Applied Mathematics and Nonlinear Sciences 5(2) (2020) 307–310

Applied Mathematics and Nonlinear Sciences

https://www.sciendo.com

Any closed 3-manifold supports A-flows with 2-dimensional expanding attractors

V. Medvedev, E. Zhuzhoma †.

National Research University Higher School of Economics, 25/12 Bolshaya Pecherskaya,
603005, Nizhni Novgorod, Russia

Submission Info

Communicated by Lyudmila Sergeevna Efremova
Received January 27th 2020
Accepted June 19th 2020
Available online November 16th 2020

Abstract

We prove that given any closed 3-manifold M^3, there is an A-flow f^t on M^3 such that the non-wandering set $NW(f^t)$ consists of 2-dimensional non-orientable expanding attractor and trivial basic sets.

Keywords: A-flow, expanding attractor
AMS 2010 codes: Primary 37D05; Secondary 37B35, 34C40.

1 Introduction

A-flows were introduced by Smale [15] (see basic definitions bellow). This class of flows contains structurally stable flows including Morse-Smale flows and Anosov flows. Recall that a Morse-Smale flow has a non-wandering set consisting of finitely many hyperbolic periodic trajectories and hyperbolic singularities, while any Anosov flow has hyperbolic structure on the whole supporting manifold. A-flows have hyperbolic non-wandering sets that are the topological closure of periodic trajectories. In a sense, A-flows with nontrivial and trivial pieces (basic sets) of non-wandering sets take an intermediate place between Morse-Smale and Anosov flows. We see that A-flows form an important class containing flows with regular and chaotic dynamics.

Due to Smale's Spectral Theorem, a non-wandering set of A-flow is a disjoint union of closed transitive invariant pieces called basic sets. A basic set is called *trivial* if it is either an isolated fixed point or isolated periodic trajectory.

A nontrivial basic set Ω is called *expanding* if its topological dimension coincides with the dimension of unstable manifold at each point of Ω. Due to Williams [16], an expanding attractor consists of unstable manifolds of its points. Moreover, the unstable manifolds of points of expanding attractor form a lamination whose leaves are planes and cylinders. In addition, this lamination is locally homeomorphic to the product of Cantor set and

†Corresponding author.
Email address: zhuzhoma@mail.ru

ISSN 2444-8656 doi:10.2478/AMNS.2020.2.00053

Euclidean space of dimension at least two. Thus, the minimal topological dimension of expanding attractor equals two. Therefore, the minimal dimension of manifold supporting two-dimension expanding attractors equals three. It is natural to study the following question : what manifolds admit A-flows with 2-dimensional expanding attractors ?

In the paper, we consider closed 3-manifolds supporting A-flows with 2-dimensional expanding attractors. The main result of the paper is the following statement.

Theorem 1. *Given any closed 3-manifold M^3, there is an A-flow f^t on M^3 such that the non-wandering set $NW(f^t)$ consists of a two-dimensional non-orientable expanding attractor and trivial basic sets.*

This result contrasts with the case for 3-dimensional A-diffeomorphisms. To be precise, it follows from [8, 9, 12, 17] that if a closed 3-manifold M^3 admits an A-diffeomorphism with 2-dimensional expanding attractor, then $\pi_1(M^3) \neq 0$. Note that though there are Anosov flows with 2-dimensional expanding attractors [1], the A-flow to be constructed in Theorem 1 will not be necessarily Anosov. In fact, Margulis [11] proved that the fundamental group $\pi_1(M^3)$ of closed 3-manifold M^3 supporting Anosov flows has an exponential growth. In the end of the paper, we discuss the result and formulate some conjectures.

Acknowledgments. We would like to thank Misha Malkin. The authors are partially supported by Laboratory of Dynamical Systems and Applications NRU HSE, of the Ministry of science and higher education of the RF, grant ag. 075-15-2019-1931.

2 Basic definitions

Let f^t be a smooth flow on a closed n-manifold M^n, $n \geq 3$. A subset $\Lambda \subset M^n = M$ is *invariant* provided Λ consists of trajectories of f^t. An invariant nonsingular set $\Lambda \subset M$ is called *hyperbolic* if the sub-bundle $T_\Lambda M$ of the tangent bundle TM can be represented as a Df^t-invariant continuous splitting $E^{ss}_\Lambda \oplus E^t_\Lambda \oplus E^{uu}_\Lambda$ such that

1) $\dim E^{ss}_\Lambda + \dim E^t_\Lambda + \dim E^{uu}_\Lambda = n$;
2) E^t_Λ is the line bundle tangent to the trajectories of the flow f^t);
3) there are $C_s > 0$, $C_u > 0$, $0 < \lambda < 1$ such that

$$\|df^t(v)\| \leq C_s \lambda^t \|v\|, \quad v \in E^{ss}_\Lambda, \quad t > 0, \|df^{-t}(v)\| \leq C_u \lambda^t \|v\|, \quad v \in E^{uu}_\Lambda, \quad t > 0.$$

If $x \in \Lambda$ is a fixed point of hyperbolic Λ, then x is an isolated hyperbolic equilibrium state. The topological structure of flow near x is described by Grobman-Hartman theorem, see for example [13]. In this case $E^t_x = 0$ and $\dim E^{ss}_\Lambda + \dim E^{uu}_\Lambda = n$.

If hyperbolic Λ does not contain fixed points, then the bundles

$$E^{uu}_\Lambda \oplus E^1_\Lambda = E^u_\Lambda, \quad E^{ss}_\Lambda \oplus E^1_\Lambda = E^s_\Lambda, \quad E^{uu}_\Lambda, \quad E^{ss}_\Lambda$$

are uniquely integrable [5], [15]. The corresponding leaves

$$W^u(x), \quad W^s(x), \quad W^{uu}(x), \quad W^{ss}(x)$$

through a point $x \in \lambda$ are called *unstable*, *stable*, *strongly unstable*, and *strongly stable manifolds*.

Given a set $U \subset M^n$, denote by $f^{t_0}(U)$ the shift of U along the trajectories of f^t on the time t_0. Recall that a point x is non-wandering if given any neighborhood U of x and a number T_0, there is $t_0 \geq T_0$ such that $U \cap f^{t_0}(U) \neq \emptyset$. The *non-wandering set* $NW(f^t)$ of f^t is the union of all non-wandering point.

Denote by $Fix\,(f^t)$ the set of fixed points of flow f^t. Following Smale [15], we call f^t an *A-flow* provided its non-wandering set $NW(f^t)$ is hyperbolic and the periodic trajectories are dense in $NW(f^t) \setminus Fix\,(f^t)$. It is well known [10, 15] that that the non-wandering set $NW(f^t)$ of A-flow f^t is a disjoint union of closed, and invariant, and transitive sets called *basic sets*. Following Williams [16], we'll call a basic set Ω an *expanding attractor* provided Ω is an attractor and its topological dimension equals the dimension of unstable manifold $W^u(x)$ for every points $x \in \Omega$. A basic set Λ is called *orientable* provided the fiber bundles E^{ss}_{Λ} and E^{uu}_{Λ} are orientable. Note that if E^{ss}_{Λ} and E^{uu}_{Λ} are one-dimensional, then the orientability of Λ means that the both E^{ss}_{Λ} and E^{uu}_{Λ} can be embedded in vector fields on M^n.

Recall that an A-flow is a Morse-Smale flow provided its non-wandering set is the union of finitely many singularities and periodic trajectories. Clearly that all basic sets of Morse-Smale flow are trivial.

3 Proof of the main result

We begin with previous results which are interesting itself. Recall that any closed manifold admits a Morse-Smale flow with a source that is a repelling fixed point. In particular, any closed manifold admits a gradient-like Morse-Smale flow having at least one source and sink [14]. The following result says that any closed 3-manifold admits a Morse-Smale flow with one-dimensional repelling periodic trajectory.

Lemma 2. *Given any closed 3-manifolds M^3, there is a Morse-Smale flow f^t with repelling isolated periodic trajectory on M^3.*

Sketch of the proof. Take a gradient-like Morse-Smale flow f^t_0 with a sink, say ω, on M^3. Let $U(\omega)$ be a neighborhood of ω. Without loss of generality, we can suppose that $U(\omega)$ is a ball. Since ω is a hyperbolic fixed point, one can assume that the boundary $\partial U(\omega)$ is a smooth sphere that is transversal to the trajectories. This means that the vector field inducing the flow f^t_0 is directed inside of $U(\omega)$. Let us introduce coordinates (x,y,z) and the corresponding cylinder coordinates (ρ,ϕ,z) in $U(\omega)$ smoothly connected with the original coordinates in $U(\omega)$. Consider the system

$$\dot{\rho} = \rho \cdot (1-\rho), \quad \dot{\phi} = 1, \quad \dot{z} = -z.$$

It is easy to check that this system has an attractive hyperbolic trajectory and the saddle fixed point at the origin. Reversing time, one gets the repelling trajectory. □

Lemma 3. *There is an A-flow on $S^2 \times S^1$ such that the spectral decomposition of f^t consists of two-dimensional (non-orientable) expanding attractor Λ_a and four isolated hyperbolic repelling trajectories. Moreover, there is a neighborhood P of Λ_a homeomorphic to the solid torus $S^1 \times D^2$ such that the boundary $\partial P = S^1 \times S^1$ is transversal to the trajectories which enter inside of P as the time parameter increases.*

Proof. Take an A-diffeomorphism $f : S^2 \to S^2$ whose the spectral decomposition consists of a Plykin attractor Λ_0 and four hyperbolic sources. Due to Plykin [12], such diffeomorphism exists. Contemporary construction of Plykin attractor can be found in [7]. Since Plykin attractor is one-dimensional, Λ_0 is an expanding attractor. Without loss of generality, one can assume that f is a preserving orientation diffeomorphism (otherwise, one takes f^2).

Let $sus^t(f)$ be the dynamical suspension over f. Since f is an A-diffeomorphism, $sus^t(f)$ is an A-flow. Obviously, the spectral decomposition of f corresponds to the spectral decomposition of $sus^t(f)$. Because of f preserves orientation, the supporting manifold for $sus^t(f)$ is homeomorphic to $S^2 \times S^1$. Since Λ_0 is a one-dimensional expanding attractor, $sus^t(f)$ has a two dimensional expanding attractor denoted by Λ_a.

Take an isolated hyperbolic repelling trajectory γ that corresponds to some source of f. Since γ is a repelling trajectory, there is a neighborhood $V(\gamma)$ homeomorphic to a solid torus such that the boundary $\partial V(\gamma)$ is transversal to the trajectories of $sus^t(f)$, so that the trajectories move outside of $V(\gamma)$ as the time parameter increases. This follows that the interior of $(S^2 \times S^1) \setminus V(\gamma)$ is the neighborhood, say P, of Λ_a such that the boundary $\partial P = S^1 \times S^1$ is transversal to the trajectories which enter inside of P as a time parameter increases.

Since f is an orientation preserving diffeomorphism of the sphere S^2, f is homotopic to the identity. This implies that P is homeomorphic to the solid torus $S^1 \times D^2$. Hence, $sus^t(f) = f^t$ is a desired flow. □

PROOF OF THEOREM 1. Let M^3 be a closed 3-manifold. According to Lemma 2, there is a Morse-Smale flow φ^t with repelling isolated periodic trajectory l on M^3. Hence, there is a neighborhood $U(l)$ of l such that $U(l)$ is homeomorphic to the interior of the solid torus $S^1 \times D^2$, and the boundary $\partial U(l)$ is transversal to the trajectories of φ^t, so that the trajectories move outside of $U(l)$ as a time parameter increases. This follows that $U(l)$ can be replaced by the solid torus P satisfying Lemma 3. As a consequence, one gets the A-flow with two-dimensional (non-orientable) expanding attractor Λ_a. This completes the proof. □

4 Conclusions

It follows from the proof of the main result that the 2-dimensional expanding attractor satisfying Theorem 1 is non-orientable. We suggest the following conjecture.

Conjecture 4. *Given any closed 3-manifold M^3, there is an A-flow f^t on M^3 such that the non-wandering set $NW(f^t)$ contains an orientable two-dimensional expanding attractor.*

Two-dimensional expanding attractors are evidence of chaotic dynamics. However, it seems that the following conjecture holds because of Plykin diffeomorphism is structurally stable.

Conjecture 5. *Given any closed 3-manifold M^3, there is a structurally stable flow f^t on M^3 such that the non-wandering set $NW(f^t)$ consists of a two-dimensional expanding attractor and trivial basic sets.*

Note that a structurally stable flow automatically is an A-flow.

References

[1] Franks J., Williams B. Anomalous Anosov flows. *Lect. Notes in Math.*, **819**(1980), 158-174.
[2] Grobman D. On a homeomorphism of systems of differential equations. *Dokl. Akad. Nauk, USSR*, **128**(1959), 5, 880-881.
[3] Grobman D. Topological classification of neighborhoods of singular point in n-dimensional space. *Matem. sbornik, USSR*, **56**(1962), 1, 77-94.
[4] Hartman P. On the local linearization of differential equations. *Proc. AMS* **14**(1963), no 4, 568-573.
[5] Hirch M., Pugh C., Shub M. *Invariant Manifolds*. *Lect. Notes in Math.*, **583**(1977), Springer-Verlag.
[6] Hirch M., Palis J., Pugh C., Shub M. Neighborhoods of hyperbolic sets. *Invent. Math.*, **9**(1970), 121-134.
[7] A. Katok, B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems.* Encyclopedia of Math. and its Appl. Cambridge Univ. Press, 1994.
[8] Medvedev V., Zhuzhoma E. On non-orientable two-dimensional basic sets on 3-manifolds, *Sbornic Math.*, **193**(2002), no 6, 869-888.
[9] Medvedev V., Zhuzhoma E. On the existence of codimension one non-orientable expanding attractors. *Journ. Dyn. Contr. Syst.*, **11**(2005), no 3, 405-411.
[10] Pugh C., Shub M. The Ω-stability theorem for flows. *Invent. Math.*, **11**(1970), 150-158.
[11] Margulis G. U-flows on three-dimensional manifolds. Appendix of Anosov D.V., Sinai Ya.G. Some smooth ergodic systems, *Russian Math. Surveys*, **22**(1967), 103-168.
[12] Plykin R.V. On the geometry of hyperbolic attractors of smooth cascades. *Russian Math. Surveys*, **39**(1984), no 6, 85-131.
[13] Robinson C. *Dynamical Systems: stability, symbolic dynamics, and chaos*. Studies in Adv. Math., 2nd ed., CRC Press, Boca Raton, FL, **1999**.
[14] Smale S. Morse inequalities for a dynamical system. *Bull. Amer. Math. Soc.*, **66**(1960), 43-49.
[15] Smale S. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, **73**(1967), 747-817.
[16] Williams R. Expanding attractors. *Publ. Math. IHES*, **43**(1974), 169-203.
[17] Zhuzhoma E. Orientable basic sets of codimension 1. *Soviet Math. (Iz. VUZ)*, **26**(1082), no 5, 17-25.