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Dynamics of conformal foliations

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Abstract

The purpose of this article is to review the author's results on the existence and structure of minimal sets and attractors of conformal foliations. Results on strong transversal equivalence of conformal foliations are also presented. Connections with works of other authors are indicated. Examples of conformal foliations with exceptional, exotic and regular minimal sets which are attractors are constructed.

Keywords: conformal foliation; attractor; minimal set; global attractor, conformal transformation

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1 Introduction

The purpose of this article is to review the author's results on the existence and structure of minimal sets and attractors conformal foliations. The results are illustrated by various examples. Connections with works of other authors are indicated.

Remind that two Riemannian metrics h and g on a manifold M are called conformally equivalent if there exists a positive smooth function f on M with $h = fg$. A conformal equivalence class $[g]$ of Riemannian metrics on M is called a conformal structure on M , and the pair $(M, [g])$ is said to be a conformal manifold.

A group of conformal transformations of a Riemannian manifold (M, g) is called *inessential* if it is a group of isometries of a Riemannian manifold (M, h) with some $h \in [g]$. Otherwise, it is called *essential*.

Lichnerowicz conjectured that for $n \geq 3$ every n -dimensional compact Riemannian manifold admitting an essential group of conformal transformations is conformally equivalent to the standard n -dimensional sphere S^n . The articles by M. Obata [22], D.V. Alekseevskii [1, 2], J. Ferrand [13] and others are devoted to this conjecture.

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It was also established that if a group of conformal transformations of a non-compact Riemannian manifold M is essential then M is conformally equivalent to the n -dimensional Euclidean space. In 1996 J. Ferrand [13] gave a complete proof of the Lichnerowicz conjecture, including the case of non-compact manifolds.

C. Tarquini [26] and then C. Tarquini and C. Frances [14] posed the following question about conformal foliations:

Is every codimension $q \geq 3$ conformal foliation on a compact manifold either a Riemannian foliation or a $(Conf(\mathbb{S}^q), \mathbb{S}^q)$ -foliation?

C. Frances and C. Tarquini called the question a *foliated analog* of the Lichnerowicz conjecture. For $q \geq 3$ a conformal foliation is a $(Conf(\mathbb{S}^q), \mathbb{S}^q)$ -foliation if and only if it is transversally conformally flat. C. Frances and C. Tarquini [14] gave a positive answer to this question under some additional assumptions. We proved that in general the answer to the C. Frances and C. Tarquini question is positive (Theorem 7).

Our main results included in this article are as follows:

- 1) Different interpretations for the germ holonomy group of a leaf of a conformal foliation (Theorem 2).
- 2) A criterion for a conformal foliation to be Riemannian (Theorem 3).
- 3) The existence of an attractor which is a minimal set for every non-Riemannian conformal foliation of codimension $q \geq 3$ (Theorem 4).
- 4) A structure theorem for conformal and transversely similar foliations on closed manifolds (Theorems 7 and 8).
- 5) Generalization of the results of P. Molino on the closures of leaves of Riemannian foliations (Theorem 10) and equivalence of completeness and existence of an Ehresmann connection for a conformal foliation (Theorem 9).
- 6) The existence of a global transitive attractor for every complete non-Riemannian conformal foliation of codimension $q \geq 3$ (Theorem 12) and description of the structure of such foliations (Theorem 13).
- 7) Properties of global holonomy groups of conformal foliations (Theorems 13, 14).
- 8) The global holonomy group defined up to conjugacy in the group $Conf(\mathbb{S}^q)$ is a complete invariant of the class of strong transversely equivalent conformal foliations (Theorem 15). An analogous result for strong transversely equivalent similar foliations (Theorem 16).
- 9) The existence of a two-dimensional suspension conformal foliation in every class of transversely equivalent non-Riemannian conformal foliations (Theorem 18).

In Section 3.5 we compare our results with some results of B. Deroin and V. Kleptsyn [11].

Section 7 contains examples of conformal foliations with global attractors. In particular, we construct a conformal foliation with two global attractors, one of which is a transitive attractor but not a minimal set (Example 22).

Notations In this article we denote by M a smooth manifold of dimension n and by $C^\infty(M)$ the space of smooth functions on M . Let $\mathfrak{X}(M)$ be the $C^\infty(M)$ -module of smooth vector fields on M . Also, we denote by $\mathfrak{X}_c(M)$ the $C^\infty(M)$ -module of compactly supported smooth vector fields on M .

2 Conformal foliations and associated constructions

2.1 Conformal foliations

First, we recall some basic notions.

A diffeomorphism $f : N_1 \rightarrow N_2$ of Riemannian manifolds (N_1, g_1) and (N_2, g_2) is called *conformal* if there exists a smooth function λ on N_1 with $f^*g_2 = \lambda g_1$. A conformal diffeomorphism f from a Riemannian manifold (N, g) to itself is also called a conformal transformation. A conformal transformation f of (N, g) is said to be a *similarity*, if $f^*g = \lambda g$ for a constant λ .

Conformal foliations were studied by some authors as foliations admitting a transverse conformal structure.

Definition 1. A codimension $q \geq 3$ smooth foliation (M, F) is called *conformal*, if there are:

- a possibly disconnected Riemannian manifold (N, g) ;
- an open cover $\{U_i \mid i \in J\}$ of M ;
- submersions $f_i : U_i \rightarrow V_i$ with connected fibers, $V_i \subset N$,

such that: if $U_i \cap U_j \neq \emptyset$, then there exists a conformal diffeomorphism $\gamma_{ij} : f_j(U_i \cap U_j) \rightarrow f_i(U_i \cap U_j)$ satisfying the condition $f_i = \gamma_{ij} \circ f_j$ on $U_i \cap U_j$.

Definition 2. A maximal (with respect to inclusion) N -cocycle $\{U_i, f_i, \gamma_{ij}\}_{i,j \in J}$, satisfying the above properties, determines a new topology on M , whose base is the set of fibers of all submersions f_i . This topology is called the leaf topology and denoted by τ .

Path-connected components of the topological space (M, τ) form a partition $F = \{L_\alpha \mid \alpha \in A\}$ of M , and (M, F) is called the conformal foliation with leaves L_α determined by the N -cocycle $\{U_i, f_i, \gamma_{ij}\}_{i,j \in J}$.

It is also said that (M, F) is *modeled on the conformal geometry* $(N, [g])$.

Definition 3. If each γ_{ij} is a local similar transformation of (N, g) , then (M, F) is called a *transversely similar Riemannian foliation*.

Definition 4. If each γ_{ij} is an isometry, then (M, F) is called a *Riemannian foliation*.

Let $Conf(\mathbb{S}^q)$ be the Lie group of all conformal transformations of the q -dimensional sphere \mathbb{S}^q .

Definition 5. If a foliation (M, F) is defined by the N -cocycle $\{U_i, f_i, \gamma_{ij}\}_{i,j \in J}$, where $N = \mathbb{S}^q$ and each γ_{ij} is the restriction of a transformation $f \in Conf(\mathbb{S}^q)$, then (M, F) is referred to as a $(Conf(\mathbb{S}^q), \mathbb{S}^q)$ -foliation.

A conformal transformation of the complex plane is conformal in the above sense. Therefore, every transversely complex analytic foliation of complex codimension one is a conformal foliation of real codimension two in the sense of Definition 1. The converse is also true, i.e., every conformal foliation of codimension two can be considered as a transversely complex analytic foliation of complex codimension one.

Conformal foliations of codimension two have been studied by several authors, using methods of complex analysis. For example, in [6] P. Baird and M. Eastwood gave a detailed twistor description of conformal foliations of codimension two in the 3-dimensional Euclidean space. Holonomy diffeomorphisms of these foliations preserve the conformal equivalence class of Riemannian metrics on orthogonal complements to their leaves.

Since every 2-dimensional manifold is locally conformally flat, every conformal foliation of codimension two is transversely conformally flat. A transversal conformal manifold $(N, [g])$ is locally conformally flat if and only if, in codimension $q > 3$, its Weyl conformal curvature tensor vanishes or, in codimension $q = 3$, its Schouten tensor vanishes.

Like S. Morita [21], in contrast to the case of conformal foliations of codimension two, we study conformal foliations of codimension $q, q \geq 3$, by differential geometric methods, considering them as Cartan foliations.

2.2 The foliated bundle over a conformal foliation

The construction of the foliated bundle is essentially used by us. Foliated bundles were introduced in works of P. Molino, F. Kamber and Ph. Tondeur.

Recall that a foliation admitting an effective Cartan geometry of type (G, H) [10] as a transverse structure is said to be a Cartan foliation of the same type. Remark that any conformal foliation may be considered as a Cartan foliation of type (G, H) , where $G = \text{Conf}(\mathbb{S}^q)$ is the full group of conformal transformations of \mathbb{S}^q and H is the stabilizer of G at an arbitrary point $b \in \mathbb{S}^q$. Let \mathfrak{g} and \mathfrak{h} be the Lie algebras of the Lie groups G and H , respectively. Then we have the following statement [27].

Proposition 1. *Let (M, F) be a conformal foliation of codimension $q \geq 3$. Then there are the following objects:*

1. *a principal H -bundle $\pi : \mathcal{R} \rightarrow M$;*
2. *an H -invariant foliation $(\mathcal{R}, \mathcal{F})$ which is mapped by π into (M, F) ;*
3. *a \mathfrak{g} -valued 1-form ω on \mathcal{R} , satisfying the properties:*
 - (i) *$\omega(A^*) = A$ for any $A \in \mathfrak{h}$, where A^* is the fundamental vector field corresponding to A ;*
 - (ii) *$R_a^* \omega = \text{Ad}_G(a^{-1}) \omega$ for any $a \in H$, where Ad_G is the adjoint representation of the Lie group G in its Lie algebra \mathfrak{g} ;*
 - (iii) *the Lie derivative $L_X \omega$ vanishes for every vector field X tangent to the leaves of $(\mathcal{R}, \mathcal{F})$.*

The H -bundle $\pi : \mathcal{R} \rightarrow M$ is said to be *foliated*. The foliation $(\mathcal{R}, \mathcal{F})$ is called the *lifted* foliation, and $(\mathcal{R}, \mathcal{F})$ is a transversely parallelizable foliation, i.e., an *e*-foliation.

2.3 Holonomy groups of a conformal foliation

We consider conformal foliations as Cartan foliations and use the construction of the principal foliated bundle. Due to this, we can apply the results of our previous work [27].

In foliation theory, the germ holonomy groups are usually used. By analogy with [27], we give the following interpretations for the germ holonomy group of a leaf of a conformal foliation.

As above, let $G = \text{Conf}(\mathbb{S}^q)$ be the Lie group of conformal transformations of the sphere \mathbb{S}^q and let H be the stabilizer of G at an arbitrary point in \mathbb{S}^q .

Theorem 2. *Let (M, F) be a conformal foliation of codimension $q \geq 3$, \mathcal{R} be the foliated H -bundle over (M, F) with the projection $\pi : \mathcal{R} \rightarrow M$ and $(\mathcal{R}, \mathcal{F})$ be the lifted foliation. Then, for every leaf $L = L(x)$, $x \in M$, of (M, F) the restriction $\pi|_{\mathcal{L}} : \mathcal{L} \rightarrow L$ to the leaf $\mathcal{L} = \mathcal{L}(u)$, $u \in \pi^{-1}(x)$, of the lifted foliation $(\mathcal{R}, \mathcal{F})$ is a regular covering map, and the germ holonomy group $\Gamma(L, x)$ of L is isomorphic to each of the following groups:*

- (i) *the subgroup $H(\mathcal{L}) := \{a \in H \mid R_a(\mathcal{L}) = \mathcal{L}\}$ of H ;*
- (ii) *the group of deck transformations of the covering $\pi|_{\mathcal{L}} : \mathcal{L} \rightarrow L$.*

Note that the group $H(\mathcal{L})$ is defined up to conjugacy in the group H .

3 Minimal sets and attractors of conformal foliations

3.1 Minimal sets of foliations

A *saturated set* of a foliation is a union of leaves. Recall that a *minimal set* of a foliation (M, F) is a nonempty closed saturated subset of M without proper subsets satisfying these properties.

Existence of minimal sets and description of their structure are one of the central problems in the foliation theory. A. H. Aranson, V. Z. Grines [4, 5] and J. Levitt [19] obtained a topological classification of nontrivial minimal sets of flows and foliations on closed surfaces of genus $p \geq 2$. Minimal sets of Riemannian foliations were studied by P. Molino [20], A. Haefliger [15] and E. Salem [23] without using the term minimal set.

Every foliation on a compact manifold has a minimal set. This is wrong for non-compact manifolds. Foliations without minimal sets (on non-compact manifolds) are constructed in the works of J.-C. Beniere and G. Meigniez [7], T. Inaba [16] and M. Kulikov [18]. Moreover, J.-C. Beniere and G. Meigniez proved the existence of foliations without minimal sets on every non-compact manifold of dimension $n \geq 2$, different from surfaces of finite kind [7].

A minimal set \mathcal{M} is called *regular*, if it is a submanifold of M . Let $\{U_i | i \in N\}$ be an at most countable locally finite open cover of M by foliated coordinate neighborhoods. Let T_i be a submanifold in U_i , which intersects transversely each leaf of the induced foliation (U_i, F_{U_i}) , and let $T = \sqcup_{i \in N} T_i$ be the complete transversal. A minimal set \mathcal{M} , having no interior points, is called:

- *exceptional* if $\mathcal{M} \cap T$ is a Cantor set;
- *exotic* if $\mathcal{M} \cap T$ is not a totally disconnected topological space.

In Section 7 we construct examples of conformal foliations with exotic, exceptional, and regular minimal sets.

3.2 A criterion for a conformal foliation to be Riemannian

Definition 6. The holonomy group of a leaf L of a conformal foliation is said to be *relatively compact* or *inessential* if the corresponding subgroup $H(\mathcal{L})$ of the Lie group H is relatively compact.

Since $H(\mathcal{L})$ is defined up to conjugacy in H , then this definition is correct.

We established the following criterion for a conformal foliation to be Riemannian [29, Theorem 3].

Theorem 3. Let (M, F) be a codimension $q \geq 3$ conformal foliation modeled on a conformal geometry $(N, [g])$. Then there exists a Riemannian metric $d \in [g]$ such that (M, F) is a Riemannian foliation modeled on (N, d) if and only if every holonomy group of this foliation is relatively compact, i.e. inessential.

Remark 1. According to [31, Theorem 3], Theorem 3 holds for transversely similar Riemannian foliations of codimension $q \geq 2$. Note that it is true also for $q = 1$.

3.3 The existence of attractors of conformal foliations

Let (M, F) be a foliation.

Definition 7. A nonempty closed saturated subset \mathcal{M} of M is called an *attractor* of (M, F) if there exists a saturated open neighborhood $\text{Attr}(\mathcal{M})$ of \mathcal{M} such that the closure of every leaf belonging to $\text{Attr}(\mathcal{M}) \setminus \mathcal{M}$ contains \mathcal{M} .

If, in addition, $\text{Attr}(\mathcal{M}) = M$, then the attractor \mathcal{M} is called *global*.

In the case when there exists a dense leaf in \mathcal{M} , the attractor \mathcal{M} is referred to be *transitive*.

Using the criterion for conformal foliation to be Riemannian (Theorem 3) and some results from local conformal geometry, we prove the following theorem on two alternative possibilities for conformal foliations.

Theorem 4. Let (M, F) be a codimension $q \geq 3$ conformal foliation. Then:

- either it is Riemannian,

- or it has an attractor \mathcal{M} which is the closure of a leaf L with essential holonomy group, $\mathcal{M} = \bar{L}$, and a minimal set. Moreover, the restriction $(\text{Attr}(\mathcal{M}), F)$ of the foliation to the attraction basin is a $(\text{Conf}(\mathbb{S}^q), \mathbb{S}^q)$ -foliation.

Recall that a foliation (M, F) is called proper if all its leaves are embedded submanifolds of M . A leaf L is called closed if L is a closed subset of M .

Corollary 5. *Every proper non-Riemannian conformal foliation of codimension $q \geq 3$ has a closed leaf with essential holonomy group that is an attractor.*

Corollary 6. *A non-Riemannian conformal foliation (M, F) of codimension $q \geq 3$ has a minimal set that is an attractor of this foliation.*

Remark 2. Theorem 4 is proved without assumption that the foliated manifold M is compact.

3.4 An analog of the Lichnerowicz conjecture for conformal foliations

The following theorem [28, Theorem 4] gives a positive answer to the C. Frances and C. Tarquini question stated in Introduction in the general case.

Theorem 7. *Any codimension $q \geq 3$ conformal foliation (M, F) on a compact manifold M is*

- either a Riemannian foliation;
- or a $(\text{Conf}(\mathbb{S}^q), \mathbb{S}^q)$ -foliation with finitely many minimal sets. All these sets are attractors given by the closures of leaves with essential holonomy group, and each leaf of F belongs to the basin of at least one of these attractors.

For the proof of Theorem 7, we applied the results of D.V. Alekseevskii [1] and J. Ferrand [12] concerning with local differential geometry as well as Riemannian geometry on non-Hausdorff manifolds.

Let us emphasize that transversely similar foliations form a subclass of conformal foliations. Since any Riemannian manifold of dimension 1 or 2 is locally conformally flat, every transversely similar Riemannian foliation of codimension $q = 1$ or 2 is a $(\text{Sim}(\mathbb{E}^q), \mathbb{E}^q)$ -foliation. By using similar arguments as in the proof of [28, Theorem 4], we prove the following statement.

Theorem 8. *Every transversely similar Riemannian foliation (M, F) of codimension $q \geq 1$ on a compact manifold M is*

- either a Riemannian foliation
- or a $(\text{Sim}(\mathbb{E}^q), \mathbb{E}^q)$ -foliation with finitely many minimal sets. They are all attractors formed by the closures of leaves with essential holonomy group, and each leaf of the foliation belongs to the basin of at least one of them.

3.5 Comparison with the results B. Deroin and V. Kleptsyn [11]

Theorem 7 strengthens (in an appropriate smoothness class and for $q \geq 3$) the first part of the main theorem of B. Deroin and V. Kleptsyn [11], which claims that every conformal foliation of arbitrary codimension $q \geq 1$ on a compact manifold either admits a transversal invariant measure or has finitely many minimal sets satisfying some properties. As emphasized in the introduction of [11], E. Ghys conjectured that in the absence of attractors of the conformal foliation, the invariant transverse measure is given by a Riemannian metric. It follows from our Theorem 7 that this conjecture of E. Ghys holds for conformal foliations of codimension $q \geq 3$.

It is known (see, for example [1]) that there are nontrivial second order conformal diffeomorphisms f of the sphere \mathbb{S}^q , $q \geq 2$, such that the differential f_{*x} at a fixed point x is an orthogonal transformation of the tangent

Euclidean vector space $T_x \mathbb{S}^q$ at x . These conformal transformations are essential. The 1-dimensional conformal foliation (M, F) formed by the suspension of such a conformal diffeomorphism f does not admit a transversely invariant measure and does not have hyperbolic holonomy; therefore, (M, F) does not satisfy [11, Cor. 1.3]. Thus, unlike us, in [11] the authors do not consider conformal foliations with such holonomies.

Note that in [28, 29] we use the methods of local and global differential geometry, including foliated principal bundles, geometries on non-Hausdorff manifolds as well as Ehresmann connections for foliations. While B. Deroin and V. Kleptsyn apply methods of random dynamical systems, including Lyapunov exponents of harmonic measures. In [11], a Laplace operator along leaves is considered and the existence of a transverse invariant measure is investigated.

4 Completeness and Ehresmann connections for conformal foliations

4.1 Smooth distributions on manifolds

We will use the definition of a smooth distribution given by I. Androulidakis and G. Skandalis [3, §1.1]. Let us recall this definition.

Let \mathfrak{M} be a $C^\infty(M)$ -submodule of $\mathfrak{X}_c(M)$ and let U be an open subset of M . Set $i_U : U \rightarrow M$ the inclusion map. For any vector field $X \in \mathfrak{X}(M)$ we denote $X|_U := X \circ i_U$. The restriction of \mathfrak{M} to U is the $C^\infty(M)$ -submodule of $\mathfrak{X}_c(M)$ generated by the vector fields $f \cdot X|_U$, where $f \in C^\infty(U)$ and $X \in \mathfrak{M}$. This restriction is denoted by $\mathfrak{M}|_U$.

The module \mathfrak{M} is said to be *locally finitely generated* if, for every $x \in M$ there exist an open neighbourhood U of x and a finite number of vector fields ξ_1, \dots, ξ_k belonging to $\mathfrak{X}(M)$ such that $\mathfrak{M}|_U = C^\infty(U)_c \cdot \xi_1 + \dots + C^\infty(U)_c \cdot \xi_k$. It is said that the vector fields ξ_1, \dots, ξ_k generate the restriction $\mathfrak{M}|_U$ of \mathfrak{M} to U .

Definition 8. ([3]) A smooth distribution on M is a locally finitely generated $C^\infty(M)$ -submodule \mathfrak{M} of the $C^\infty(M)$ -module $\mathfrak{X}_c(M)$. It is also denoted by the pair (M, \mathfrak{M}) .

If \mathfrak{M} is a constant rank distribution, then \mathfrak{M} is a projective submodule of the $C^\infty(M)$ -module $\mathfrak{X}_c(M)$, i.e., the space of smooth sections of some vector bundle. We consider constant rank distributions.

Remark 3. A foliation (M, F) in the sense of [3] is a smooth involutive distribution on M , i.e. F is a locally finitely generated $C^\infty(M)$ -submodule of $\mathfrak{X}_c(M)$ for which $[F, F] \subset F$. Due to the Frobenius theorem, this definition is equivalent to the commonly used definition of C^∞ -smooth foliation [25].

4.2 Completeness of conformal foliations

Let (M, F) be a smooth foliation of codimension $q \geq 1$ on an n -dimensional manifold M . Let \mathfrak{M} be a q -dimensional distribution on M . We say that \mathfrak{M} is *transverse* to (M, F) if the distributions \mathfrak{M} and TF define transverse subbundles of the tangent bundle to M .

Let (M, F) be a conformal foliation of codimension $q \geq 3$ on a smooth n -dimensional manifold M . We keep notation introduced in Section 2.2. Denote by \mathcal{R} the foliated bundle over (M, F) with the projection $\pi : \mathcal{R} \rightarrow M$ and by ω the \mathfrak{g} -valued 1-form on \mathcal{R} . Let $(\mathcal{R}, \mathcal{F})$ be the lifted e -foliation. Consider a q -dimensional distribution \mathfrak{M} on M . Denote by $\pi^*\mathfrak{M}$ the distribution on \mathcal{R} generated by vector fields $Y \in \mathfrak{X}(\mathcal{R})$ for which $\pi_*Y \in \mathfrak{M}$.

Definition 9. A conformal foliation (M, F) of codimension $q \geq 3$ is referred to as *complete*, if there exists a smooth transverse q -dimensional distribution \mathfrak{M} on M such that every vector field $Y \in \pi^*\mathfrak{M}$, for which $\omega(Y) = \text{const} \in \mathfrak{g}$, is complete.

4.3 Ehresmann connection for foliations

The notion of an Ehresmann connection for foliations was introduced by R.A. Blumenthal and J.J. Hebda in [8]. R.A. Blumenthal and J.J. Hebda [8] considered a smooth distribution on a smooth manifold M as a

smooth subbundle of the tangent bundle TM . We clarify the notion of Ehresmann connection for foliations due to the new definition of a distribution in sense I. Androulidakis and G. Skandalis which we have accepted.

Consider a smooth foliation (M, F) of codimension q , $q \geq 1$. Let \mathfrak{M} be a smooth q -dimensional distribution in sense I. Androulidakis and G. Skandalis, and \mathfrak{M} is transverse to (M, F) .

All maps considered here are assumed to be piecewise smooth. The integral curves of the foliation (M, F) are called vertical; the integral curves of the distribution \mathfrak{M} are called horizontal. Emphasize that a vertical curve lies in some leaf of (M, F) .

A map $H : I_1 \times I_2 \rightarrow M$, where $I_1 = I_2 = [0, 1]$, is called a *vertical-horizontal homotopy* if for each fixed $t \in I_2$, the curve $H|_{I_1 \times \{t\}}$ is horizontal, and for each fixed $s \in I_1$, the curve $H|_{\{s\} \times I_2}$ is vertical. The pair of curves $(H|_{I_1 \times \{0\}}, H|_{\{0\} \times I_2})$ is called the base of H .

A pair of curves (σ, h) with a common starting point $\sigma(0) = h(0)$, where $\sigma : I_1 \rightarrow M$ is a horizontal curve, and $h : I_2 \rightarrow M$ is a vertical curve, is called admissible. If for each admissible pair of curves (σ, h) there exists a vertical-horizontal homotopy with the base (σ, h) , then the distribution \mathfrak{M} is called an Ehresmann connection for the foliation (M, F) . Note that there exists at most one vertical-horizontal homotopy with a given base. Let H be a vertical-horizontal homotopy with the base (σ, h) . We say that $\tilde{\sigma} = H|_{I_1 \times \{1\}}$ is the result of the transfer of the horizontal curve σ along the vertical curve h with respect to the Ehresmann connection \mathfrak{M} . Similarly the curve $\tilde{h} = H|_{\{1\} \times I_2}$ is called the transfer of the curve h along σ with respect to \mathfrak{M} . Thus, an Ehresmann connection allows us to transfer horizontal curves along respective vertical curves.

The following theorem is proved in [29, Theorem 4].

Theorem 9. *For any non-Riemannian conformal foliation (M, F) of codimension $q \geq 3$ the following two conditions are equivalent:*

- (i) *There exists an Ehresmann connection for (M, F) .*
- (ii) *The conformal foliation (M, F) is complete.*

Remark 4. Like Theorem 8, Theorem 9 holds for transversely similar Riemannian foliations of codimension $q \geq 1$.

5 Minimal sets and global attractors of conformal foliations

5.1 Minimal sets of Riemannian foliations

The following theorem was proved by us in [29, Theorem 1], see also [32]. For compact manifolds this theorem was earlier proved by P. Molino [20]. In the case when a bundle-like Riemannian metric on M with respect to (M, F) is complete, it was proved by E. Salem [23] without using the term "minimal set".

Theorem 10. *Let (M, F) be a Riemannian foliation with an Ehresmann connection. Then the closure \bar{L} of any leaf L is a smooth embedded submanifold of M which is a minimal set of the foliation. The closures of leaves of the foliation (M, F) form a Riemannian foliation with singularities.*

In particular, if (M, F) is a proper foliation, then all its leaves are closed, and the leaf space is a smooth q -dimensional orbifold.

5.2 Global attractors of a countable subgroup of $\text{Conf}(\mathbb{S}^q)$

Let Ψ be a homeomorphism group of a topological space B . A nonempty closed invariant subset $\mathcal{K} \subset B$ is called an *attractor* of Ψ if there exists an open invariant subset $W \subset B$ such that the orbit closure $Cl(\Psi.z)$ of any point $z \in W \setminus \mathcal{K}$ contains \mathcal{K} . An attractor \mathcal{K} of Ψ is *global* if $W = B$.

We recall that a minimal set of a homeomorphism group Ψ of a topological space B is a nonempty closed invariant subset \mathcal{K} with respect to Ψ containing no proper subsets with this property. A finite minimal set is called trivial.

Let \mathcal{K} be a minimal set of a diffeomorphism group Ψ of a manifold B . The minimal set \mathcal{K} with empty interior is called *exceptional* if \mathcal{K} is a Cantor set, and *exotic* if \mathcal{K} is not a totally disconnected topological subspace of B .

It is well known that the conformal group $\text{Conf}(\mathbb{S}^q)$, $q \geq 2$, is isomorphic to the Möbius group $\text{Mob}_q(\mathbb{R})$ [24].

The limit set $\Lambda(\Psi)$ of an arbitrary subgroup Ψ of $\text{Conf}(\mathbb{S}^q)$ coincides with the intersection of the closures of all non-one-point orbits of this group, that is, $\Lambda(\Psi) = \bigcap \text{Cl}(\Psi.z)$, where $z \in \mathbb{S}^q$ and $\Psi.z \neq z$. If the limit set $\Lambda(\Psi)$ is finite, then either it is empty or it consists of one or two points. In this case, the group Ψ is called *elementary*.

Using some facts about subgroups of $\text{Conf}(\mathbb{S}^q)$ [17], we get the following statement.

Proposition 11. *Let Ψ be a countable essential subgroup of $\text{Conf}(\mathbb{S}^q)$. Then Ψ has a global attractor which coincides with the limit set $\Lambda(\Psi)$ of Ψ .*

5.3 Global attractors of complete conformal foliations

The following theorem is the main result of our works [28] and [29]. Emphasize that the Ehresmann connection for (M, F) is one of the main technical tools used in the proof.

Theorem 12. *A complete non-Riemannian conformal foliation (M, F) of codimension q , $q \geq 3$, has a global attractor \mathcal{M} such that one of the following conditions holds:*

- \mathcal{M} is either a closed leaf or a union of two leaves.
- \mathcal{M} is either a nontrivial minimal set, or contains a closed leaf. The induced foliation (M_0, F_{M_0}) on $M_0 = M \setminus \mathcal{M}$ is Riemannian with Ehresmann connection. The closure \bar{L} in M of any $L \subset M_0$ is equal to $\mathcal{L} \cup \mathcal{M}$, where \mathcal{L} is an embedded submanifold of M , and \mathcal{L} coincides with the closure of L in M_0 .

Remark 5. Sufficient conditions for the existence of a global attractor for transversely similar pseudo-Riemannian foliations are obtained in [31].

5.4 The global holonomy group

The following theorem [29, Theorem 5] describes the global structure of the foliated manifold and establishes a relationship between the attractors of conformal and transversely similar foliations and the attractors of their global holonomy groups.

Theorem 13. *Let (M, F) be a non-Riemannian complete conformal foliation (M, F) . Then one of the following two statements holds.*

- 1) *The foliation (M, F) is a complete transversely similar foliation with the only minimal set \mathcal{M} , which is a global attractor and contains all leaves with essential holonomy group.*
- 2) *The foliation (M, F) is covered by a bundle $r: \hat{M} \rightarrow \mathbb{S}^q$, where $f: \hat{M} \rightarrow M$ is a regular covering map. A group homomorphism*

$$\chi: \pi_1(M, x) \rightarrow \text{Conf}(\mathbb{S}^q)$$

of the fundamental group $\pi_1(M, x)$ to the group $\text{Conf}(\mathbb{S}^q)$ of conformal transformations of \mathbb{S}^q is induced, and the group of covering transformations of the covering $f: \hat{M} \rightarrow M$ is isomorphic to the group $\Psi := \chi(\pi_1(M, x))$. Moreover, in this case one of the following two statements holds.

- (i) *The foliation (M, F) has a global attractor \mathcal{M} which is either one closed leaf or two closed leaves.*
- (ii) *There exists a global attractor \mathcal{M} that coincides with the closure of a leaf with essential holonomy group, and $\mathcal{M} = f(r^{-1}(\mathcal{K}))$ where $\mathcal{K} = \Lambda(\Psi)$ is the limit set of Ψ .*

In both cases 1) and 2) the induced foliation (M_0, F_{M_0}) on the complement $M_0 := M \setminus \mathcal{M}$ is a Riemannian foliation with Ehresmann connection. The closures of its leaves in M_0 form a Riemannian foliation with singularities, and the closure \bar{L} in M of a leaf L of (M_0, F_{M_0}) equals $\mathfrak{L} \cup \mathcal{M}$, where \mathfrak{L} is an embedded submanifold of M . The set \mathfrak{L} coincides with the closure of the leaf L in M_0 .

Remark 6. Just like Theorem 8, the statement of the Theorem 13 holds for transversely similar Riemannian foliations of every codimension $q \geq 1$.

Definition 10. The group $\Psi := \chi(\pi_1(M, x))$ specified in Theorem 13, is called the *global holonomy group* of the conformal foliation (M, F) .

The following statement is proved in a constructive way for $q \geq 3$ [29, Theorem 7]. For $q = 2$, the proof is similar.

Theorem 14. Every countable subgroup Ψ of $\text{Conf}(\mathbb{S}^q)$, where $q \geq 2$, can be realized as the global holonomy group of some two-dimensional conformal foliation (M, F) of codimension q .

If Ψ is finitely generated, then such a foliation (M, F) exists on a closed manifold M of dimension $(q + 2)$.

6 Strong transverse equivalence of foliations

6.1 Invariants with respect to strong transverse equivalence of conformal foliations

The notion of strong transverse equivalence of foliations was introduced and investigated by us in [30] under the name "transverse equivalence".

A continuous map $p : X \rightarrow Y$ has the covering homotopy property with respect to a topological space K if, for any continuous map $G_0 : K \rightarrow X$ and any homotopy $H_t : K \rightarrow Y$, $t \in [0, 1]$, such that $p \circ G_0 = H_0$, there exists an extension of G_0 to a homotopy $G_t : K \rightarrow X$ satisfying the equality $p \circ G_t = H_t$, $t \in [0, 1]$.

We recall that a *Serre fibration* is a continuous surjective map having the covering homotopy property with respect to any finite polyhedron. It is known that for Serre fibrations it is possible to construct the exact homotopy sequence for a fibration. It is also well known that any locally trivial fibration is a Serre fibration. If a Serre fibration is a submersion, then it is called a smooth Serre fibration.

Definition 11. Foliation (M, F) and (M', F') are strong transversely equivalent if there exist a foliation (\mathbb{M}, \mathbb{F}) and surjective submersions $p : \mathbb{M} \rightarrow M$ and $p' : \mathbb{M} \rightarrow M'$, which are Serre fibrations, with connected fibers such that

$$\mathbb{F} := \{p^{-1}(L) \mid L \in F\} = \{p'^{-1}(L') \mid L' \in F'\}. \quad (1)$$

In comparison with the notion of transverse equivalence introduced by P. Molino [20, p. 63], Definition 11 contains an additional requirement that the submersions $p : \mathbb{M} \rightarrow M$ and $p' : \mathbb{M} \rightarrow M'$ are Serre fibrations, i.e., have the covering homotopy property. As shown in [30, Proposition 4.1], the strong transverse equivalence is indeed an equivalence relation.

Using this additional requirement, we proved that the strong transverse equivalence of foliations covered by fibrations can be realized by a foliation covered by a fibration. This fact was used in the proofs of the following theorems in [30]. As we have shown, in general, this is not true for transverse equivalent foliations in the sense of P. Molino.

Theorem 15. Two complete conformal, but not transversally similar foliations (M_1, F_1) and (M_2, F_2) of codimension $q \geq 3$ are strong transversely equivalent if and only if their global holonomy groups Ψ_1 and Ψ_2 coincide (up to conjugacy in $\text{Conf}(\mathbb{S}^q)$).

Theorem 16. Two complete non-Riemannian transversally similar foliations (M_1, F_1) and (M_2, F_2) of codimension $q \geq 1$ are strong transversely equivalent if and only if their global holonomy groups Ψ_1 and Ψ_2 coincide (up to conjugacy in $\text{Sim}(\mathbb{E}^q)$).

The structure Lie algebra \mathfrak{g}_0 of a complete non-Riemannian conformal foliation (M, F) of codimension $q \geq 3$ is isomorphic to the Lie algebra of the Lie group $\overline{\Psi}$, where $\overline{\Psi}$ is the closure of the global holonomy group Ψ of this foliation in the Lie group $\text{Conf}(\mathbb{S}^q)$. If (M, F) is a transversally similar Riemannian foliation of codimension $q \geq 1$, then \mathfrak{g}_0 is the Lie algebra of the closure $\overline{\Psi}$ of Ψ in the Lie group $\text{Sim}(\mathbb{E}^q)$. In particular, the structure Lie algebra \mathfrak{g}_0 is equal to zero if and only if Ψ is a discrete subgroup.

From Theorems 15 and 16 we obtain the following assertion.

Proposition 17. *The Lie group Ψ , the structure Lie algebra \mathfrak{g}_0 and the limit set $\Lambda(\Psi)$ of the global holonomy group Ψ of a complete non-Riemannian conformal foliation (M, F) are invariants with respect to the strong transverse equivalence.*

6.2 Suspension foliations

A. Haefliger introduced the construction of a suspension foliation. Let B and T be smooth connected manifolds, and $\rho : \pi_1(B, b) \rightarrow \text{Diff}(T)$ be a group homomorphism. Let $G := \pi_1(B, b)$ and $\Psi := \rho(G)$. Consider a universal covering map $\widehat{p} : \widehat{B} \rightarrow B$. Define a right action of the group G on the product $\widehat{B} \times T$ as

$$\Theta : \widehat{B} \times T \times G \rightarrow \widehat{B} \times T, \quad \Theta(x, t, g) = (xg, \rho(g^{-1})(t)),$$

where $\widehat{B} \rightarrow \widehat{B} : x \rightarrow xg$ is the deck transformation of the covering \widehat{p} induced by an element $g \in G$, which acts on \widehat{B} on the right. The map $p : M := (\widehat{B} \times T)/G \rightarrow B = \widehat{B}/G$ is a locally trivial bundle over B with standard fiber T . It is associated with the principal bundle $\widehat{p} : \widehat{B} \rightarrow B$ with the structural group G . Let $\Theta_g := \Theta|_{\widehat{B} \times t \times g}$. Since

$$\Theta_g(\widehat{B} \times t) = \widehat{B} \times \rho(g^{-1})(t) \quad \forall t \in T,$$

the action of G preserves the trivial foliation $F := \{\widehat{B} \times t \mid t \in T\}$ on the product $\widehat{B} \times T$. Therefore the quotient map

$$f_0 : \widehat{B} \times T \rightarrow (\widehat{B} \times T)/G = M$$

induces a smooth foliation F on M whose leaves are transversal to fibers of the bundle $p : M \rightarrow B$. The pair (M, F) is called a *suspension foliation* and is denoted by $\text{Sus}(T, B, \rho)$. The group $\Psi := \rho(G)$ of diffeomorphisms of the manifold T is said to be the *structural group* of the suspension foliation (M, F) . When T is simply connected, Ψ is called the *global holonomy group* of (M, F) .

Using suspension foliations, we proved the following theorem. Note that the same method is used in the construction of Example 21.

Theorem 18. *Every strong transverse equivalence class of a non-Riemannian conformal foliation contains a two-dimensional conformal foliation, which is a suspension foliation.*

7 Examples

Let B_k be a smooth closed 3-manifold, which is homeomorphic to the connected sum $\#_{i=1}^k S^1 \times S^2$ of k copies of $S^1 \times S^2$. Its fundamental group $\pi_1(B_k, b) = \langle g_1, \dots, g_k \rangle$ is the free group of rank k .

Example 19. *Recall first the definition of the Menger curve. Let $S = I \times I$ be the unit square. Let us divide it into 9 squares of size $1/3 \times 1/3$, then remove the open square $(1/3, 2/3) \times (1/3, 2/3)$. Then repeat this procedure with each of the remaining eight squares and continue by induction. As a result, a countable family of open squares will be removed from S and we get a compact subset of S , called the Sierpinski carpet. Let $Q = I \times I \times I$ be the unit cube. Consider a copy \mathcal{S}_j of the Sierpinski carpet on each face P_j of Q , $j = \overline{1, 6}$. Let $p_j : Q \rightarrow P_j$ be the orthogonal projection. Then the set $\mathcal{M} := \cap p_j^{-1}(\mathcal{S}_j)$ is called the Menger curve. As well-known, the topological dimension of the Menger curve \mathcal{M} is equal to 1.*

For $k \geq 6$ and $m \geq 3$, consider the group

$$\Gamma_{km} = \langle s_i, i = 1, \dots, k \mid s_i^m = 1, [s_i, s_{i+1}] = 1 \rangle.$$

Bourdon proved [9] that if $\sin \pi/k < 1/\sqrt{m}$, then there is an exact representation $\alpha_{km} : \Gamma_{km} \rightarrow \text{Conf}(\mathbb{S}^{2m-2})$ such that the group $\Psi_{km} = \alpha_{km}(\Gamma_{km})$ is a Fuchsian group (i.e. a finitely generated discrete subgroup of $\text{Conf}(\mathbb{S}^{2m-2})$), and the limit set $\Lambda(\Psi_{km})$ of Ψ_{km} is homeomorphic to the Menger curve. Thus $\psi_i := \alpha(s_i)$ are generators of Ψ_{km} .

Using the fact that $\sin x < x$ for all $x > 0$, it is easy to check that the condition $\sin \pi/k < 1/\sqrt{m}$ holds if $k > \pi\sqrt{m}$. Let $k(m) = [\pi\sqrt{m}]$ be the integer part of $\pi\sqrt{m}$. Then, for any fixed natural $m \geq 3$, there exists a countable family of Fuchsian groups Ψ_{km} , $k > k(m)$, such that $\Lambda(\Psi_{km})$ is homeomorphic to the Menger curve. For example $k(3) = 5$, $k(4) = 6$, $k(5) = k(6) = 7$, $k(7) = k(8) = 8$, $k(9) = k(10) = 9$.

Thus, $\Lambda(\Psi_{km})$ is an exotic minimal set of the group Ψ_{km} in S^{2m-2} .

Let $B_k := \prod_{i=1}^k S^1 \times S^2$ be as above. Consider a group homomorphism $\rho_{km} : \pi_1(B_k, b) \rightarrow \Psi_{km}$ defined on generators by $\rho_{km}(g_i) := \alpha_{km}(s_i)$, $i = \overline{1, k}$. Let $(M_{km}, F_{km}) := \text{Sus}(\mathbb{S}^{2m-2}, B_k, \rho_{km})$, $m \geq 3$, be the corresponding suspension foliation. Since Ψ_{km} is a subgroup of $\text{Conf}(\mathbb{S}^{2m-2})$, (M_{km}, F_{km}) is a complete conformal foliation of codimension $q = 2m - 2 \geq 4$. Hence, according to Theorem 13, $\mathcal{M} := f(r^{-1}(\Lambda(\Psi_{km})))$ is an exotic minimum set and a global attractor of the foliation (M_{km}, F_{km}) .

Example 20. Let $\mathcal{B}_1^+, \dots, \mathcal{B}_k^+, \mathcal{B}_1^-, \dots, \mathcal{B}_k^-$ be a finite set of disjoint closed topological balls with smooth boundary in the sphere \mathbb{S}^q , and $q \neq 4$. Let $\psi_i \in \text{Conf}(\mathbb{S}^q)$ be a conformal transformation such that $\psi_i(\text{int}(\mathcal{B}_i^+)) = \text{ext}(\mathcal{B}_i^-)$. The group Ψ generated by ψ_1, \dots, ψ_k is called the Schottky group. As well-known (see, for example, [17]), the Schottky group Ψ is the free group of rank k , i.e. $\Psi = \langle \psi_1, \dots, \psi_k \rangle$, and Ψ has a minimal set $\Lambda(\Psi)$ homeomorphic to a Cantor subset of the segment $[0, 1]$. Therefore, the topological dimension $\Lambda(\Psi)$ is equal to zero.

Let $B_k := \prod_{i=1}^k S^1 \times S^2$ be as above. Define a group isomorphism $\rho_k : \pi_1(B_k, b) \rightarrow \Psi$, setting $\rho_k(g_i) = \psi_i$, $i = \overline{1, k}$. The suspension foliation $(M_k, F_k) := \text{Sus}(\mathbb{S}^q, B_k, \rho_k)$ is a complete conformal foliation and, according to Theorems 13, (M_k, F_k) has a global attractor, representing an exceptional minimal set.

If $f_k : \tilde{M}_k \rightarrow M_k$ is the universal covering map, then the induced foliation $\tilde{F}_k = f_k^* F_k$ is given by the fibers of a locally trivial bundle $r_k : \tilde{M}_k \rightarrow \mathbb{S}^q$. As (M_k, F_k) is a suspended foliation, (M_k, F_k) admits an Ehresmann connection \mathfrak{M}_k .

Let $M_k^0 := M_k \setminus \mathcal{M}$. Therefore, $M_k^0 = f_k(r_k^{-1}(\mathbb{S}^q \setminus \Lambda(\Psi)))$. Since the group Ψ acts freely and properly discontinuously on the open dense subset $\mathbb{S}^q \setminus \Lambda(\Psi)$ in \mathbb{S}^q , the set M_k^0 is an open saturated dense submanifold of the compact manifold M_k . Hence, the leaf space $M_k^0/F_{M_k^0}$ is the smooth q -dimensional manifold which is homeomorphic to $(\mathbb{S}^q \setminus \Lambda(\Psi))/\Psi$. The induced foliation $(M_k^0, F_{M_k^0})$ is Riemannian, and the restriction $\mathfrak{M}_k|_{M_k^0}$ is an Ehresmann connection for $(M_k^0, F_{M_k^0})$. Therefore, Theorem 10 implies that $(M_k^0, F_{M_k^0})$ is formed by fibres of a Riemannian submersion $h : M_k^0 \rightarrow N_k$, where N_k as well as $(\mathbb{S}^q \setminus \Lambda(\Psi))/\Psi$ is homeomorphic to the connected sum of k copies of $\mathbb{S}^{q-1} \times \mathbb{S}^1$.

Note that the closure $\overline{L_\alpha}$ in M of a leaf $L_\alpha \subset M_k^0$ is equal to $L_\alpha \cup \mathcal{M}$, what agrees with Theorem 12.

Example 21. Let \mathbb{E}^q be the q -dimensional Euclidean space and \mathbb{E}^k , $2 \leq k < q$, be its k -dimensional subspace. Let e_i , $i = \overline{1, k}$, be a basis of vector space \mathbb{R}^k corresponding to \mathbb{E}^k . Consider a subgroup Ψ of the group of similarities $\text{Sim}(\mathbb{E}^q)$ generated by transformations ψ_j , $j = \overline{1, k+1}$, where $\psi_j(z) := z + e_j$ for $j = \overline{1, k}$ and $\psi_{k+1}(z) := \lambda \cdot z$ for some number $\lambda \in (0, 1)$ and all $z \in \mathbb{E}^q$. Note that Ψ is essential group of similar transformations, and the limit set $\Lambda(\Psi)$ of Ψ is equal to \mathbb{E}^k . Moreover, $\Lambda(\Psi)$ is a minimal set and a global attractor of Ψ .

Denote by S_m^2 the smooth two-dimensional sphere with m , $m := k + 1$, handles. The fundamental group of S_m^2 is equal to $\{a_i, b_i \mid i = \overline{1, m}; a_1 b_1 a_1^{-1} b_1^{-1} \dots a_m b_m a_m^{-1} b_m^{-1} = 1\}$.

Let $B := S_m^2$, $T := \mathbb{E}^q$ and define the group homomorphism $\rho : \pi_1(B, b) \rightarrow \text{Sim}(\mathbb{E}^q)$ by setting it on generators: $\rho(a_i) := \psi_i$, $i = \overline{1, m}$, $\rho(b_i) := \text{Id}_{\mathbb{E}^q}$ where $\text{Id}_{\mathbb{E}^q}$ is the identical map of \mathbb{E}^q . Then $(M, F) := \text{Sus}(\mathbb{E}^q, S_m^2, \rho)$ is a suspension two-dimensional non-Riemannian transversely similar foliation. Let $f : \mathbb{R}^2 \times \mathbb{E}^q \rightarrow M$ be the

universal covering map. Then (M, F) is covered by the trivial bundle $r : \mathbb{R}^2 \times \mathbb{E}^q \rightarrow \mathbb{E}^q$ and has the global holonomy group Ψ . By properties of suspension foliations, the $(q+2)$ -dimensional manifold M is the space of a locally trivial bundle $p : M \rightarrow \mathbb{S}_m^2$ with a standard fibre \mathbb{E}^q over the base \mathbb{S}_m^2 . Therefore M is not compact.

Thus, we get a transversely similar foliation (M, F) of codimension q , $q \geq 3$, with a regular minimal set $\mathcal{M} := f(r^{-1}(\mathbb{E}^k))$, and \mathcal{M} is a global attractor and a minimal set of (M, F) . According to [27, Theorem 9], \mathcal{M} and M are homotopy equivalent.

Let $M_0 := M \setminus \mathcal{M}$ and L_α be an arbitrary leaf in M_0 . Emphasize that the induced foliation (M_0, F_{M_0}) is an improper Riemannian foliation without holonomy, admitting an Ehresmann connection. According to Theorem 12, the closure $\overline{L_\alpha}$ is equal to $\mathcal{L}_\alpha \cup \mathcal{M}$, where \mathcal{L}_α is the closure of L_α in M_0 . Note that if $k < q-1$, then M_0 is connected and dense in M . In the case when $k = q-1$, the submanifold M_0 is dense and has two connected components denoted by M_0^+ and M_0^- , hence \mathcal{L}_α belongs to M_0^+ or M_0^- , respectively. We emphasize that all \mathcal{L}_α are pairwise diffeomorphic, that is, \mathcal{L}_α does not depend on α . Therefore, the closure $\overline{L_\alpha}$ of any leaf $L_\alpha \subset M_0$ is also independent of α .

Example 22. Fix a point $a \in \mathbb{S}^q$. Let $f : \mathbb{S}^q \rightarrow \mathbb{R}^q \cup \{\infty\}$ be the homeomorphism defined via the stereographic projection $\mathbb{S}^q \setminus \{a\} \rightarrow \mathbb{R}^q$, and $f(a) = \{\infty\}$. Identify \mathbb{S}^q with $\mathbb{R}^q \cup \{\infty\}$ through f . The map f induces a group isomorphism $f : \text{Conf}_a(\mathbb{S}^q) \rightarrow \text{Sim}(\mathbb{E}^q)$ of the stationary subgroup $\text{Conf}_a(\mathbb{S}^q)$ of the group $\text{Conf}(\mathbb{S}^q)$ at point a onto the similarity group $\text{Sim}(\mathbb{E}^q)$.

Let $\Psi \subset \text{Sim}(\mathbb{E}^q)$ be the similarity group defined in Example 21. Hence $\tilde{\Psi} := \tilde{f}^{-1}(\Psi)$ is a countable subgroup of $\text{Conf}_a(\mathbb{S}^q)$, and $\tilde{\psi}_j := \tilde{f}^{-1}(\psi_j)$, $j = \overline{1, k+1}$, are generators of $\tilde{\Psi}$.

Let $B := \mathbb{S}_m^2$ be as above, $T := \mathbb{S}^q$ and define the group homomorphism $\rho : \pi_1(B, b) \rightarrow \text{Conf}(\mathbb{S}^q)$ by setting it on generators: $\rho(a_i) := \tilde{\psi}_i$, $i = \overline{1, k+1}$, $\rho(b_i) := \text{Id}_{\mathbb{S}^q}$. Then we get the suspension foliation $(M, F) := \text{Sus}(\mathbb{S}^q, \mathbb{S}_m^2, \rho)$ which is non-Riemannian conformal foliation of codimension q , and $\tilde{\Psi}$ is its global holonomy group.

The universal covering manifold for M is the product $\mathbb{R}^2 \times \mathbb{S}^q$. Denote by $h : \mathbb{R}^2 \times \mathbb{S}^q \rightarrow M$ the universal covering map. Let $r : \mathbb{R}^2 \times \mathbb{S}^q \rightarrow \mathbb{S}^q$ be the canonical projection.

Since the global holonomy group $\tilde{\Psi}$ of (M, F) has a fixed point a , the leaf $L := h(r^{-1}(a))$ is a unique closed leaf and a global regular attractor of (M, F) . Emphasize that L is diffeomorphic to the base $B := \mathbb{S}_m^2$.

According to the definition of a limit set, $\Lambda(\tilde{\Psi}) = f^{-1}(\mathbb{R}^k \cup \{\infty\})$. Therefore, $\Lambda(\tilde{\Psi})$ is the canonically embedded k -dimensional sphere in \mathbb{S}^q containing the point a . As it was shown in Example 21, $\Lambda(\Psi)$ is the global attractor of the respective transversely similar foliation, hence $\Lambda(\tilde{\Psi})$ is a global attractor of the conformal foliation (M, F) . Since $\Lambda(\tilde{\Psi})$ contains a fixed point a , $\Lambda(\tilde{\Psi})$ is not a minimal set. We emphasize that for every point $z \in \Lambda(\tilde{\Psi}) \setminus \{a\}$ the orbit $\tilde{\Psi}.z$ is dense in $\Lambda(\tilde{\Psi})$. This implies that

$$\mathcal{M} := h(r^{-1}(\Lambda(\tilde{\Psi})))$$

is a global transitive regular attractor of the conformal foliation (M, F) .

Thus we construct a conformal foliation having two global regular attractors L and \mathcal{M} , $L \subset \mathcal{M}$, such that L is a closed leaf and \mathcal{M} is transitive attractor which is not a minimal set.

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References

- [1] D.V. Alekseevskii (1972), Groups of conformal transformations of Riemannian spaces, Math. USSR-Sb., **18**(2), 285–301.

- [2] D.V. Alekseevskii (1973), S^n and E^n are the only Riemannian spaces that admit an essential conformal transformation, *Uspekhi Mat. Nauk*, **28**(5), 225–226.
- [3] I. Androulidakis and G. Skandalis (2009), The holonomy groupoid of a singular foliation, *J. Reine Angew. Math.*, **626**, 1–37.
- [4] S.Kh. Aranson, V.Z. Grines (1978), On the representation of minimal sets of currents on two-dimensional manifolds by geodesics, *Math. USSR-Izv.*, **12**(1), 103–124.
- [5] S.Kh. Aranson, V.Z. Grines (1986), Topological classification of flows on closed two-dimensional manifolds, *Russian Math. Surveys*, **41**(1), 183–208.
- [6] P. Baird and M. Eastwood (2013), CR geometry and conformal foliations, *Ann. Global Anal. Geom.*, **44**(1), 73–90.
- [7] J.-C. Beniere, G. Meigniez (1999), Flows without minimal set, *Erg. Th. and Dyn. Sys.*, **19**(1), 1–30.
- [8] R.A. Blumenthal, J.J. Hebda (1984), Ehresmann connections for foliations, *Indiana Univ. Math. J.*, **33**(4), 597–611.
- [9] M. Bourdon (1997), Sur la dimension de Hausdorff de l'ensemble limite d'une famille de sous-groupes convexes co-compacts, *C. R. Acad. Sci. Paris Ser. I Math.*, **325**(10), 1097–1100.
- [10] A. Cap, J. Slovak (2009), *Parabolic Geometries I: Background and General Theory*, AMS Publishing House, *Math. Surveys Monogr.*, **154**, 1–628.
- [11] B. Deroin, V. Kleptsy (2007), Random Conformal Dynamical Systems. *Geometric and functional analysis*, **17**(4), 1043–1105.
- [12] J. Ferrand (1977), Sur un lemme d'Alekseevskii relatif aux transformations conformes, *C. R. Acad. Sc. Paris*, **284**, 121–123.
- [13] J. Ferrand (1996), The action of conformal transformations on a Riemannian manifold, *Math. Ann.*, **304**(2), 277–291.
- [14] C. Frances, C. Tarquini (2007), Autour du theoreme de Ferrand-Obata, *Ann. Glob. Anal. Geom.*, **21**(1), 51–62.
- [15] A. Haefliger (1985), Pseudogroups of local isometries, *Research Notes in Math.*, **131**, 174–197.
- [16] T. Inaba (1999), An example of a flow on a non-compact surface without minimal set, *Erg. Th. and Dyn. Sys.*, **19**(1), 31–33.
- [17] M. Kapovich (2007), *Kleinian groups in higher dimensions, Geometry and dynamics of groups and spaces*, *Progr. Math.*, **265**, Birkhauser, Basel 2007, 487–564.
- [18] M.S. Kulikov (2004), Schottky-type groups and minimal sets of horocycle and geodesic flows, *Sb. Math.*, **195**(1), 35–64.
- [19] G. Levitt (1983), Foliations and laminations on hyperbolic surfaces, *Topology*, **22**(2), 119–135.
- [20] P. Molino (1988), *Riemannian Foliations*, *Progr. Math.*, **263**, Birkhauser, Boston, MA.
- [21] S. Morita (1979), On characteristic classes of conformal and projective foliations, *J. Math. Soc. Japan*, **31**(4), 693–718.
- [22] M. Obata (1971), The conjectures on conformal transformations of Riemannian manifolds, *J. Differential Geom.*, **6**(2), 247–258.
- [23] E. Salem (1988), *Riemannian foliations and pseudogroups of isometries, Application D in: P. Molino, Riemannian foliations*, *Progr. Math.*, **263**, Birkhauser, Boston, MA, 1988.
- [24] R.W. Sharpe (1997), *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*, *Grad. Texts in Math.*, **166**, New York: Springer-Verlag.
- [25] I. Tamura (1992), *Topology of foliations: An Introduction*, *Transl. of Math. Monographs*, **97**, Publisher: AMS.
- [26] C. Tarquini (2004), Feuilletages conformes, *Ann. Inst. Fourier*, **52**(2), 453–480.
- [27] N.I. Zhukova (2007), Minimal sets of Cartan foliations, *Proc. Steklov Inst. Math.*, **256**(1), 105–135.
- [28] N.I. Zhukova (2011), Attractors and an analog of the Lichnerowicz conjecture for conformal foliations, *Siberian Math. J.*, **52**(3), 436–450.
- [29] N.I. Zhukova (2012), Global attractors of complete conformal foliations. *Sb. Math.*, **203**(3), 380–405.
- [30] N.I. Zhukova (2015), Transverse Equivalence of Complete Conformal Foliations *J. Math. Sci.*, **208**(1), 115–130.
- [31] N.I. Zhukova (2018), The existence of attractors of Weyl foliations modelled on pseudo-Riemannian manifolds, *JPCS*, **990**(1), 1–15.
- [32] N.I. Zhukova (2018), The structure of Riemannian foliations with Ehresmann connection, *Zh. Sredn. Mat. Obshch.*, **20**(4), 395–407 (in Russian).