CrossMark

sciendo

Applied Mathematics and Nonlinear Sciences 5(2) (2020) 311–316

Applied Mathematics and Nonlinear Sciences

https://www.sciendo.com

On Limit Sets of Monotone Maps on Dendroids

E.N. Makhrova †

Lobachevsky State University of Nizhny Novgorod, Russia

Submission Info

Communicated by Lyudmila Sergeevna Efremova
Received December 24th 2019
Accepted March 14th 2020
Available online November 30th 2020

Abstract

Let X be a dendrite, $f : X \to X$ be a monotone map. In the papers by I. Naghmouchi (2011, 2012) it is shown that ω-limit set $\omega(x, f)$ of any point $x \in X$ has the next properties:
(1) $\omega(x, f) \subseteq \overline{Per(f)}$, where $Per(f)$ is the set of periodic points of f;
(2) $\omega(x, f)$ is either a periodic orbit or a minimal Cantor set.
In the paper by E. Makhrova, K. Vaniukova (2016) it is proved that
(3) $\Omega(f) = \overline{Per(f)}$, where $\Omega(f)$ is the set of non-wandering points of f .
The aim of this note is to show that the above results (1) – (3) do not hold for monotone maps on dendroids.

Keywords: dendroid, dendrite, monotone map, periodic point, non-wandering point, ω-limit set
AMS 2010 codes: 54H20, 54F50, 37B20, 37C25

1 Introduction

We use $\mathbb{N}$ and $\mathbb{C}$ to denote the set of natural numbers and a complex plane, respectively. The simbol $\mathbf{i}$ means an imaginary unit.

By *continuum* we mean a compact connected metric space. A topological space X is *unicoherent* provided that whenever A and B are closed, connected subsets of X such that $X = A \cup B$, then $A \cap B$ is connected. A topological space is *hereditarily unicoherent* provided that each of its closed, connected subset is unicoherent. By *a dendroid* we mean an arcwise connected hereditarily unicoherent continuum. *A dendrite* is a locally connected continuum without subsets homeomorphic to a circle. We note that a dendrite is a locally connected dendroid. Also we notice that a circle is not a unicoherent continuum. So a dendroid and a dendrite do not contain subsets homeomorphic to the circle and they are one-dimensional continua.

Let X be a dendroid with a metric d. An *arc* is any set homeomorphic to the closed interval $[0, 1]$. We notice that any two distinct points $x, y \in X$ can be joined by a unique arc with endpoints x, y (see, e.g., [1], [2]). We

† Corresponding author.
Email address: elena_makhrova@inbox.ru

sciendo
ISSN 2444-8656 doi:10.2478/AMNS.2020.2.00056 sciendo

denote by $[x,y]$ an arc joining x and y and containing these points, $(x,y) = [x,y] \setminus \{x,y\}$, $(x,y] = [x,y] \setminus \{x\}$ and $[x,y) = [x,y] \setminus \{y\}$.

The set $X \setminus \{p\}$ consists of one or more connected set. Each such set is called a component of a point p.

Definition 1. A point $p \in X$ is called to be
– an *end point of X* if the set $X \setminus \{p\}$ is connected;
– a *branch point of X* if the set $X \setminus \{p\}$ has at least three components.

If X is a dendrite then the set of branch points and the number of components of any point $p \in X$ are at most countable (see [1, §51]). These statements are not true for dendroids.

Let $f : X \to X$ be a continuous map of a dendroid X. *ω-limit set* of a point $x \in X$ is the set

$$\omega(x, f) = \{z \in X : \exists n_j \in \mathbb{N}, n_j \to \infty, \lim_{j\to\infty} f^{n_j}(x) = z\}.$$

Definition 2. A point $x \in X$ is said to be
– *a periodic point of f* if $f^n(x) = x$ for some $n \in \mathbb{N}$. When $n = 1$, we say that x is *a fixed point of f*;
– *a recurrent point of f* if $x \in \omega(x, f)$;
– *a non-wandering point of f* if for any neighborhood $U(x)$ of a point x there is a number $n \in \mathbb{N}$ so that $f^n(U(x)) \cap U(x) \neq \emptyset$.

Let $Fix(f)$, $Per(f)$, $Rec(f)$, $\Omega(f)$ denote the set of fixed points of f, the set of periodic points of f, the set of recurrent points of f, the set of non-wandering points of f respectively. It is well known that

$$Fix(f) \subseteq Per(f) \subseteq Rec(f) \subseteq \bigcup_{x\in X} \omega(x, f) \subseteq \Omega(f).$$

Definition 3. [1, §46] Let $f : X \to X$ be a continuous map of a dendroid X. A map f is said to be monotone if for any connected subset $C \subset f(X)$, $f^{-1}(C)$ is connected.

Let $f : X \to X$ be a monotone map. Denote by f^n the n-iterate of f; that is, f^0 = identity and $f^n = f \circ f^{n-1}$ if $n \geq 1$. We note that f^n is monotone for every $n \in \mathbb{N}$.

For monotone maps on dendrites the next statements are true.

Theorem 1. [3] *Let $f : D \to D$ be a monotone map of a dendrite D. Then for any point $x \in D$, $\omega(x, f) \subseteq \overline{Per(f)}$.*

Theorem 2. [4] *Let $f : D \to D$ be a monotone map of a dendrite D. Then $\Omega(f) = \overline{Per(f)}$.*

Theorem 3. [5] *Let $f : D \to D$ be a monotone map of a dendrite D. Then for any point $x \in D$, $\omega(x, f)$ is either a periodic orbit or a minimal Cantor set.*

In the note we show that Theorems 1 – 3 do not true for monotone maps on dendroids. Theorem 4 shows that Theorems 1, 2 do not hold for such maps.

Theorem 4. *There are a dendroid X_1 and a monotone map $f_1 : X_1 \to X_1$ such that*
(4.1) $\omega(x, f_1) \nsubseteq \overline{Per(f_1)}$ for some point $x \in X_1$;
(4.2) $\Omega(f_1) \neq \overline{Per(f_1)}$.

The next Theorem shows that Theorem 3 does not true for monotone maps on dendroids.

Theorem 5. *There are a dendroid X_2 and a monotone map $f_2 : X_2 \to X_2$ such that for some point $x \in X_2$, $\omega(x, f_2)$ is a nondegenerate closed interval belonging to the set $Fix(f_2)$.*

We note that there are continuous skew products of maps of an interval with a closed set of periodic points such that some their trajectories have a nondegenerate closed intervals as ω-limits sets (see, e.g., [6] – [11]).

2 Proof of Theorem 4

I. *Construction of the dendroid* X_1.

Let K be a Cantor set on the closed interval $[0,1]$, a point $p(\frac{1}{2}, \frac{1}{2}+\mathbf{i}) \in \mathbb{C}$. We set

$$X_1 = \bigcup_{e \in K} [p,e].$$

Note that X_1 is a dendroid which is not a locally connected continuum in any point $x \in X_1 \setminus \{p\}$.

II. *Construction of the map* $f_1 : X_1 \to X_1$.

We need the auxiliary map named binary adding machine.

Definition 4. Let $\Sigma = \{(j_1, j_2, \ldots)\}$ be the set of sequences, where $j_i \in \{0,1\}$. We put a metric d_Σ on Σ given by

$$d_\Sigma((k_1,k_2,\ldots),(j_1,j_2,\ldots)) = \sum_{i=1}^{+\infty} \frac{\delta(k_i,j_i)}{2},$$

where $\delta(k_i, j_i) = 1$, if $k_i \neq j_i$ and $\delta(k_i, j_i) = 0$, if $k_i = j_i$. The addition in Σ is defined as follows:

$$(k_1,k_2,\ldots)+(j_1,j_2,\ldots) = (l_1,l_2,\ldots),$$

where $l_1 = k_1 + j_1 \,(mod\, 2)$ and $l_2 = k_2 + j_2 + r_1 \,(mod\, 2)$, with $r_1 = 0$, if $k_1 + j_1 < 2$ and $r_1 = 1$, if $k_1 + j_1 = 2$. We continue adding the sequences in this way.

The adding machine map $\sigma : \Sigma \to \Sigma$ is defined as follows: for any $(j_1, j_2, j_3, \ldots) \in \Sigma$,

$$\sigma((j_1,j_2,j_3,\ldots)) = (j_1,j_2,j_3,\ldots)+(1,0,0,\ldots).$$

Lemma 6. [12], [13] *1. Σ is a Cantor set;*
2. $\sigma : \Sigma \to \Sigma$ is a homeomorphism;
3. $Per(\sigma) = \emptyset$;
4. $Rec(\sigma) = \Sigma$.

To define a map $f_1 : X_1 \to X_1$ we need two auxiliary maps.

1. Let $h : K \to \Sigma$ be any homeomorphism. We define a map $\tau : X_1 \to X_1$ as follows:
$\tau : [p,e] \to [p, h^{-1} \circ \sigma \circ h(e)]$ be a linear homeomorphism so that $\tau(p) = p$, $\tau(e) = h^{-1} \circ \sigma \circ h(e)$.

According to lemma 6 we get the next properties of τ:
1.1. τ is a homeomorphism;
1.2. $Per(\tau) = Fix(\tau) = \{p\}$;
1.3. $x \in Rec(\tau) \setminus Per(\tau)$ for any point $x \in X_1 \setminus \{p\}$.

2. Let e be any point from K and $\varphi : [p,e] \to [0,1]$ be any linear homeomorphism so that $\varphi(p) = 1$, $\varphi(e) = 0$.

We define a second auxiliary map $g : X_1 \to X_1$ by the following way: for any point $e \in K$
$g : [p,e] \to [p,e]$ be a homeomorphism such that $g(x) = \varphi^{-1} \circ x^2 \circ \varphi(x)$ for any point $x \in [p,e]$. Then a map g has the next properties:
2.1. g is a homeomorphism;
2.2. $Per(g) = Fix(g) = \{p\} \cup K$;
2.3. for any point $e \in K$ and an arbitrary point $x \in (p,e]$, $\omega(x,g) = \{e\}$.

Now we set $f_1 = g \circ \tau : X_1 \to X_1$. By properties of maps τ and g, we get the following statements:
1) f_1 is a homeomorphism and so f_1 is a monotone map;
2) $Per(f_1) = Fix(f_1) = \{p\}$;
3) for any point $x \in X_1 \setminus \{p\}$, $\omega(x, f_1)$ is a minimal Cantor set K, that is $\omega(x, f_1) = K$. Hence, $\omega(x, f_1) \not\subseteq \overline{Per(f_1)}$.
4) $\Omega(f_1) = \{p\} \cup K$. So $\Omega(f_1) \neq \overline{Per(f_1)}$.

Theorem 4 is proved.

3 Proof of Theorem 5

I. *Construction of the dendroid* X_2.

We define a sequence $\{s_k\}_{k\geq 1}$ by the following way:

$$s_0 = 0,\ s_k = s_{k-1} + 2(2^k - 1),\ \text{for } k \geq 1. \tag{1}$$

We set

$$I_j = \left[\frac{1}{2^j}; \frac{1}{2^j} + \mathbf{i}\right],\ \text{for } j \in \{s_k\}_{k\geq 0}. \tag{2}$$

For any number $n \in \mathbb{N} \setminus \{s_k\}_{k\geq 1}$ there is a natural number $k \geq 0$ such that $s_k < n < s_{k+1}$. It follows from (1) that for any $k \geq 0$ every interval $(s_k; s_{k+1})$ contains $2^{k+2} - 3$ natural numbers. For every $k \geq 0$ and any number $1 \leq j \leq 2^{k+2} - 3$ we define a vertical segmet I_{s_k+j} by the following way:

$$I_{s_k+j} = \begin{cases} \left[\frac{1}{2^{s_k+j}}; \frac{1}{2^{s_k+j}} + (1 - \frac{j}{2^{k+1}})\mathbf{i}\right], & \text{if } 1 \leq j \leq 2^{k+1} - 1; \\ \left[\frac{1}{2^{s_k+j}}; \frac{1}{2^{s_k+j}} + \frac{j + 2 - 2^{k+1}}{2^{k+1}}\mathbf{i}\right], & \text{if } 2^{k+1} \leq j \leq 2^{k+2} - 3. \end{cases} \tag{3}$$

It follows from (2) and (3), that for any number $n \in \mathbb{N} \cup \{0\}$ we defined a segment I_n. Now we set

$$X_2 = [0,1] \cup [0,\mathbf{i}] \cup \bigcup_{n=0}^{\infty} I_n.$$

A continuum X_2 is a dendroid, but it is not a dendrite because X_2 is not a locally connected continuum in any point $x \in (0,\mathbf{i}]$. You can see a dendroid homeomorphic to X_2 on figure 1.

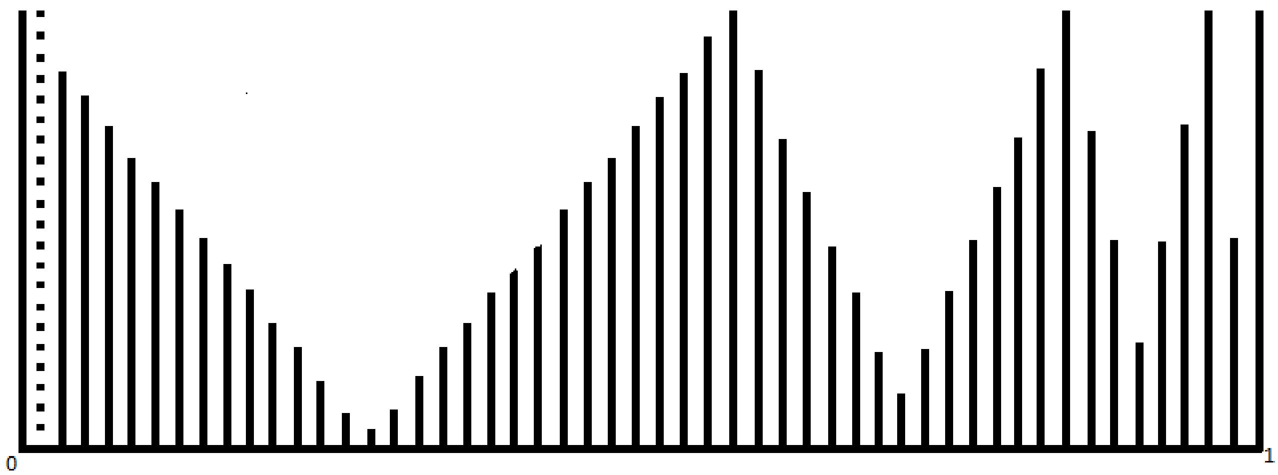

Fig. 1 Dendroid homeomorphic to X_2.

II. *Construction of the map* $f_2 : X_2 \to X_2$.

We define a monotone map $f_2 : X_2 \to X_2$ as follows:

(i) $f_2(z) = z$, if $z \in [0,\mathbf{i}]$;

(ii) $f_2(z) = z/2$, if $z \in [0,1]$;

(iii) $f_2 : I_j \to I_{j+1}$ be a linear homeomorphism such that $f_2(I_j) = I_{j+1}$ for any number $j \geq 0$.

III. *Properties of* f_2.

1. f_2 is a homeomorphism.

2. $Per(f_2) = Fix(f_2) = [0, \mathbf{i}]$.

3. We show that f_2 is a continuous map.

It is evident that f_2 is a continuous map in any point $z \in X_2 \setminus [0, \mathbf{i}]$. We'll prove a continuity of f_2 in any point $z \in [0, \mathbf{i}]$. Let $U(z)$ be an arbitrary neighborhood of a point z and let $\varepsilon > 0$ be a diameter of $U(z)$. We take any number $k \geq 1$ so that $I_{s_k} \cap U(z) \neq \emptyset$. Then by (3) and (iii) for any $j \geq s_k$ and for any point $x \in I_j$

$$|\mathrm{Im}\, f_2(x) - \mathrm{Im}\, x| \leq \frac{1}{2^{k+1}}, \tag{4}$$

where Im $*$ is the imaginary part of a complex number *. By (ii) and (iii),

$$|\mathrm{Re}\, f_2(x) - \mathrm{Re}\, x| = \frac{1}{2^{j+1}} \leq \frac{1}{2^{k+1}}, \tag{5}$$

where Re $*$ is a the real part of a complex number *.

It follows from (4) and (5) that for any $j \geq s_k$ and any point $x \in I_j$

$$|f_2(x) - x| \leq \sqrt{\frac{1}{2^{2(k+1)}} + \frac{1}{2^{2(k+1)}}} = \frac{1}{2^{2k+1}}. \tag{6}$$

Let $U_1(z) \subset U(z)$ be a neighborhood of a point x with diameter $\varepsilon/2^{k+1}$. Then by (6) $f_2(U_1(z)) \subseteq U(z)$, that is f_2 is a continuous map in a point z.

4. We show that $\omega(1 + \mathbf{i}, f_2) = [0, \mathbf{i}]$.

Let z be any point from $[0, \mathbf{i}]$ and $U(z)$ be an arbitrary neighborhood of a point z of diameter d. We take any natural number k_1 so that

$$\frac{1}{2^{k_1}} < \frac{d}{2}.$$

Now we take any natural number $K \geq k_1$ such that $I_{s_K} \cap U(z) \neq \emptyset$. According to the choice of k_1 and (4) there is a natural number $j \geq 1$ so that

$$\mathrm{Im}\, f_2^j\left(\frac{1}{2^{s_K}} + \mathbf{i}\right) \in \left(\mathrm{Im}\, z - \frac{d}{2}, \mathrm{Im}\, z + \frac{d}{2}\right).$$

It follows from here that $f_2^{s_K + j}(1 + \mathbf{i}) \in U(z)$. So, $z \in \omega(1 + \mathbf{i}, f_2)$.

Thus, $\omega(1 + \mathbf{i}, f_2) = [0, \mathbf{i}] = Fix(f_2)$. Theorem 5 is proved.

References

[1] Kuratovski K. (1968) *Topology, vol.2* (New York: Academic Press).

[2] Nadler S. B. (1992) Continuum Theory: An Introduction (Monographs and Textbooks in Pure and Applied Mathematics, 158). Marcel Dekker, New York.

[3] Naghmouchi I. (2011) Dynamics of monotone graph, dendrite and dendroid maps *International Journal of Bifurcation & Chaos* **21** 3205–3215.

[4] Makhrova E. N., Vaniukova K. S. (2016) On the set of non-wandering of monotone maps on local dendrites *Journal of Physics: Conference Series* **692**, 012012.

[5] Naghmouchi I. (2012) Dynamical properties of monotone dendrite maps *Topology and its Applications* **159** 144–149.

[6] Balibrea F., García Guirao J.L., Muñoz Casado J.I. (2001) Description of ω-limit sets of a triangular map on I^2 *Far East J. Dyn. Syst.* **3** 87–101.

[7] Balibrea F., García Guirao J.L., Muñoz Casado J.I. (2002) A triangular map on I^2 whose ω-limit sets are all compact interval of $0 \times I$ *Discrete Contin. Dyn. Syst.* **8** 983–994.

[8] Efremova L.S. (2017) Dynamics of skew products of interval maps *Russian Math. Surveys* **72** 101–178.

[9] Efremova L. S. (2010) Differential properties and attracting sets of a simplest skew product of interval maps *Sb. Math.* **201** 873–907.
[10] Kočen Z. (1999) The problem of classification of triangular maps with zero topological entropy *Ann. Math. Sil.* **13** 181–192.
[11] Kolyada S.F. (1992) On dynamics of triangular maps of the square *Ergodic Theory Dynam. Systems* **12** 749–768.
[12] Buescu J., Stewart I. (1995) Liapunov stability and adding machines *Ergodic Theory and Dynamical Systems* **15** 271–290.
[13] Block L., Keesling J. (2004) A characterization of adding machine maps *Topology and its Applications* **140** 151–161.