1}{4m+1} + \frac{2q}{p}]. \end{aligned}$$

Thus from (20) and (21) we obtain

$$\overline{Ric}(T,T) = \frac{\overline{Scal}p}{q(4m+2)}[\frac{1}{4m+1} + \frac{2q}{p}] + 2d\sigma(U,V) - \frac{(4m+1)q+p}{q}\sigma(U) + \sigma(V). \tag{24}$$

Also for unit T vector field, from (23) $\overline{\mathscr{GH}}(T)$ is given by

$$\overline{\mathscr{GH}}(T) = g(\overline{R}(T,GT)GT,T) = -\frac{q}{p}[\overline{Ric}(GT,GT)g(T,T) - \overline{Ric}(T,GT)g(GT,T)$$

$$
\begin{aligned}
&+g(GT,GT)\overline{Ric}(T,T)-g(T,GT)\overline{Ric}(GT,T)] \\
&+\frac{\overline{Scal}}{(4m+2)}[\frac{1}{4m+1}+\frac{2q}{p}][g(GT,GT)g(T,T)-g(T,GT)g(GT,T)] \\
&=-\frac{q}{p}[\overline{Ric}(GT,GT)+\overline{Ric}(T,T)]+\frac{\overline{Scal}}{(4m+2)}[\frac{1}{4m+1}+\frac{2q}{p}] \\
&=-\frac{2q}{p}\overline{Ric}(T,T)+\frac{\overline{Scal}}{(4m+2)}[\frac{1}{4m+1}+\frac{2q}{p}].
\end{aligned}
$$

From (24) we get

$$
\begin{aligned}
\overline{\mathscr{G}\mathscr{H}}(T)=&-\frac{4q}{p}d\sigma(U,V)+\frac{(8m+2)q+2p}{p}\sigma(U) \\
&-\frac{2q}{p}\sigma(V)-\frac{\overline{Scal}}{4m+1}[\frac{1}{4m+1}+\frac{2q}{p}].
\end{aligned}
$$

Similarly the holomorphic sectional curvature is

$$
\begin{aligned}
\overline{k}(T,JT)=g(\overline{R}(T,JT)JT,T)=&-\frac{q}{p}[\overline{Ric}(JT,JT)g(T,T)-\overline{Ric}(T,JT)g(JT,T) \\
&+g(JT,JT)\overline{Ric}(T,T)-g(T,JT)\overline{Ric}(JT,T)]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\overline{Scal}}{(4m+2)}[\frac{1}{4m+2}+\frac{2q}{p}][g(JT,JT)g(T,T)-g(T,JT)g(JT,T)] \\
&=-\frac{q}{p}[\overline{Ric}(JT,JT)+\overline{Ric}(T,T)]+\frac{\overline{Scal}}{(4m+2)}[\frac{1}{4m+2}+\frac{2q}{p}] \\
&=-\frac{2q}{p}\overline{Ric}(T,T)+\frac{\overline{Scal}}{(4m+2)}[\frac{1}{4m+1}+\frac{2q}{p}].
\end{aligned}
$$

Thus from (24) we get

$$
\overline{k}(T,JT)=-\frac{4q}{p}d\sigma(U,V)+\frac{(8m+2)q+2p}{p}\sigma(U)-\frac{2q}{p}\sigma(V)-\frac{\overline{Scal}}{4m+1}[\frac{1}{4m+1}+\frac{2q}{p}].
$$

Therefore we obtain $\overline{\mathscr{G}\mathscr{H}}(T)=\overline{k}(T,JT)$. There is a contradiction from 18 and so our assumption is not true. By following same steps one can shown the non-existence of conformal and concircular flatness. □

6 Iwasawa Manifold Admitting a Semi Symmetric Metric Connection

An Iwasawa manifold is an important example of a compact complex manifold which does not admit any Kähler metric [6]. Fernandez and Gray [6] proved that an Iwasawa manifold has indefinite Kähler structure has symplectic forms each of which is Hermitian with respect to a complex structure .

The Iwasawa manifold is the compact quotient space $\Gamma\setminus H_{\mathbb{C}}$ formed from the right cosets of the discrete subgroup Γ given by the matrices whose entries z_1,z_2,z_3 are Gaussian integers where $H_{\mathbb{C}}$ is given by

$$
H_{\mathbb{C}}=\left\{\begin{pmatrix}1 & c_{12} & c_{13}\\ 0 & 1 & c_{23}\\ 0 & 0 & 1\end{pmatrix}:c_{12},c_{13},c_{23}\in\mathbb{C}\right\}\simeq\mathbb{C}^3.
$$

Like real Heisenberg group is an example of contact manifolds (see [3]), complex Heisenberg group has complex almost contact structure. This structure was given by Baikoussis et al. [1] and normality of the structure was obtained by Korkmaz [12]. Also this manifold is the initial point of the work of Korkmaz and it distinguishes Korkmaz's normality from IK-normality.

Blair and Turgut Vanlı [14] worked on corrected energy of Iwasawa manifolds and also Turgut Vanlı and Unal [15] obtained some curvature results. In this section we examine Iwasawa Manifold with a semi symmetric metric connection.

Let $\{e_1, e_1^*, e_2, e_2^*, U, V\}$ be an orthonormal frame of Iwasawa manifold which is given by

$$\begin{aligned} e_1 &= 2\left(\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3} + y_2\frac{\partial}{\partial y_3}\right), \\ e_1^* &= 2\left(\frac{\partial}{\partial y_1} - y_2\frac{\partial}{\partial x_3} + x_2\frac{\partial}{\partial y_3}\right) \\ e_2 &= 2\frac{\partial}{\partial x_2} \quad , e_2^* = 2\frac{\partial}{\partial y_2}, \\ U &= 2\frac{\partial}{\partial x_3} \quad , \ V = -2\frac{\partial}{\partial y_3}. \end{aligned} \tag{25}$$

Then for

$$G = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & y_2 & 0 & 0 \\ 0 & 0 & y_2 & -x_2 & 0 & 0 \end{bmatrix} \text{ and } H = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -y_2 & x_2 & 0 & 0 \\ 0 & 0 & x_2 & y_2 & 0 & 0 \end{bmatrix}$$

we have

$$\begin{aligned} &Ge_1 = -e_2, \ \ Ge_1^* = e_2^*, \quad Ge_2 = e_1, \quad Ge_2^* = -e_1^* \\ &He_1 = -e_2^*, \ \ He_1^* = -e_2, \ \ He_2 = e_1^*, \ \ He_2^* = e_1 \\ &Je_1 = -e_1^*, \ \ Je_1^* = e_1, \quad Je_2 = -e_2^*, \ \ Je_2^* = e_2. \end{aligned}$$

Also the Lie derivatives and Lie bracket of basis vectors are given by

$$[e_1, e_2] = -2U, \ [e_1, e_2^*] = 2V, \ [e_1^*, e_2] = 2V, \ [e_1^*, e_2^*] = 2U \tag{26}$$

and, from Kozsul formula we get

$$\nabla_{e_j} e_j = \nabla_{e_j} e_j = \nabla_{e_j} e_{j^*} = \nabla_{e_j^*} e_j^* = 0. \tag{27}$$

Also by using (26)and (27) we get

$$\begin{aligned} \nabla_{e_2} U &= -\nabla_{e_2^*} V = -e_1 & \qquad \nabla_{e_2^*} U &= \nabla_{e_2} V = e_1^* \\ \nabla_{e_1} U &= -\nabla_{e_1^*} V = e_2 & \qquad \nabla_{e_1} V &= \nabla_{e_1^*} U = -e_2^* \\ -\nabla_{e_1} e_2 &= \nabla_{e_1^*} e_2^* = U & \qquad \nabla_{e_1} e_2^* &= \nabla_{e_1^*} e_2 = V. \end{aligned}$$

In addition for $e_i, e_j \in \mathscr{H}$ we have $Ric(e_i, e_i) = Ric(e_i^*, e_i^*) = -4$ and $Ric(U,U) = Ric(V,V) = 4$. By using above equations and from the definition of the semi symmetric metric connection $\overline{\nabla}$ we get following corollary.

Corollary 10. *On the Iwasawa manifold admitting $\overline{\nabla}$ we get*

$$\begin{aligned}
\overline{\nabla}_{e_i}e_j &= \nabla_{e_i}e_j - \delta_i^j(U+V)\\
\overline{\nabla}_{e_i^*}e_j^* &= \nabla_{e_i^*}e_j^* - \delta_i^j(U+V)\\
\overline{\nabla}_{e_i^*}e_j &= \nabla_{e_i^*}e_j, \qquad \overline{\nabla}_{e_i}e_j^* = \nabla_{e_i}e_j^*\\
\overline{\nabla}_U e_i &= \nabla_U e_i, \qquad \overline{\nabla}_V e_i = \nabla_V e_i
\end{aligned}$$

where $e_i, e_j \in \mathscr{H}$.

We obtain curvatures of Iwasawa manifold admitting $\overline{\nabla}$ as follow:

$$\begin{array}{lll}
\overline{R}(e_1,e_1^*)e_1 = 2e_1^* - e_2^* - e_2 & \overline{R}(e_1,e_2)e_1 = 5e_2 + e_1^* & \overline{R}(e_1^*,e_2^*)e_1 = -e_2 - e_1^*\\
\overline{R}(e_1,e_1^*)e_1^* = -2e_1 - e_2 + e_2^* & \overline{R}(e_1,e_2)e_1^* = -e_2^* - e_1 & \overline{R}(e_1^*,e_2^*)e_1^* = 5e_2^* + e_1\\
\overline{R}(e_1,e_1^*)e_2 = -2e_2^* + e_1 + e_1^* & \overline{R}(e_1,e_2)e_2 = -5e_1 + e_2^* & \overline{R}(e_1^*,e_2^*)e_2 = e_1 - e_2^*\\
\overline{R}(e_1,e_1^*)e_2^* = 2e_2 + e_1 - e_1^* & \overline{R}(e_1,e_2)e_2^* = e_1^* - e_2 & \overline{R}(e_1^*,e_2^*)e_2^* = -5e_1^* + e_2\\
\overline{R}(e_1,e_1^*)U = -2V & &
\end{array}$$

$$\begin{array}{lll}
\overline{R}(e_1^*,e_2)e_1 = e_1^* + e_2^* & \overline{R}(e_1,e_2^*)e_1 = 5e_2^* + e_1^* & \overline{R}(e_2,e_2^*)e_1 = -2e_1^* - e_2 - e_2^*\\
\overline{R}(e_1^*,e_2)e_1^* = 5e_2 - e_1 & \overline{R}(e_1,e_2^*)e_1^* = e_2 - e_1 & \overline{R}(e_2,e_2^*)e_1^* = 2e_1 - e_2 + e_2^*\\
\overline{R}(e_1^*,e_2)e_2 = -5e_1^* + e_2^* & \overline{R}(e_1,e_2^*)e_2 = e_2^* - e_1^* & \overline{R}(e_2,e_2^*)e_2 = e_1 + e_1^* + 2e_2^*\\
\overline{R}(e_1^*,e_2)e_2^* = -e_2 - e_1 & \overline{R}(e_1,e_2^*)e_2^* = -5e_1 - e_2 & \overline{R}(e_2,e_2^*)e_2^* = e_1 - 2e_2^* - e_1^*
\end{array}$$

$$\begin{array}{ll}
\overline{R}(e_1,U)V = \overline{R}(e_1^*,V)U = e_1 + e_1^* & \overline{R}(e_1,e_2)U = \overline{R}(e_1,e_2)V = \overline{R}(e_1,e_2^*)U = 0\\
\overline{R}(e_2,U)V = \overline{R}(e_2^*,V)U = e_2 + e_2^* & \overline{R}(e_1^*,e_2)U = \overline{R}(e_1^*,e_2)V = \overline{R}(e_2^*,e_1)U = 0\\
\overline{R}(e_1,V)U = -\overline{R}(e_1^*,U)V = e_1 - e_1^* & \overline{R}(e_1^*,e_2^*)U = \overline{R}(e_1^*,e_2^*)V = \overline{R}(e_1,e_2^*)V = 0\\
\overline{R}(e_2,V)U = -\overline{R}(e_2^*,U)V = e_2 - e_2^* & \overline{R}(e_1,e_1^*)V = \overline{R}(e_2,e_2^*)U = 2U
\end{array}$$

$$\begin{array}{llll}
\overline{R}(e_2,U)e_1 = -U & \overline{R}(e_2,V)e_1 = -V & \overline{R}(e_2^*,U)e_1 = U & \overline{R}(e_2^*,V)e_1 = V\\
\overline{R}(e_2,U)e_1^* = U & \overline{R}(e_2,V)e_1^* = V & \overline{R}(e_2^*,U)e_1^* = U & \overline{R}(e_2^*,V)e_1^* = V\\
\overline{R}(e_2,U)e_2 = -V & \overline{R}(e_2,V)e_2 = -U & \overline{R}(e_2^*,U)e_2 = V & \overline{R}(e_2^*,V)e_2 = -U\\
\overline{R}(e_2,U)e_2^* = -V & \overline{R}(e_2,V)e_2^* = U & \overline{R}(e_2^*,U)e_2^* = -V & \overline{R}(e_2^*,V)e_2^* = -U.
\end{array}$$

As we know on Iwasawa manifold $\sigma = 0$. Thus from (4) we get $\overline{\sigma}(T) = v(T) - u(T)$ and from (19) we get

$$\begin{aligned}
\overline{Ric}(T,W) = Ric(T,W) &+ 4(u(T)u(W) + v(T)u(W) + u(T)v(W) + v(T)v(W)\\
&+ g(GT,W) + g(HT,W)) - 8g(T,W).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\overline{Ric}(e_1,e_1) &= Ric(e_1,e_1) + 4(u(e_1)u(e_1) + v(e_1)u(e_1) + u(e_1)v(e_1) + v(e_1)v(Y)\\
&+ g(Ge_1,e_1) + g(He_1,e_1)) - 8g(e_1,e_1)\\
&= -4 + 4(g(-e_2,e_1) + g(-e_2^*,e_1) - 8\\
&= -12.
\end{aligned}$$

Similarly we have $\overline{Ric}(e_1^*,e_1^*) = \overline{Ric}(e_2,e_2) = \overline{Ric}(e_2^*,e_2^*) = -12$ ve $\overline{Ric}(U,U) = \overline{Ric}(V,V) = 0$. By direct computation we get $\overline{Scal} = -48$. On the other hand for sectional curvatures of Iwasawa manifold admitting $\overline{\nabla}$

$$
\begin{aligned}
&k(e_3,e_3^*)=k(e_1,e_1^*)=k(e_1,e_2^*)=k(e_1^*,e_2)=k(e_2,e_2^*)=0\\
&k(e_1,e_3)=k(e_1^*,e_3)=k(e_2,e_3)=k(e_2^*,e_3)=1,\\
&k(e_1,e_3^*)=k(e_1^*,e_3^*)=k(e_2,e_3^*)=k(e_2^*,e_3^*)=1
\end{aligned}
$$

$$k(e_1,e_2)=3 \text{ ve } k(e_1^*,e_2^*)=1.$$

From 11 we get

$$
\begin{aligned}
&\overline{k}(e_3,e_3^*)=\overline{k}(e_1,e_1^*)=\overline{k}(e_1,e_2^*)=\overline{k}(e_1^*,e_2)=\overline{k}(e_2,e_2^*)=-2\\
&\overline{k}(e_1,U)=\overline{k}(e_1^*,U)=\overline{k}(e_2,U)=\overline{k}(e_2^*,U)=-1,\\
&\overline{k}(e_1,V)=\overline{k}(e_1^*,V)=\overline{k}(e_2,V)=\overline{k}(e_2^*,V)=-1,\\
&\overline{k}(e_1,e_2)=1 \text{ ve } \overline{k}(e_1^*,e_2^*)=-1.
\end{aligned}
$$

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