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## A modified invariant subspace method for solving partial differential equations with non-singular kernel fractional derivatives

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### Abstract

In this work, the well known invariant subspace method has been modified and extended to solve some partial differential equations involving Caputo-Fabrizio (CF) or Atangana-Baleanu (AB) fractional derivatives. The exact solutions are obtained by solving the reduced systems of constructed fractional differential equations. The results show that this method is very simple and effective for constructing explicit exact solutions for partial differential equations involving new fractional derivatives with nonlocal and non-singular kernels, such solutions are very useful to validate new numerical methods constructed for solving partial differential equations with CF and AB fractional derivatives.

**Keywords:** Modified invariant subspace method, Caputo-Fabrizio fractional derivative, Atangana-Baleanu fractional derivative, Exact solution, Partial differential equations.

**AMS 2010 codes:** 26A33; 35R11; 47A15.

## 1 Introduction

Fractional calculus provides an important characteristic to describe the complicated physical phenomena with memory effects. For this reason, the fractional calculus is becoming increasingly used as a modeling tool in physics, engineering and control processing in various fields of sciences such as fluid dynamics, plasma physics, mathematical biology and chemical kinetics, diffusion, etc [1–4]. Due to their properties, fractional derivatives and integrals make this kind of calculus a good candidate to describe such phenomena. Some fundamental definitions of fractional derivatives were given by Riemann-Liouville and Liouville-Caputo [5–9]. Recently, Caputo and Fabrizio defined a new fractional derivative without singular kernel [10] named Caputo-Fabrizio derivative with specific properties, the derivative of a constant is zero and the initial conditions used in the fractional differential equations having a physical interpretation. Later, Atangana and Baleanu proposed

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another fractional derivative with non-local and non-singular kernel named Atangana-Baleanu derivative [11]. Besides, seeking exact solutions of fractional partial differential equations is not an easy task, and it's remain a relevant problem. Therefore, many powerful methods have been proposed for solving analytically the fractional partial differential equations. Such methods include; Homotopy Perturbation Method [12], Homotopy Perturbation coupled with Sumudu Transform [13], Adomian Meecomposition Method [14], Variational Iteration Method [15], Fractional Iteration Method [16], etc.

On the author hand, recent investigations show that the invariant subspace method, developed by V.A. Galaktionov and S.R. Svirshchevski [17], is an effective tool to construct exact solutions of some fractional partial differential equations with Caputo fractional derivative. R.Sahadevan and P.Prakash [18] used invariant subspace method to derive exact solutions of certain time fractional nonlinear partial differential equations, Hashemi [19] also adopted the same method to solve partial differential equations with conformable derivatives, Choudhary et al. [20] used this technique to explore solutions of some fractional differential equations, etc.

In the present paper, we present a modified version of the invariant subspace method which does not require any use of the Laplace transformation. We then make use of this novel technique to solve some fractional partial differential equations using fractional operators of Caputo-Fabrizio and also Atangana-Baleanu type. The exact solutions of these equations are obtained by solving the reduced systems constructed from the studied equations.

The laout of the paper is organized as follows: In section 2, we present some basic definitions of fractional derivatives and integrals. Section 3 describes the modified invariant subspace method. Construction of exact solutions to some partial differential equations with Caputo-Fabrizio and Atangan-Baleanu derivatives is presented in section 4. Finally, concluding remarks are given in section 5.

## 2 Fractional Calculus tools

In this section, we present some important definitions and mathematical concepts on fractional derivatives with nonsingular kernels and related tools.

**Definition 1.** The Mittag-Leffler function  $E_\alpha$  [21], is defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad (1)$$

where  $z$  is a complex variable,  $\alpha \in \mathbb{C}$  and  $\Re(\alpha) > 0$ .

This function arises naturally in the solution of fractional order integral equations or fractional order differential equations. It interpolate between a purely exponential law and power-law like behavior of phenomena governed by ordinary kinetic equations and their fractional counterparts.

On the other hand, Caputo and Fabrizio [10] developed a new fractional derivative as follows

**Definition 2.** Let  $u$  be a function in  $H^1(a, b)$ ,  $b > a$  et  $0 < \alpha < 1$  then, the new Caputo-Fabrizio derivative of fractional order  $\alpha$  is defined as [10]

$${}^{CF}D_t^\alpha u(x, t) = \frac{M(\alpha)}{1 - \alpha} \int_0^t \frac{\partial}{\partial \tau} u(x, \tau) \exp \left[ -\alpha \frac{(t - \tau)}{1 - \alpha} \right] d\tau, \quad (2)$$

where  $M(\alpha)$  is a normalization function satisfying  $M(0) = M(1) = 1$ .

From [10], we recall that if the function  $u$  does not belong to  $H^1(a; b)$  then, CF derivative can be writted as

$${}^{CF}D_t^\alpha u(x, t) = \frac{\alpha M(\alpha)}{1 - \alpha} \int_0^t (u(x, t) - u(x, \tau)) \exp \left[ -\alpha \frac{(t - \tau)}{1 - \alpha} \right] d\tau. \quad (3)$$

The fractional integral operator associated to the CF fractional derivative is expressed as

$${}^{CF}I_t^\alpha u(x, t) = \frac{1-\alpha}{M(\alpha)} u(x, t) + \frac{\alpha}{M(\alpha)} \int_0^t u(x, \tau) d\tau. \quad (4)$$

It's clear that the Caputo-Fabrizio derivative has no singular kernel, since the kernel is based on exponential function.

Recently, Atangana and Baleanu proposed a new fractional derivative which has non-local and non-singular kernel based on the generalized Mittag-Leffler function. More recently, they claimed that there are two general definitions of their derivative in the Riemann-Liouville and Caputo sense. Moreover, this fractional derivative has a fractional integral as an anti-derivative of their operators.

The Atangana-Baleanu fractional derivative in Caputo sense (ABC) is given by

**Definition 3.** The AB fractional derivative of order  $\alpha$  in Caputo sense is given by [11]

$${}^{ABC}D_t^\alpha u(x, t) = \frac{B(\alpha)}{1-\alpha} \int_0^t \frac{\partial}{\partial \tau} u(x, \tau) E_\alpha \left[ -\alpha \frac{(t-\tau)^\alpha}{1-\alpha} \right] d\tau, \quad (5)$$

where  $B(\alpha)$  is a normalization function and  $B(0) = B(1) = 1$  and  $0 < \alpha < 1$ .

The AB fractional integral operator of order  $\alpha$  is given by [11]

**Definition 4.** The Atangana-Baleanu fractional integral of order  $\alpha$  is defined as [11]

$${}^{AB}I_t^\alpha u(x, t) = \frac{1-\alpha}{B(\alpha)} u(x, t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t u(x, \tau) (t-\tau)^{\alpha-1} d\tau. \quad (6)$$

### 3 Description of the Modified Method

This section is devoted to describe the invariant subspace method. Such method has been firstly used in [17] to construct particular exact solutions for partial differential equations of the form

$$\frac{\partial u}{\partial t} = F(u, u_{1x}, u_{2x}, \dots, u_{kx}), \quad k \in \mathbb{N}, \quad (7)$$

where  $u = u(x, t)$ ,  $u_{ix} = \frac{\partial^i u}{\partial x^i}$  is the  $i^{th}$  order derivative of  $u$  with respect to the space variable  $x$  and  $F$  is a nonlinear differential operator.

Recently, Gazizov and Kasatkin [22] showed that the invariant subspace method can be applied also to equations with time fractional derivative.

In fact, consider the time fractional partial differential equation of the form

$$D_t^\alpha u(x, t) = F[u], \quad (8)$$

where  $F[u] = F(u, u_{1x}, u_{2x}, \dots, u_{kx})$  and  $D_t^\alpha$  is the time fractional derivative.

The modified invariant subspace method is based on the following basic definitions and results [22].

**Definition 5.** Let  $f_1(x), \dots, f_n(x)$  be an  $n$  linearly independent functions and  $W_n$  is the  $n$ -dimensional linear space namely  $W_n = \langle f_1(x), \dots, f_n(x) \rangle$ .  $W_n$  is said to be invariant under the given operator  $F$  if  $F[u] \in W_n$  whenever  $u \in W_n$ .

**Proposition 1.** Let  $W_n$  be an invariant subspace of  $F$ . A function  $u(x, t) = \sum_{i=1}^n f_i(x)u_i(t)$  is a solution of equation (8) if and only if the expansion coefficients  $u_i(t)$  satisfy the following system of fractional ordinary differential equations

$$\begin{cases} D_t^\alpha u_1 = F_1(u_1, \dots, u_n), \\ D_t^\alpha u_2 = F_2(u_1, \dots, u_n), \\ \vdots \\ D_t^\alpha u_n = F_n(u_1, \dots, u_n), \end{cases}$$

where  $F_1, \dots, F_n$  are given by

$$F(c_1 f_1(x) + \dots + c_n f_n(x)) = F_1(c_1, \dots, c_n) f_1(x) + \dots + F_n(c_1, \dots, c_n) f_n(x). \quad (9)$$

*Remark 1.* The important question concerning the modified invariant subspace method was how to obtain the corresponding invariant subspace of a given differential operator. The answer of this question is given by the following proposition, for more details we refer the reader to [22].

**Proposition 2.** Let  $f_1(x), \dots, f_n(x)$  form the fundamental set of solutions of a linear  $n$ th-order ordinary differential equation

$$T[y] = y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_{n-1}(x)y' + a_n(x)y = 0, \quad (10)$$

and  $F[y] = F(x, y, y', \dots, y^k)$  a given differential operator of order  $k \leq n-1$ , then the subspace  $W_n = \langle f_1(x), \dots, f_n(x) \rangle$  is invariant with respect to  $F$  if and only if

$$T[F[y]] = 0, \quad (11)$$

whenever  $y$  satisfies the equation (10).

## 4 Applications

### 4.1 Fractional partial differential equations with Caputo-Fabrizio derivative

In this section, we apply the modified invariant subspace method to construct exact solutions of some partial differential equations with Caputo-Fabrizio derivative in time.

#### • Example 1:

Consider the following time-fractional partial differential equation

$${}^{CF}D_t^\alpha u(x, t) = u_{xx}(x, t) + t^2 u_x(x, t), \quad (12)$$

where  $t > 0, x \in \mathbb{R}$  and  $0 < \alpha < 1$ .

Setting  $F[u] := {}^{CF}D_t^\alpha u(x, t)$ , it is obvious that Eq.(12) admits the following invariant subspace

$$W_1 = \mathcal{L}\{1, x\}, \quad (13)$$

Since

$$F[c_1(t) + c_2(t)x] = c_2(t)t^2 \in W_1. \quad (14)$$

Therefore, the exact solution of Eq.(12) can be written as

$$u(x, t) = c_1(t) + c_2(t)x, \quad (15)$$

where  $c_1(t)$  and  $c_2(t)$  satisfy the following system of FDEs

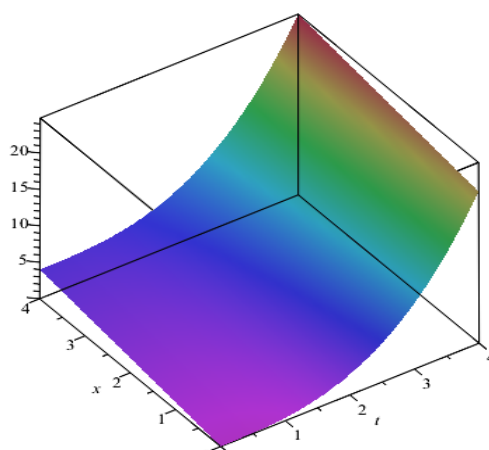
$$\begin{cases} {}^{CF}D_t^\alpha c_1(t) = c_2(t)t^2, \\ {}^{CF}D_t^\alpha c_2(t) = 0. \end{cases} \quad (16)$$

From the second equation of (16), we find that the function  $c_2(t)$  is a constant and then we assume that  $c_2(t) = 1$ . Thus, the first equation of (16) has the following solution

$$c_1(t) = \frac{(1-\alpha)t^2}{M(\alpha)} + \frac{1}{3} \frac{\alpha t^3}{M(\alpha)}. \quad (17)$$

Therefore, Eq.(12) has an exact solution of the form

$$u(x,t) = \frac{(1-\alpha)t^2}{M(\alpha)} + \frac{1}{3} \frac{\alpha t^3}{M(\alpha)} + x. \quad (18)$$



**Fig. 1** Profile of the solution (18) for  $\alpha = 0.9$ .

### • Example 2:

Consider now, the following time-fractional partial differential equation

$${}^{CF}D_t^\alpha u(x,t) = \sin(t)u_{xx}(x,t), \quad (19)$$

where  $t > 0, x \in \mathbb{R}$  and  $0 < \alpha < 1$ .

It is easy to check that

$$W_2 = \mathcal{L}\{1, x^2\}, \quad (20)$$

is an invariant subspace of Eq(19), seeing that

$$F[c_1(t) + c_2(t)x^2] = 2c_2(t)\sin(t) \in W_2. \quad (21)$$

consequently, an exact solution of Eq.(19) can be written as

$$u(x,t) = c_1(t) + c_2(t)x^2, \quad (22)$$

where  $c_1(t)$  and  $c_2(t)$  are unknown functions to be determined.

Substituting Eq.(22) in Eq.(19) yields:

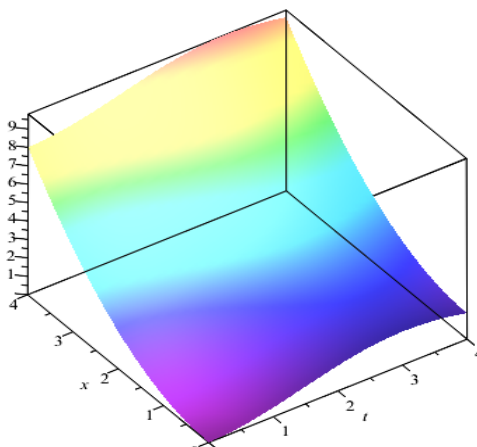
$$\begin{cases} {}^{CF}D_t^\alpha c_1(t) = 2c_2(t) \sin(t), \\ {}^{CF}D_t^\alpha c_2(t) = 0. \end{cases} \quad (23)$$

The second equation of (23) shows that the function  $c_2(t)$  is a constant and we infer that  $c_2(t) = \frac{1}{2}$ . Accordingly, the first equation of (23) can be expressed as

$$c_1(t) = \frac{(1-\alpha) \sin(t)}{M(\alpha)} + \frac{\alpha (1 - \cos(t))}{M(\alpha)}. \quad (24)$$

Finally, we obtain an exact solution of Eq.(19) as

$$u(x, t) = \frac{(1-\alpha) \sin(t)}{M(\alpha)} + \frac{\alpha (1 - \cos(t))}{M(\alpha)} + \frac{1}{2}x^2. \quad (25)$$



**Fig. 2** Profile of the solution (25) for  $\alpha = 0.9$ .

### • Example 3:

Now we deal with the nonlinear time-fractional partial differential equation

$${}^{CF}D_t^\alpha u(x, t) = tu_x^2(x, t) + u_{xx}(x, t), \quad (26)$$

where  $t > 0, x \in \mathbb{R}$  and  $0 < \alpha < 1$ .

Eq.(26) admits an invariant subspace defined through

$$W_3 = \mathcal{L}\{1, x\}, \quad (27)$$

as

$$F[c_1(t) + c_2(t)x] = tc_2^2(t) \in W_3. \quad (28)$$

As a deduction, an exact solution of Eq.(26) can take the form

$$u(x, t) = c_1(t) + c_2(t)x, \quad (29)$$

Substituting Eq.(29) in Eq.(26) and equating coefficients of different powers of  $x$ , we get

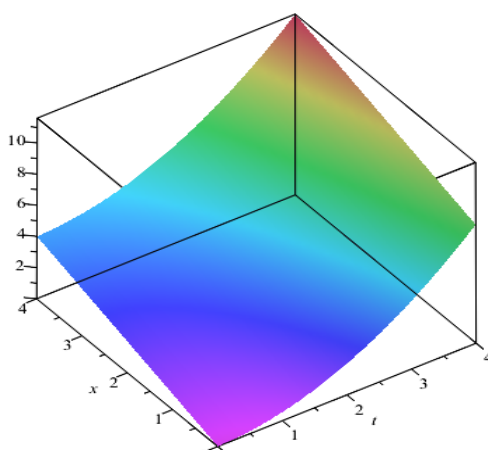
$$\begin{cases} {}^{CF}D_t^\alpha c_1(t) = t c_2^2(t), \\ {}^{CF}D_t^\alpha c_2(t) = 0. \end{cases} \quad (30)$$

Solving second equation of (30) gives  $c_2(t) = 1$ . Therefore, the solution of the first equation of (30) is given by:

$$c_1(t) = \frac{(1-\alpha)t}{M(\alpha)} + \frac{1}{2} \frac{\alpha t^2}{M(\alpha)}. \quad (31)$$

it then follows that an exact solution of Eq.(26) is given by

$$u(x,t) = \frac{(1-\alpha)t}{M(\alpha)} + \frac{1}{2} \frac{\alpha t^2}{M(\alpha)} + x. \quad (32)$$



**Fig. 3** Profile of the solution (32) for  $\alpha = 0.9$ .

#### • Example 4:

Let us consider the following equation

$${}^{CF}D_t^{2\alpha} u(x,t) = \frac{1}{2} x^2 u_{xx}^2(x,t) + t u_x(x,t), \quad (33)$$

where  $t > 0, x \in \mathbb{R}, \alpha \in [0, 1]$  and  $\alpha \neq \frac{1}{2}$ .

It is clear that the above equation Eq.(33) admits an invariant subspace

$$W_4 = \mathfrak{L}\{1, x\}, \quad (34)$$

by cause of

$$F[c_1(t) + c_2(t)x] = c_2(t)t \in W_4. \quad (35)$$

In an analogous way, the exact solution of Eq.(33) has the form

$$u(x,t) = c_1(t) + c_2(t)x, \quad (36)$$

where  $c_1(t)$  and  $c_2(t)$  satisfy the following system of FDEs

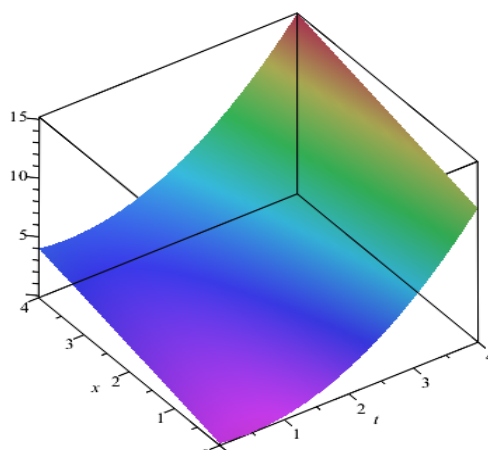
$$\begin{cases} {}^{CF}D_t^{2\alpha} c_1(t) = c_2(t)t, \\ {}^{CF}D_t^{2\alpha} c_2(t) = 0. \end{cases} \quad (37)$$

Similarly, we find that the function  $c_2(t)$  is a constant and then we assume that  $c_2(t) = 1$ . Therefore, the first equation of (37) has the following solution:

$$c_1(t) = \frac{(1-2\alpha)t}{M(\alpha)} + \frac{\alpha t^2}{M(\alpha)}. \quad (38)$$

This leads eventually to an exact solution to the system Eq.(33) as:

$$u(x,t) = \frac{(1-2\alpha)t}{M(\alpha)} + \frac{\alpha t^2}{M(\alpha)} + x. \quad (39)$$



**Fig. 4** Profile of the solution (39) for  $\alpha = 0.9$ .

## 4.2 Fractional partial differential equations with Atangana-Baleanu derivative

In what follows, we discuss four examples of getting exact solutions to some partial differential equations with Atangana-Baleanu fractional derivative.

### • Example 1:

Consider the time-fractional partial differential equation:

$${}^{ABC}D_t^\alpha u(x,t) = u_{xx}(x,t) + t^2 u_x(x,t), \quad (40)$$

where  $t > 0, x \in \mathbb{R}$  and  $0 < \alpha < 1$ .

which admits an invariant subspace defined through

$$W_1 = \mathcal{L}\{1, x\}, \quad (41)$$

by virtue of

$$F[c_1(t) + c_2(t)x] = c_2(t)t^2 \in W_1.$$



It then follows that the form of exact solution for Eq.(40) is

$$u(x, t) = c_1(t) + c_2(t)x, \quad (42)$$

Substituting Eq.(42) in Eq.(40) and equating different powers of  $x$  to zero yields

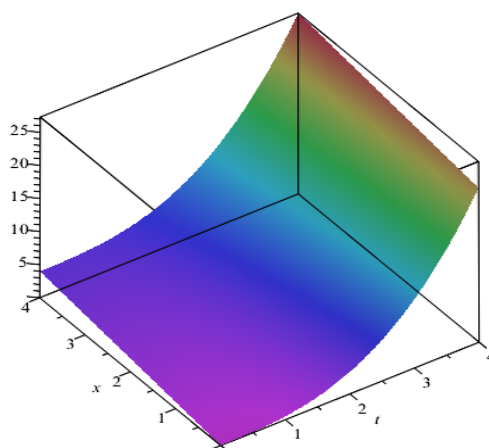
$$\begin{cases} {}^{ABC}D_t^\alpha c_1(t) = c_2(t)t^2, \\ {}^{ABC}D_t^\alpha c_2(t) = 0. \end{cases} \quad (43)$$

From the second equation of (43), we find that the function  $c_2(t)$  is a constant, then we assume that  $c_2(t) = 1$ . We then conclude that the solution of the first equation of (43) is expressed as

$$c_1(t) = \frac{(1-\alpha)t^2}{B(\alpha)} + \frac{2\alpha t^{\alpha+2}}{\Gamma(\alpha)B(\alpha)(\alpha^2 + 2\alpha + 2)}. \quad (44)$$

Consequently, the exact solution of Eq.(40) reads

$$u(x, t) = \frac{(1-\alpha)t^2}{B(\alpha)} + \frac{2\alpha t^{\alpha+2}}{\Gamma(\alpha)B(\alpha)(\alpha^2 + 2\alpha + 2)} + x. \quad (45)$$



**Fig. 5** Profile of the solution (45) for  $\alpha = 0.9$ .

### • Example 2:

Let us second consider the following time-fractional partial differential equation

$${}^{ABC}D_t^\alpha u(x, t) = \sin(t)u_{xx}(x, t), \quad (46)$$

where  $t > 0, x \in \mathbb{R}, 0 < \alpha < 1$ .

It is easy to check that Eq.(46) admits an invariant subspace as

$$W_2 = \mathcal{L}\{1, x^2\}, \quad (47)$$

Since

$$F[c_1(t) + c_2(t)x^2] = 2c_2(t)\sin(t) \in W_2. \quad (48)$$

Hence, an exact solution of Eq(46) has the following form

$$u(x, t) = c_1(t) + c_2(t)x^2, \quad (49)$$

In a similar way, substitution of Eq.(49) in Eq.(46) gives

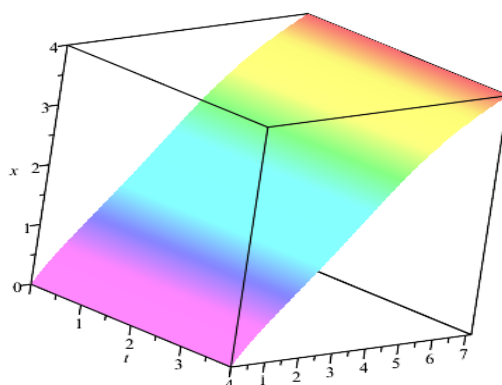
$$\begin{cases} {}^{ABC}D_t^\alpha c_1(t) = 2c_2(t) \sin(t), \\ {}^{ABC}D_t^\alpha c_2(t) = 0. \end{cases} \quad (50)$$

From second equation of (50) it comes  $c_2(t)$  is a constant then we assume that  $c_2(t) = \frac{1}{2}$ . Therefore, the solution of the first equation of (50) is

$$c_1(t) = \frac{(1 - \alpha) \sin(t)}{B(\alpha)} + \frac{-\sqrt{t} \text{LommelSI}(\frac{3}{2} + \alpha, \frac{1}{2}, t) + t^{1+\alpha}}{\Gamma(\alpha) B(\alpha) (1 + \alpha)}. \quad (51)$$

Finally, we obtain an exact solution of Eq.(46) as

$$u(x, t) = \frac{(1 - \alpha) \sin(t)}{B(\alpha)} + \frac{-\sqrt{t} \text{LommelSI}(\frac{3}{2} + \alpha, \frac{1}{2}, t) + t^{1+\alpha}}{\Gamma(\alpha) B(\alpha) (1 + \alpha)} + \frac{1}{2}x^2. \quad (52)$$



**Fig. 6** Profile of the solution (52) for  $\alpha = 0.9$ .

### • Example 3:

Consider now the partial differential equation

$${}^{ABC}D_t^\alpha u(x, t) = tu_x^2(x, t) + u_{xx}(x, t), \quad (53)$$

where  $t > 0, x \in \mathbb{R}$  and  $0 < \alpha < 1$ .

Equation (26) admits an invariant subspace of the form

$$W_3 = \mathfrak{L}\{1, x\}, \quad (54)$$

as far as

$$F[c_1(t) + c_2(t)x] = tc_2^2(t) \in W_3. \quad (55)$$

Then we can form an exact solution of Eq.(26) as

$$u(x, t) = c_1(t) + c_2(t)x, \quad (56)$$

where  $c_1(t)$  and  $c_2(t)$  satisfy the following system of FDEs

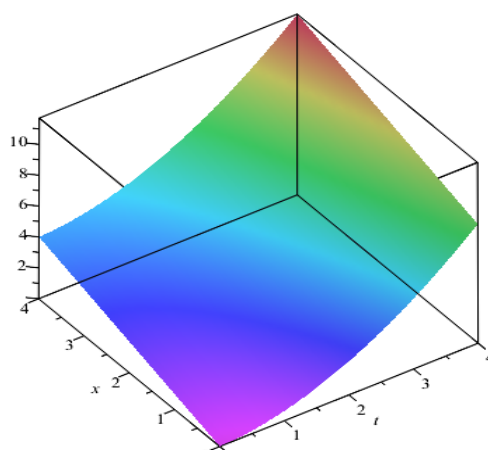
$$\begin{cases} {}^{ABC}D_t^\alpha c_1(t) = t c_2^2(t), \\ {}^{ABC}D_t^\alpha c_2(t) = 0. \end{cases} \quad (57)$$

Solving second equation of (30), we get  $c_2(t) = 1$ . Therefore, the solution of the first equation of (57) is constructed as

$$c_1(t) = \frac{(1-\alpha)t}{B(\alpha)} + \frac{t^{1+\alpha}}{\Gamma(\alpha)B(\alpha)(1+\alpha)}. \quad (58)$$

We finally obtain an exact solution of Eq.(53) as

$$u(x, t) = \frac{(1-\alpha)t}{B(\alpha)} + \frac{t^{1+\alpha}}{\Gamma(\alpha)B(\alpha)(1+\alpha)} + x. \quad (59)$$



**Fig. 7** Profile of the solution (59) for  $\alpha = 0.9$ .

#### • Example 4:

We finally consider the nonlinear time-fractional partial differential equation

$${}^{ABC}D_t^{2\alpha} u(x, t) = \frac{1}{2} x^2 u_{xx}^2(x, t) + t u_x(x, t), \quad (60)$$

where  $t > 0, x \in \mathbb{R}, \alpha \in [0, 1]$  and  $\alpha \neq \frac{1}{2}$ .

It is easy to check that the above Eq.(60) admits an invariant subspace as

$$W_4 = \mathcal{L}\{1, x\}, \quad (61)$$

Since

$$F[c_1(t) + c_2(t)x] = c_2(t)t \in W_4. \quad (62)$$

Therefore, the exact solution of Eq.(33) has the form

$$u(x, t) = c_1(t) + c_2(t)x. \quad (63)$$

The functions  $c_1(t)$  and  $c_2(t)$  satisfy the following system of FDEs

$$\begin{cases} {}^{CF}D_t^{2\alpha} c_1(t) = c_2(t)t, \\ {}^{CF}D_t^{2\alpha} c_2(t) = 0. \end{cases} \quad (64)$$

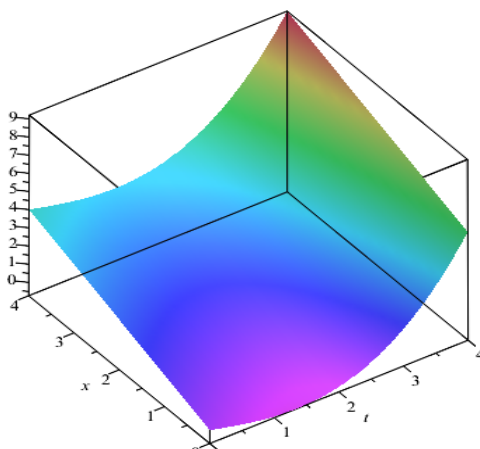
From (64), it can be inferred that  $c_2(t)$  is a constant, let us assume that  $c_2(t) = 1$ .

The first equation of (64) has then the following solution

$$c_1(t) = \frac{(1-2\alpha)t}{B(\alpha)} + \frac{1}{2} \frac{t^{2\alpha+1}}{\Gamma(\alpha)B(\alpha)(2\alpha+1)}. \quad (65)$$

Accordingly, we get an exact solution of Eq.(60) as

$$u(x, t) = \frac{(1-2\alpha)t}{B(\alpha)} + \frac{1}{2} \frac{t^{2\alpha+1}}{\Gamma(\alpha)B(\alpha)(2\alpha+1)} + x. \quad (66)$$



**Fig. 8** Profile of the solution (66) for  $\alpha = 0.9$ .

## 5 Conclusion

The modified invariant subspace method was used to seek exact solutions to a class of nonlinear equations with fractional derivatives having nonsingular kernels. Several examples illustrated the effectiveness of the invariant subspace theory for exploring solution of various structures. It is also worth mentioning that the present method does not need any use of laplace transform. Furthermore, some graphical representations are given to show the profiles of the obtained solutions. We stress here that those solutions are very useful to test the efficiency of newly suggested numerical methods for solving partial differential equations with Caputo-Fabrizio or Atangana-Baleanu fractional derivatives.

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