sciendo Applied Mathematics and Nonlinear Sciences 5(2) (2020) 267–278

Applied Mathematics and Nonlinear Sciences

https://www.sciendo.com

Besicovitch cascades and bounded partial quotients

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Submission Info

Communicated by Lyudmila Sergeevna Efremova
Received January 23rd 2020
Accepted April 4th 2020
Available online November 16th 2020

Abstract

The article continues a series of works studying cylindrical transformations having discrete orbits (Besicovitch cascades). For any $\gamma \in (0,1)$ and any $\varepsilon > 0$ we construct a Besicovitch cascade over some rotation with bounded partial quotients, and with a γ–Hölder function, such that the Hausdorff dimension of the set of points in the circle having discrete orbits is greater than $1-\gamma-\varepsilon$.

Keywords: Besicovitch, cylinder transformation, Hausdorff dimension, Hölder function, discrete orbit, partial quotients
AMS 2010 codes: 37B05, 37C45

1 Besicovitch cascades

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be a circle of length 1, $T_\rho : \mathbb{T} \to \mathbb{T}$ be an irrational circle rotation

$$T_\rho x = x + \rho \pmod 1,$$

and let $f : \mathbb{T} \to \mathbb{R}$ be a continuous function with zero mean. We consider a *cylindrical transformation* $T_{\rho,f} : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ with a cocycle f:

$$T_{\rho,f}(x,y) = (T_\rho x, y + f(x)),$$

which is a skew product over the circle rotation. (D.V. Anosov [1] also called it a *cylindrical cascade*.) The iterations of a point (x,y) are described by

$$T^r_{\rho,f}(x,y) = (T^r_\rho x, y + f^r(x)), \quad r \in \mathbb{Z},$$

where $f^r(x)$ is a Birkhoff sum

$$f^r(x) = \begin{cases} f(x) + f(T_\rho x) + \cdots + f(T_\rho^{r-1}x) & \text{for } r > 0, \\ 0 & \text{for } r = 0, \\ -f(T_\rho^{-1}x) - \cdots - f(T_\rho^{r+1}x) - f(T_\rho^r x) & \text{for } r < 0. \end{cases}$$

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sciendo ISSN 2444-8656 doi:10.2478/AMNS.2020.2.00050 sciendo

Cylindrical transformation was studed by H. Poincaré [14] as a model for flat transformations. In particular, he studied the ω-limit points of orbits. Using these mappings, he, as well as other authors, constructed examples of flows in spaces of higher dimension with various topological properties. Cylindrical transformation have various applications in ergodic theory: see, for example, [13], [2], [15].

It turned out that there are cascades having unbounded orbits, and there are those only with bounded ones. If f is a *coboundary* over T_ρ in the class of continuous functions, i.e., there exists $F \in C(\mathbb{T})$ such that

$$f(x) = F(T_\rho x) - F(x) \tag{1}$$

for all x, then the orbit of each point (x_0, y_0) is contained within a closed invariant curve

$$\{(x,y)\colon\, y = y_0 + F(x) - F(x_0), \quad x \in \mathbb{T}\},$$

and therefore it is bounded. In this case, the cylinder $\mathbb{T}\times\mathbb{R}$ splits into such curves, and $T_{\rho,f}$ is isomorphic to the rotation of the cylinder around the axis.

Later L.G. Shnirel'man [16] and A.S. Besicovitch [3] found examples of topologically transitive cylindrical cascades, i.e., cascades having dense orbits in the cylinder (such orbits are also called topologically transitive). At that time, they did not know whether all the orbits in this case could be topologically transitive.

In 1955 W.H. Gottschalk, G.A. Hedlund [8] showed that $T_{\rho,f}$ *is topologically transitive if and only if* f *is not a coboundary over* T_ρ *and has zero mean.*

The coboundaries and the *cohomological* (or homological in earlier terminology) equation (1) are of particular importance in the theory of dynamical systems. They are used to establish isomorphism of skew products, special flows, to study the spectral properties of metric automorphisms, etc. Cohomological equations can be considered in various classes of functions (see, for example, [1]). The Birkhoff sums for coboundaries have a convenient description

$$f^r(x) = F(T_\rho^r x) - F(x). \tag{2}$$

In 1951 A.S. Besicovitch [4] proved that a cylindrical cascade cannot be minimal, it has nontransitive orbits, and for any irrational circle rotation T_ρ, there exists a continuous f such that $T_{\rho,f}$ is topologically transitive and has discrete orbits (which are closed invariant sets). The point $(x,y) \in \mathbb{T}\times\mathbb{R}$ has a discrete orbit if and only if

$$\lim_{|r|\to+\infty} f^r(x) = \infty.$$

The condition

$$\int_{\mathbb{T}} f(x)\,dx = 0$$

is essential. Otherwise, by ergodic theorem, $f^r(x) \to \infty$ as $r \to \pm\infty$, and so each orbit is discrete.

The presence of discrete orbits, as well as the property of a function to be a coboundary, substantially depends on the properties of ρ. Recall the basic notions. Every irrational $\rho \in (0,1)$ may be represented as an infinite continued fraction

$$\rho = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cdots}},$$

or briefly,

$$\rho = [k_1, k_2, \ldots],$$

and the natural numbers k_n are called *partial quotients*. The fraction $p_n/q_n = [k_1,\ldots,k_n]$ is called the *convergent* of continued fraction. It is known [9], that the convergents p_n/q_n are determined recursively:

$$p_{n+1} = k_{n+1}p_n + p_{n-1},\ n \ge 1,\ p_0 = 0,\ p_1 = 1,$$
$$q_{n+1} = k_{n+1}q_n + q_{n-1},\ n \ge 1,\ q_0 = 1,\ q_1 = k_1.$$

ρ is called Diophantine if there exist $C > 0$, $\theta > 0$ such that $q_{n+1} < Cq_n^{1+\theta}$ for any n. Otherwise, it is called Liouville.

In 2010, K. Frączek and M. Lemańczyk [7] began to study the properties of a set of discrete orbits depending on the function f and the rotation number ρ. (Following them, a transitive cascade with discrete orbits is called the Besicovitch cascade, and the set $B \subset \mathbb{T} \times \{0\}$ of circle points having discrete orbits is called the Besicovitch set.)

The Besicovitch set B is invariant under T_ρ, T_ρ is uniquely ergodic with the only invariant Lebesgue measure, and, therefore, B has a null Lebesgue measure.

Obviously, if f has bounded variation, then T_f is not Besicovitch, because the sequence $T_{\rho,f}^{q_n}(x,y)$ is bounded for denominators q_n of ρ, as $\|f^{q_n}\|_{C(\mathbb{T})} \leqslant \mathrm{Var}(f)$.

K. Frączek and M. Lemańczyk showed that the Besicovitch cascade with continuous function f can be constructed for any transitive T_ρ. For ρ satisfying some Diophantine condition, the γ-Hölder function f was obtained, so that $T_{\rho,f}$ is Besicovitch. In this construction, γ depends on the Diophantine parameter and $\gamma < 1/2$ in any case. They also showed that, under additional conditions, the Hausdorff dimension of the Besicovitch set can be at least $1/2$.

Thus, it was established in [7] that both the admissible degree of continuity of the function and the Hausdorff dimension of the Besicovitch set depend on the properties of rotation.

We also note the result of E. Dymek [5], who showed that for any irrational ρ, one can construct a continuous cascade for which the Besicovitch set has a Hausdorff dimension 1.

A number of examples were constructed in [11], [12], demonstrating a closer relationship between the Hölder exponent of the function f and the obtained estimate of the Hausdorff dimension for the Besicovitch set. In [11], using angles with property $q_{n+1} \asymp q_n^{1+\theta}$, $\theta > 0$, it was shown that *for any $\gamma \in (0,1)$ and any $\varepsilon > 0$, there exist a γ-Hölder function f and a circle rotation T_ρ such that the cylindrical transformation $T_{\rho,f}$ is Besicovitch, and the Hausdorff dimension of the Besicovitch set in the circle is greater than $1-\gamma-\varepsilon$.*

In [12], using angles with relatively slowly varying, but infinitely large partial quotients, it was managed to achieve inequality $\dim_H(B) \geqslant 1-\gamma$ and also to construct the Besicovitch cascade with a function that is γ-Hölder with any exponent $\gamma \in (0,1)$.

Here we prove the following theorem.

Theorem 1. *For any $\gamma \in (0,1)$ and any $\varepsilon > 0$, there exists a γ-Hölder function f and a circle rotation T_ρ with bounded quotients such that the cylindrical transformation $T_{\rho,f}$ is Besicovitch, and the Hausdorff dimension of the Besicovitch set in the circle is greater than $1-\gamma-\varepsilon$.*

Before proceeding to the proof of the theorem, we formulate two problems.

Problem 1. Is $1-\gamma$ the upper bound for $\dim_H(B)$? It is unknown whether there exists a γ-Hölder Besicovitch cascade, for which the Hausdorff dimension of the Besicovitch set is greater than $1-\gamma$.

Problem 2. [7] Is it possible to construct a Besicovitch cascade with Hölder function over Liouville rotation of a circle?

The next three sections are devoted to the proof of the theorem.

2 The main construction

The construction is based on the design proposed in [11], but the operation with rotations having bounded partial quotients required a slight modification and more subtle estimates.

So, let $\rho = [k_1, \dots, k_n, \dots]$ be an irrational number, and $\{p_n/q_n\}$ be the sequence of convergents to ρ. Let $\delta_n = |\rho - p_n/q_n|$. It is well known that

$$\frac{1}{q_n(q_{n+1}+q_n)} < \delta_n < \frac{1}{q_n q_{n+1}} \tag{3}$$

and

$$\rho = \frac{p_n}{q_n} + (-1)^n \delta_n.$$

We will construct the γ-Hölder cocycle f as the sum of a series of Lipschitz functions f_n corresponding to fractions p_n/q_n:

$$f = \sum_{n=1}^{\infty} f_n, \tag{4}$$

where f_n is $1/q_n$–periodical function defined by expression

$$f_n(x) = \begin{cases} \frac{a_n}{2\delta_n}(x+\delta_n)^2, & x \in [-\delta_n, 0], \\ \frac{a_n\delta_n}{2} + a_n x, & x \in \left[0, \frac{1}{2t_nq_n} - \delta_n\right], \\ \frac{a_n}{2t_nq_n} - \frac{a_n\delta_n}{4} - \frac{a_n}{\delta_n}\left(x - \frac{1}{2t_nq_n} + \frac{\delta_n}{2}\right)^2, & x \in \left[\frac{1}{2t_nq_n} - \delta_n, \frac{1}{2t_nq_n}\right], \\ f_n\left(\frac{1}{t_nq_n} - \delta_n - x\right), & x \in \left[\frac{1}{2t_nq_n}, \frac{1}{t_nq_n}\right], \\ 0, & x \in \left[\frac{1}{t_nq_n}, \frac{1}{2q_n} - \delta_n\right], \\ -f_n\left(x - \frac{1}{2q_n}\right), & x \in \left[\frac{1}{2q_n} - \delta_n, \frac{1}{q_n} - \delta_n\right]. \end{cases}$$

In this expression, in addition to the rotation number parameters, the quantities t_n and a_n are used. We assume, that for any number n

$$t_n > 2. \tag{5}$$

$1/t_n$ shows, what part of the period is occupied by the support of f_n, and $a_n > 0$ is the Lipschitz constant for f_n.

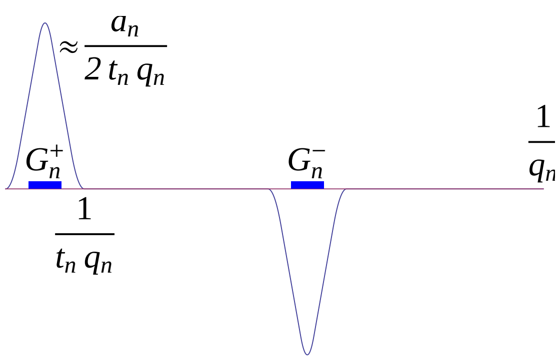

Fig. 1 The graf of $f_n(x)$

Note. Since the addition of the coboundary to the function $f = \sum_{i=1}^{\infty} f_i$ does not change the Besikovich set, we can consider the terms and the convergents p_n/q_n, and also estimate the partial sums $\sum_{i=n_0}^{n} f_i^r$, starting from some number n_0. For simplicity, in this case we shift the numbering, assuming $n_0 = 1$.

Simultaneously, we define the subset of Besicovitch set $D \subset \mathbb{T}$, each point of which has a discrete orbit:

$$\begin{aligned}
&G_n^+ = \bigcup_{k=0}^{q_n-1} \left(\frac{k}{q_n} + \left[\frac{1}{4t_nq_n}, \frac{3}{4t_nq_n} \right] \right), \qquad G_n^- = \frac{1}{2q_n} + G_n^+, \\
&D_s^+ = \bigcap_{n=s}^{\infty} G_n^+, \qquad D_s^- = \bigcap_{n=s}^{\infty} G_n^-, \\
&D^+ = \bigcup_{s=1}^{\infty} D_s^+, \qquad D^- = \bigcup_{s=1}^{\infty} D_s^-, \\
&D = D^+ \cup D^-.
\end{aligned} \tag{6}$$

Lemma 2. *If $a_n \leqslant \mathrm{const}(t_nq_n)^{1-\gamma}$ for any n and some $\gamma \in (0,1)$, then the series (4) converges, and the function f is γ–Hölder.*

The detailed proof of this lemma is given in [12]. The more general construction of functions with various continuity properties was also formulated and justified there. Earlier such construction of «almost Lipschitz function» was used by the author in [10].

The convergence of series (4) follows from the inequality $\|f_n\|_C \leqslant \dfrac{\mathrm{const}}{(t_nq_n)^\gamma}$ and the fact that denominators q_n grow not slower than exponentially.

In this article we set

$$a_n = (t_nq_n)^{1-\gamma}. \tag{7}$$

The conditions of the next lemma are modified compared with those in the previous papers [11] and [12]. These changes are adapted to dealing with bounded partial quotients.

Lemma 3. *If the sequences $\{q_n\}$, $\{t_n\}$ and $\{a_n\}$ satisfy the conditions*

$$\frac{q_{n+1}}{t_nq_n} = m \geqslant 6, \quad m \in \mathbb{N}, \tag{8}$$

$$\frac{a_n}{t_n} \to +\infty, \tag{9}$$

$$\frac{a_nq_{n+1}}{t_n^2q_n} : \frac{a_{n+1}}{t_{n+1}} < \frac{1}{6}, \tag{10}$$

then for any $x \in D^+$

$$\lim_{r\to+\infty} f^r(x) = +\infty, \qquad \lim_{r\to-\infty} f^r(x) = -\infty.$$

For $x \in D^-$, the situation is inverse.

Proof. **1.** At first, we will try to describe the mechanism of «pushing» the orbit of a point to infinity. The terms making up f, in turn, «are responsible» for the growth of Birkhoff sums on the set D^+. Due to the good agreement with the circle rotation, the Birkhoff sum $f_n^r(x)$ of the n-th term for $x \in G_n^+$ to a moment about $r = q_n/4$ grows up to a level of the order a_n/t_n, and at a moment of the order of $r = \frac{q_{n+1}}{t_n}$ ceases to grow at a level no more than $L_n \approx \frac{a_nq_{n+1}}{4t_n^2q_n}$, and this value keeps constant up to the moment of at list $q_{n+1}/4$, when the next term increases enough to «overtake» the current one. This fact explains the second and third assumptions of the lemma.

After the moment about $r = q_{n+1}/2$, the Birkhoff sum $f_n^r(x)$ decreases and may become negative, but the growth of the sums f^r on the set $G_{n+1}^+ \cap G_n^+$ is provided by the next summand f_{n+1}^r. As each function f_n is a coboundary, we can control the values of all previous terms, and the following terms are positive on D^+.

2. We will show that f_n is a coboundary. For this let us define a $1/q_n$–periodic function $F_n\colon \mathbb{T}\to\mathbb{R}$ such that

$$f_n(x) = F_n(T_\rho x) - F_n(x). \tag{11}$$

We put

$$\widehat{F}_n(x) = \begin{cases} \dfrac{a_n}{2\delta_n}x^2, & x\in\left[0,\dfrac{1}{2t_nq_n}\right],\\ L_n - \widehat{F}_n\left(\dfrac{1}{t_nq_n}-x\right), & x\in\left[\dfrac{1}{2t_nq_n},\dfrac{1}{t_nq_n}\right],\\ L_n, & x\in\left[\dfrac{1}{t_nq_n},\dfrac{1}{2q_n}\right],\\ L_n - \widehat{F}_n\left(x-\dfrac{1}{2q_n}\right), & x\in\left[\dfrac{1}{2q_n},\dfrac{1}{q_n}\right], \end{cases}$$

where

$$L_n = 2\widehat{F}_n(1/(2t_nq_n)) = \frac{a_n}{4\delta_n t_n^2 q_n^2}.$$

It easy to verify that

$$f_n(x) = \widehat{F}_n(x+\delta_n) - \widehat{F}_n(x).$$

Then we put

$$F_n(x) = \begin{cases} \widehat{F}_n(x) & \text{for even } n,\\ L_n - \widehat{F}_n(x+\delta_n) & \text{for odd } n. \end{cases}$$

For even n, when $\rho = p_n/q_n + \delta_n$, we immediately have (11). For odd n, when $\rho = p_n/q_n - \delta_n$,

$$F_n(x+\rho) - F_n(x) = -\widehat{F}_n(x+\delta_n+\rho) + \widehat{F}_n(x+\delta_n) = \widehat{F}_n(x+\delta_n) - \widehat{F}_n(x+p_n/q_n) = f_n(x).$$

For the constant L_n, by (3) and (8), the estimates

$$\frac{m}{4}\cdot\frac{a_n}{t_n} < L_n < \frac{m}{4}\cdot\frac{a_n}{t_n}\left(1+\frac{q_n}{q_{n+1}}\right). \tag{12}$$

are valid.

3. Since for any integer r and any $x\in\mathbb{T}$

$$f_n^r(x) = F_n(x+r\rho) - F_n(x), \quad 0\leqslant F_n(x)\leqslant L_n,$$

we have

$$\|f_n^r\|_C \le L_n.$$

Moreover,

$$|f_n^r(x)| \leqslant \frac{3}{4}L_n \quad \text{ for any } x\in D_1^{\pm} \text{ and any } r\in\mathbb{Z}, n\in\mathbb{N}. \tag{13}$$

Indeed, if $x\in D_1^{\pm}$, then $x\in G_n^{\pm}$, hence $\frac{1}{4}L_n \leqslant F_n(x) \leqslant \frac{3}{4}L_n$, and so

$$0-\frac{3}{4}L_n \leqslant f_n^r = F_n(T_\rho x) - F_n(x) \leqslant L_n - \frac{1}{4}L_n.$$

It follows from (8) and (10) that

$$\frac{a_n}{t_n} : \frac{a_{n+1}}{t_{n+1}} < \frac{1}{6m}, \tag{14}$$

therefore, by (13), we obtain

$$L_1+\cdots+L_n<\frac{m}{4}\sum_{i=1}^{n}\frac{a_i}{t_i}\left(1+\frac{q_i}{q_{i+1}}\right)<\frac{m}{4}\left(1+\frac{1}{\overline{K}_n}\right)\frac{6m}{6m-1}\frac{a_n}{t_n},$$

where, according to (5) and (8),

$$\overline{K}_n=\min_{i\leqslant n}\frac{q_{i+1}}{q_i}>12.$$

Given the last three inequalities, we obtain

$$L_1+\cdots+L_n<\frac{m}{3}\cdot\frac{a_n}{t_n}. \tag{15}$$

4. Now let be $x\in G_n^+$. Then

$$f_n^{r+1}(x)\geqslant f_n^r(x)\geqslant 0,\quad f_n^{-r}(x)\leqslant 0\quad \text{for all } r\in[0,q_{n+1}/4], \tag{16}$$

$$|f_n^{\pm r}(x)|\geq\frac{a_n}{18t_n}\quad \text{for all } r\in[q_n/4,q_{n+1}/4]. \tag{17}$$

From (16) it follows that if $x\in D_1^+=\bigcap_{n=1}^{\infty}G_n^+$, then for all $r\in[0,q_{n+1}/4]$

$$f_m^r(x)\geqslant 0\quad \text{for}\quad m\geqslant n. \tag{18}$$

To verify (16), note that by the periodicity of function f_n and the set G_n^+,

$$f_n(x+j/q_n+\rho)=f_n(x+(-1)^n\delta_n),$$

therefore, we can assume that $x\in[1/(4t_nq_n),3/(4t_nq_n)]$. For even n, and given that $(q_{n+1}/4)\delta_n<1/(4q_n)$, $t_n>2$ and $r\leqslant\frac{q_{n+1}}{4}-1$,

$$\frac{1}{4t_nq_n}\leqslant x+r\delta_n\leqslant\frac{3}{4t_nq_n}+r\delta_n<\frac{1}{2q_n}-\delta_n,$$

so all steps $T_\rho^r x$ within $r\leqslant\frac{q_{n+1}}{4}-1$ occur in the region where f_n is non-negative, which implies (16) and non-decreasing Birkhoff sums $f_n^r(x)$ with increasing r in the indicated range.

The cases of odd n and $r<0$ are similar.

To get (17), we consider $r=\lceil q_n/4\rceil$. As above, we can assume that $x\in[1/(4t_nq_n),3/(4t_nq_n)]$, and

$$f_n^r(x)=\sum_{k=0}^{r-1}f_n(x+(-1)^nk\delta_n).$$

By (3) and (8), points of the form $\{x+(-1)^nk\delta_n,\quad 0\leqslant k\leqslant q_n/4\}$ fill evenly a segment of length shorter than $1/(4mt_nq_n)$ in the segment $\left[\frac{1-1/m}{4t_nq_n},\frac{3+1/m}{4t_nq_n}\right]$, and therefore

$$f_n^r(x)\geqslant\left(1-\frac{1}{2m}\right)\cdot\frac{a_n}{16t_n}.$$

Roughing up the estimate, we obtain (17).

5. Now we can estimate Birkhoff sum $f^r(x)$ for $x\in D_1^+\subset G_n^+$ and $r\in[q_n/4,q_{n+1}/4]$:

$$\begin{aligned}f^r(x)&=\sum_{k=1}^{n-1}f_k^r(x)+f_n^r(x)+\sum_{k=n+1}^{\infty}f_k^r(x)>\quad\{\text{ by (13), (18)}\}\\&>-\frac{3}{4}(L_1+\cdots+L_{n-1})+f_n^r(x)>\quad\{\text{ by (15), (17)}\}\quad>-\frac{m}{4}\cdot\frac{a_{n-1}}{t_{n-1}}+\frac{1}{18}\frac{a_n}{t_n}.\end{aligned}$$

According to (8) and (10), we have $m \cdot \frac{a_{n-1}}{t_{n-1}} < \frac{a_n}{6t_n}$, so

$$f^r(x) > \frac{a_n}{t_n}\left(\frac{1}{18} - \frac{1}{24}\right) = \frac{1}{72} \cdot \frac{a_n}{t_n} \to \infty$$

for $n \to \infty$, and therefore, for $r \to +\infty$.

The cases $r \to -\infty$ and $x \in D_1^-$ are similar. Cases $x \in D_s^{\pm}$, $s > 1$ require a shift in the numbering of terms. Lemma 3 is proved.

3 Rotation of the circle

We consider an irrational number ρ with constant, starting from some position, partial quotients

$$\rho = [Q, K, K, \ldots],$$

where Q is a rational number. In the standard representation with integer partial quotients, this means that partial quotients $k_n = K$ for all numbers n large enough. Thus, we define the sequence $\{q_n\}$ (and also $\{t_n\}$ and $\{a_n\}$). After that we fix the Hölder constant $\gamma \in (0,1)$ and the constant

$$m \geqslant 6$$

appearing in the construction of f. It will give us, by (8), $t_n = \frac{q_{n+1}}{mq_n}$. To completely determine the cylindrical cascade $T_{\rho,f}$, we define

$$a_n = (t_n q_n)^{1-\gamma}. \tag{19}$$

So, we have three parameters K, γ and m, and we substitute them to the conditions of Lemma 3.

We have, starting from some number n,

$$p_{n+1} = Kp_n + p_{n-1}, \quad q_{n+1} = Kq_n + q_{n-1}.$$

Since the sequence $\{q_n\}$ satisfies the difference equation, for n large enough we have

$$q_n = C_1\lambda^n + C_2(-\lambda)^{-n},$$

where $\lambda = \frac{K}{2} + \sqrt{\frac{K^2}{4} + 1} \approx K + \frac{1}{K}$ is the root of characteristic equation $\lambda^2 - \lambda - 1 = 0$. Also we have

$$[K, K, K, \ldots] = \frac{1}{\lambda}, \quad \rho = [Q, K, K, \ldots] = \frac{1}{Q + 1/\lambda}.$$

We say that $a_n \approx b_n$ if $a_n = b_n\left(1 + O(1/\lambda^{-2n})\right)$ for $n \to \infty$. In this notation

$$q_n \approx C\lambda^n, \qquad \frac{q_{n+1}}{q_n} \approx \lambda. \tag{20}$$

Using equalities $\frac{q_{n+1}}{t_n q_n} = m$ (condition (8)) and $a_n = (t_n q_n)^{1-\gamma}$, we have

$$t_n = \frac{q_{n+1}}{mq_n} \approx \frac{\lambda}{m}, \tag{21}$$

$$a_n = (t_n q_n)^{1-\gamma} \approx \left(\frac{C\lambda}{m}\right)^{1-\gamma} \lambda^{n(1-\gamma)}, \tag{22}$$

$$\frac{a_n}{t_n} \approx C^{1-\gamma}\left(\frac{m}{\lambda}\right)^{\gamma} \lambda^{n(1-\gamma)}, \tag{23}$$

$$\frac{a_n q_{n+1}}{t_n^2 q_n} : \frac{a_{n+1}}{t_{n+1}} = \frac{q_{n+1}}{t_n q_n} \cdot \left(\frac{a_n}{t_n} : \frac{a_{n+1}}{t_{n+1}}\right) \approx \frac{m}{\lambda^{1-\gamma}}. \tag{24}$$

Lemma 4. *If the parameters γ, m, K are such that*

$$m \geqslant 6, \quad \lambda^{1-\gamma} > 6m, \tag{25}$$

then the conditions of Lemma 3 are satisfied.

Proof. We used the condition (8) to determine t_n. According to (24), the condition (10) of Lemma 3 holds for n large enough. By (21), the inequality $t_n > 6$ holds for n large enough. Also, by (23), the condition (9), i. e., $\frac{a_n}{t_n} \to \infty$ for $n \to \infty$, is satisfied.

4 Hausdorff dimension of Besicovitch Set

In this section, we estimate the Hausdorff dimension of the Besicovitch set B, i. e., set of points on the circle $\mathbb{T} \times \mathbb{R}$ having discrete orbits. In section 2, the subset $D \subset B$ of Besicovitch set was constructed (see (6)):

$$\lim_{r \to \pm\infty} f^r(x) = \infty \qquad \text{for any } x \in D.$$

Thus, the lower bound for $\dim_H D$ is also for $\dim_H B$. As for the upper bound for $\dim_H D$, the paper does not prove that it is such for $\dim_H B$, but such a hypothesis exists.

D is the union of a countable set of sets of Cantor type. It is sufficient to estimate $\dim_H(D_1^+)$ because

$$\dim_H(D) = \dim_H(D_1^+).$$

Lemma 5. *The Hausdorff dimension of Besicovitch subset D satisfies*

$$\frac{\ln\left(\frac{m}{2} - 1\right)}{\ln\lambda} \leqslant \dim_H D \leqslant \frac{\ln\left(\frac{m}{2} + 1\right)}{\ln\lambda}. \tag{26}$$

As a result,

$$\dim_H B \geqslant \frac{\ln\left(\frac{m}{2} - 1\right)}{\ln\lambda}. \tag{27}$$

Recall, that $D_1^+ = \bigcap_{n=1}^{\infty} G_n^+$ (see (6)). Let us call the segments of G_n^+ as n-th level segments.

We get the upper bound for $\dim_H D$ by definition (see [6], section 2.1). Define the covers W_n of D_1^+ by segments inductively. $W_1 = G_1^+$ (we consider the cover as the union of segments). The cover W_n consists of those segments of G_n^+ that intersect W_{n-1} in more than one point. The length of each of them is equal to $1/(2t_n q_n)$. By virtue of (8), each nth-level segment may intersect $m/2$ or $m/2+1$ segments of $(n+1)$-th level. If m_n is the number of nth–level segments included into W_n, than $m_n \leqslant \left(\frac{m}{2} + 1\right) m_{n-1}$, or

$$m_n \leqslant m_1 \left(\frac{m}{2} + 1\right)^{n-1}.$$

Thus, we have the cover of D_1^+ by segments of length $1/(2t_n q_n) \approx \frac{C}{m} \cdot \lambda^{n+1}$ (by (20) and (5)). Then we have the estimate

$$H_n^d = m_n \cdot 1/(2t_n q_n)^d < \text{const} \left(\frac{\frac{m}{2} + 1}{\lambda^d}\right)^{n-1}.$$

If $\lambda^d > \left(\frac{m}{2} + 1\right)$, then $\lim_{n \to \infty} H_n^d = 0$. Therefore, for Hausdorff dimension $d = \dim_H D_n^+$, the inequality

$$\lambda^d \leqslant \left(\frac{m}{2} + 1\right)$$

is necessary, and we get the upper bound for Hausdorff dimension.

We obtain the lower bound for Hausdorff dimension by the method proposed in [6], Example 4.6. Now we denote by s_n the minimal number of nth-level segments, contained in one $(n-1)$th-level segment. Let ε_n be the minimal distance between n-th level segments. Then

$$\dim_H(D_1^+) \geqslant \liminf_{n\to\infty} \frac{\ln(s_2\dots s_n)}{-\ln(s_{n+1}\varepsilon_{n+1})}. \tag{28}$$

By (8), each n-th level segment contains entirely $m/2$ or $(m/2-1)$ segments of the form $[(i-1)/q_{n+1}, i/q_{n+1}]$, therefore, each nth-level segment entirely contains at least $(m/2-1)$ $(n+1)$-th level segments, where from $s_n \geqslant m/2-1$, we have $\varepsilon_n = \dfrac{1}{q_n}\left(1-\dfrac{1}{2t_n}\right)$. Substituting these inequalities and (20) to (29), we obtain

$$\dim_H(D_1^+) \geqslant \lim_{n\to\infty} \frac{(n-1)\ln\left(\frac{m}{2}-1\right)}{\ln q_{n+1} + \ln s_{n+1} - \ln\left(1-\dfrac{1}{2t_n}\right)} = \frac{\ln\left(\frac{m}{2}-1\right)}{\ln\lambda}, \tag{29}$$

and we get the lower bound in (26).

Lemma 6. *For given $\gamma\in(0,1)$ and any $\varepsilon>0$, there exist an even integer $m\geqslant 6$, and the constant partial quotients K defining ρ, such that*

$$\lambda^{1-\gamma} > 6m \quad \text{and} \quad \frac{\ln\left(\frac{m}{2}-1\right)}{\ln\lambda} > 1-\gamma-\varepsilon. \tag{30}$$

Proof. Recall, that $\lambda = \dfrac{K}{2}+\sqrt{\dfrac{K^2}{4}+\dfrac{1}{4}} \approx K+\dfrac{1}{K}$. Since

$$\lim_{\lambda\to+\infty} \frac{\lambda^{1-\gamma}/6}{2\left(\lambda^{1-\gamma-\varepsilon}+1\right)} = +\infty,$$

the double inequality

$$\lambda^{1-\gamma}/6 < m < 2\left(\lambda^{1-\gamma-\varepsilon}+1\right),$$

which is equivalent to (30), is resolvable in the class of even m for any λ (and hence K) large enough.

This lemma completes the proof of Theorem 1.

In fact, for given $\gamma\in(0,1)$ and $\varepsilon>0$, we choose parameters $m\geqslant 6$ and K satisfying (30). Using these parameters and some rational Q, we define $\rho = [Q,K,K,\dots]$ and thus a sequence of convergents $\{p_n/q_n\}$. Now, according to (8), the sequences $\{t_n\}$ and $\{a_n = (t_nq_n)^{1-\gamma}\}$ are also defined.

Then, using the main construction, we define the function f and the set D.

According to Lemma 4, all the conditions of Lemma 3 are satisfied, therefore all points $x\in D$ run away to infinity under iterations of the cylindrical cascade $T_{\rho,f}$. So $T_{\rho,f}$ is a Besicovitch cascade, and D is a Besicovitch subset. By Lemma 2, f is γ-Hölder.

As m and K satisfy (25), from Lemma 5 we get the estimate

$$\dim_H B > 1-\gamma-\varepsilon.$$

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