Applied Mathematics and Nonlinear Sciences

Mathematical Algorithm for Solving Two–Body Problem

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Abstract

In this paper, computational algorithm with the aid of Mathematica software is specifically designed for the gravitational two–body problem. Mathematical module is established to find the position and velocity vectors. Application of this module for different kind of orbits (elliptic, parabolic and hyperbolic) leads to accurate results, which proved module efficiency and to be skillful. The classical power series method is to be utilized as the methodology.

1 Introduction

The classical two–body problem is the dynamical system which describes the motion of two objects. Of course, it is the simplest and only integrable system within frame of the classical Newtonian potential between two–bodies is applied. There are many applications in mathematical astronomy or in engineering and physical sciences can be analyzed within the frame work of two–body systems, it can be used in both quantum mechanics (particles motion) and celestial mechanics (Stars motion)

The multivariate of the perturbed forces in inner and outer space change the two–body from a simple and integrable system to one is complex and is not integrable. Thereby the analytical solutions will be invalid in most real applications. The analysis of two–body problem under the effect of many perturbed forces have received a comprehensive an extended study in the literature space dynamics. For example, the perturbed two–body problem by radiation pressure force and many types of drag forces or both together has been investigated [6,9–12]. Furthermore, the analytical solutions of the satellite motion within frame of non– sphericity and zonal harmonics perturbations effect have been studied by [7, 8]. [4] have studied also the dynamics of anisotropic

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Kepler problem with small anisotropy. They proved that at every energy level the anisotropic dynamical system has two periodic orbits.

The perturbed two–body system is not limited to the perturbations of either radiation pressure and drag force or the non–sphericity of the bodies. But the other bodies can be taken as a perturbed forces for the motion of two–body system [5]. Another consequence of two–body problem is that the three-body problem can be reduced to the system of two–body, if we consider either the third body has mass equals zero or if it moves to infinity while the other primary does not. There are a considerable work have been constructed to find the periodic solution for many perturbed two–body systems, see for details [1, 2, 13]. In addition the two–body problem within frame of the corrections law of Newtonian inverse square law of gravitation can be studied as a perturbed model [3].

The motion of planets and asteroids is a very important field of interest in astronomy, and space dynamics. It can be formulated as a dynamical system of differential equations based on the Newton's laws of gravitation [14]. The law states that everybody attracts every other body along a line intersecting the two bodies with a force equals to $F = Gm_1m_2/r^2$, where F is the force between the two bodies, G is the gravitational constant, m_1 is the mass o f the first body, m_2 is the mass of the second body, and r the distance between the centres of masses of the bodies [15].

Additionally, the Newtonian gravitation can be extended to $N-$bodies by simply summing the forces [16–18]. Moreover, various analytical and numerical methods have been used to examine such problems of celestial mechanics comprising of the two–body, three–body and the generalized $N-$body problem, see [19–26]. But we aim in this paper to computationally tackle the gravitational two–body problem with the application of the power series method [14].

2 Vector two–body equation

We start off this section by first introducing the equation of the gravitational $N-$body problem as [16]

$$\ddot{\mathbf{r}}_i = -\frac{G}{r_i^3}(m_0 + m_i)\ddot{\mathbf{r}}_i - G \sum_{j=1,j\neq i}^{N} m_j\{\frac{\mathbf{r}_i - \mathbf{r}_j}{r_{ij}^3} + \frac{\mathbf{r}_j}{r_j^3}\}, i = 1,2,...,N \tag{1}$$

where $\ddot{\mathbf{r}}_i$ is the position vector of m_i relative to m_0, m_i and m_0 are masses of ith body and central body, respectively and G is the universal constant of gravitation. Note that if all masses are equal zero except m_0, m_i, then Eq. (1) becomes

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{o} \tag{2}$$

where $\mu = G(m_0 + m_i)$. The above equation is called the classical vector two–body equation.

3 Power series solution

3.1 Lagrange's fundamental invariants

In favour of Eq. (2), the Lagrange's fundamental invariants ε, λ and ψ are defined by [14, 27, 28]

$$\varepsilon = \frac{\mu}{r^3}$$
$$\lambda = \frac{1}{r^2}(\mathbf{r} \cdot \mathbf{v}) \tag{3}$$
$$\psi = \frac{1}{r^2}(\mathbf{v} \cdot \mathbf{v})$$

where $\mathbf{v} = \dot{\mathbf{r}}$.

Lagrange's invariants in Eqs.(3) satisfy the following differential equations [14, 27, 28]

$$\dot{\varepsilon} + 3\varepsilon\lambda = 0$$
$$\dot{\lambda} + 2\lambda^2 + (\varepsilon - \psi) = 0$$
$$\dot{\psi} + 2\lambda\psi + 2\lambda\varepsilon = 0$$

3.2 Basic differential equations

Here, to analyze the gravitational two-body problem by formulate and examine the following nonlinear differential equations [14, 27, 28]

$$\ddot{q} + \varepsilon q = 0$$
$$\dot{\varepsilon} + 3\varepsilon\lambda = 0$$
$$\dot{\lambda} + 2\lambda^2 + (\varepsilon - \psi) = 0 \tag{4}$$
$$\dot{\psi} + 2\lambda\psi + 2\lambda\varepsilon = 0$$

In analyzing the previous system of nonlinear differential equations, with employ the power series method [16]. Thus, leads to the power series expansion of the four dependent variables in Eqs. (4) as follows:

$$q(t) = \sum_{n=0}^{\infty} q_n(t - t_0)^n$$
$$\varepsilon(t) = \sum_{n=0}^{\infty} \varepsilon_n(t - t_0)^n$$
$$\lambda(t) = \sum_{n=0}^{\infty} \lambda_n(t - t_0)^n \tag{5}$$
$$\psi(t) = \sum_{n=0}^{\infty} \psi_n(t - t_0)^n$$

Upon substitution of Eqs. (5) to Eqs. (4) and then solve for the coefficients of q_n, ε_n, λ_n and ψ_n, we get the following recurrence relations

$$q_{n+2} = -\frac{1}{(n+1)(n+2)} \sum_{p=0}^{n} \varepsilon_p q_{n-p}$$
$$\varepsilon_{n+1} = -\frac{3}{(n+1)} \sum_{p=0}^{n} \varepsilon_p \lambda_{n-p}$$
$$\lambda_{n+1} = \frac{1}{(n+1)} \left(\psi_n - \varepsilon_n - 2 \sum_{p=0}^{n} \lambda_p \lambda_{n-p} \right) \tag{6}$$
$$\psi_{n+1} = -\frac{2}{(n+1)} \sum_{p=0}^{n} \lambda_p (\varepsilon_{n-p} + \psi_{n-p})$$

Also with the recurrence relations above taking the following starting values

$$q_0 \equiv q(t_0), \qquad q_1 \equiv \dot{q}(t_0), \qquad \varepsilon_0 \equiv \varepsilon(t_0), \qquad \lambda_0 \equiv \lambda(t_0), \qquad \psi_0 \equiv \psi(t_0)$$

Furthermore, the values of q_0 and q_1 are known from the formulated problem, while ε_0, λ_0 and ψ_0 to be determined from $\mathbf{r}_0 \equiv \mathbf{r}(t_0)$ and $\mathbf{v}_0 \equiv \mathbf{v}(t_0)$ as follows

$$
\begin{aligned}
\varepsilon_0 &= \frac{1}{r_0^3}\mu \\
\lambda_0 &= \frac{1}{r_0^2}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) \\
\psi_0 &= \frac{1}{r_0^2}(\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2)
\end{aligned}
\tag{7}
$$

with $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$.

In the next section with a help of Eqs. (6,7), the computational values of q_n are determined for $n = 9$ using a designed algorithm which is called QPS, see Appendix A for algorithm.

3.3 f and g functions

Here, two Lagrangian coefficients functions f and g [14, 27] introduced instead of one that led to the following position vector

$$
\mathbf{r} = f\mathbf{r}_0 + g\mathbf{v}_0
\tag{8}
$$

where $\mathbf{r}_0$ and $\mathbf{v}_0$ are the initial values of the position and velocity vectors. Note also that the velocity vector is given by

$$
\mathbf{v} = \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0
\tag{9}
$$

Moreover, it is remarkable that Eqs. (8, 9) satisfies the first equation in Eq. (4) with the following initial conditions [14, 27]

$$
q(t_0) = \begin{cases} 1 & q = f \\ 0 & q = g \end{cases}
$$

$$
\frac{dq(t_0)}{dt} = \begin{cases} 0 & q = f \\ 1 & q = g \end{cases}
$$

As in the above, using the power series method to get the relation recursively. The power series expansion of these functions take the form [27]

$$
\begin{aligned}
f &= \sum_{n=0}^{\infty} f_n(t - t_0)^n \\
g &= \sum_{n=0}^{\infty} g_n(t - t_0)^n
\end{aligned}
$$

together with the starting values [14, 27]

$$
\begin{aligned}
f_0 &= 1 & f_1 &= 0 \\
g_0 &= 0 & g_1 &= 1
\end{aligned}
$$

Also in the next section, analytical computations determine for the values of f_n and g_n for $n = 10$ using a designed algorithm called FGPS, see Appendix B for the algorithm.

4 Results and Discussion

In this chapter, computational values determine for the values q_{n+2} for n=9, f_n and g_n for $n = 10$ using Mathematica software with designed computational algorithms QPS and FGPS are given in Appendix A and Appendix B, respectively.

Furthermore, a mathematical modules establish to find the scalar values of the position and velocity vectors of the two-body motion via Mathematica software with the designed computational algorithm RVPS is given in Appendix C then we apply this module for different kinds of orbits (elliptic, parabolic and hyperbolic).

4.1 Values of q_n

With the application of QPS algorithm for $n = 9$, the following symbolic expression of the q's coefficients are

$$q_0 = q_0$$
$$q_1 = q_1$$
$$q_2 = \frac{1}{2}(-q_0)\varepsilon_0$$
$$q_3 = \frac{1}{6}(-\varepsilon_0)(q_1 - (3q_0)\lambda_0)$$

$$q_4 = \frac{1}{24}\varepsilon_0\left((6q_1)\lambda_0 + q_0\left(3\left(\psi_0 - 5\lambda_0^2\right) - 2\varepsilon_0\right)\right)$$

$$q_5 = \frac{1}{120}\varepsilon_0\left(((15q_0)\lambda_0)\left(7\lambda_0^2 + 2\varepsilon_0 - 3\psi_0\right) + q_1\left(9\left(\psi_0 - 5\lambda_0^2\right) - 8\varepsilon_0\right)\right)$$

$$q_6 = \frac{1}{720}\varepsilon_0\left(((30q_1)\lambda_0)\left(14\lambda_0^2 + 5\varepsilon_0 - 6\psi_0\right) - q_0\left(22\varepsilon_0^2 + (6\varepsilon_0)\left(70\lambda_0^2 - 11\psi_0\right)\right.\right.$$
$$\left.\left. + 45\left(21\lambda_0^4 + \psi_0^2 - \left(14\lambda_0^2\right)\psi_0\right)\right)\right)$$

$$q_7 = \frac{1}{5040}\left(\varepsilon_0\left(((63q_0)\lambda_0)\left(165\lambda_0^4 + (100\varepsilon_0)\lambda_0^2 + 12\varepsilon_0^2 + 25\psi_0^2 - \left(6\left(25\lambda_0^2\right.\right.\right.\right.\right.$$
$$\left.\left.\left. + 6\varepsilon_0\right)\right)\psi_0\right) - q_1\left(172\varepsilon_0^2 + (36\varepsilon_0)\left(70\lambda_0^2 - 11\psi_0\right) + 225\left(21\lambda_0^4 + \psi_0^2\right.\right.$$
$$\left.\left.\left.\left. - \left(14\lambda_0^2\right)\psi_0\right)\right)\right)\right)$$

$$q_8 = \frac{1}{40320}\left(\varepsilon_0\left(((126q_1)\lambda_0)\left(495\lambda_0^4 + (350\varepsilon_0)\lambda_0^2 + 52\varepsilon_0^2 + 75\psi_0^2 - \left(18\left(25\lambda_0^2\right.\right.\right.\right.\right.$$
$$\left.\left.\left. + 7\varepsilon_0\right)\right)\psi_0\right) + q_0\left(-584\varepsilon_0^3 + \left(36\varepsilon_0^2\right)\left(73\psi_0 - 560\lambda_0^2\right) - (54\varepsilon_0)\left(1925\lambda_0^4\right.\right.$$
$$\left.\left. + 67\psi_0^2 - \left(1120\lambda_0^2\right)\psi_0\right) - 315\left(429\lambda_0^6 - 5\psi_0^3 + \left(135\lambda_0^2\right)\psi_0^2\right.\right.$$
$$\left.\left.\left.\left. - \left(495\lambda_0^4\right)\psi_0\right)\right)\right)\right)$$

$$q_9 = \frac{1}{362880}\left(\varepsilon_0\left(((15q_0)\lambda_0)\left(2368\varepsilon_0^3 + \left(444\varepsilon_0^2\right)\left(77\lambda_0^2 - 24\psi_0\right)\right.\right.\right.$$
$$\left. + (18\varepsilon_0)\left(7007\lambda_0^4 + 827\psi_0^2 - \left(5698\lambda_0^2\right)\psi_0\right) + 189\left(715\lambda_0^6 - 35\psi_0^3\right.\right.$$
$$\left.\left. + \left(385\lambda_0^2\right)\psi_0^2 - \left(1001\lambda_0^4\right)\psi_0\right)\right) - q_1\left(7136\varepsilon_0^3 + \left(108\varepsilon_0^2\right)\left(1785\lambda_0^2\right.\right.$$
$$\left.\left. - 232\psi_0\right) + (432\varepsilon_0)\left(1925\lambda_0^4 + 67\psi_0^2 - \left(1120\lambda_0^2\right)\psi_0\right) + 2205\left(429\lambda_0^6\right.\right.$$
$$\left.\left.\left.\left. - 5\psi_0^3 + \left(135\lambda_0^2\right)\psi_0^2 - \left(495\lambda_0^4\right)\psi_0\right)\right)\right)\right)$$

4.2 Values of f_n

With the application of FGPS for $n = 10; T = 1, 2$, the following symbolic expressions of the f's are

$$f_0 = 1$$
$$f_1 = 0$$
$$f_2 = -\frac{\varepsilon_0}{2}$$
$$f_3 = \frac{\varepsilon_0 \lambda_0}{2}$$

$$f_4 = \frac{1}{24} (-\varepsilon_0) \left(15\lambda_0^2 + 2\varepsilon_0 - 3\psi_0\right)$$

$$f_5 = \frac{1}{8} (\varepsilon_0 \lambda_0) \left(7\lambda_0^2 + 2\varepsilon_0 - 3\psi_0\right)$$

$$f_6 = \frac{1}{720} (-\varepsilon_0) \left(22\varepsilon_0^2 + (6\varepsilon_0)\left(70\lambda_0^2 - 11\psi_0\right) + 45\left(21\lambda_0^4 + \psi_0^2 - \left(14\lambda_0^2\right)\psi_0\right)\right)$$

$$f_7 = \frac{1}{80} (\varepsilon_0 \lambda_0) \left(165\lambda_0^4 + (100\varepsilon_0)\lambda_0^2 + 12\varepsilon_0^2 + 25\psi_0^2 - \left(6\left(25\lambda_0^2 + 6\varepsilon_0\right)\right)\psi_0\right)$$

$$f_8 = -\frac{1}{40320} \left(\varepsilon_0\left(584\varepsilon_0^3 + \left(36\varepsilon_0^2\right)\left(560\lambda_0^2 - 73\psi_0\right) + (54\varepsilon_0)\left(1925\lambda_0^4 + 67\psi_0^2 \right.\right.\right.$$
$$\left.\left.\left. - \left(1120\lambda_0^2\right)\psi_0\right) + 315\left(429\lambda_0^6 - 5\psi_0^3 + \left(135\lambda_0^2\right)\psi_0^2 - \left(495\lambda_0^4\right)\psi_0\right)\right)\right)$$

$$f_9 = \frac{1}{24192} \left(\left(\varepsilon_0 \lambda_0\right)\left(2368\varepsilon_0^3 + \left(444\varepsilon_0^2\right)\left(77\lambda_0^2 - 24\psi_0\right) + (18\varepsilon_0)\left(7007\lambda_0^4 \right.\right.\right.$$
$$\left.\left. + 827\psi_0^2 - \left(5698\lambda_0^2\right)\psi_0\right) + 189\left(715\lambda_0^6 - 35\psi_0^3 + \left(385\lambda_0^2\right)\psi_0^2 \right.\right.$$
$$\left.\left.\left. - \left(1001\lambda_0^4\right)\psi_0\right)\right)\right)$$

$$f_{10} = -\frac{1}{3628800} \left(\varepsilon_0\left(28384\varepsilon_0^4 + \left(48\varepsilon_0^3\right)\left(31735\lambda_0^2 - 3548\psi_0\right) \right.\right.$$
$$\left. + \left(54\varepsilon_0^2\right)\left(245245\lambda_0^4 + 6559\psi_0^2 - \left(126940\lambda_0^2\right)\psi_0\right) + (90\varepsilon_0)\left(420420\lambda_0^6 \right.\right.$$
$$\left. - 3461\psi_0^3 + \left(107514\lambda_0^2\right)\psi_0^2 - \left(441441\lambda_0^4\right)\psi_0\right) + 14175\left(2431\lambda_0^8 + 7\psi_0^4 \right.$$
$$\left.\left.\left. - \left(308\lambda_0^2\right)\psi_0^3 + \left(2002\lambda_0^4\right)\psi_0^2 - \left(4004\lambda_0^6\right)\psi_0\right)\right)\right)$$

4.3 Values of g_n

Again with the application of FGPS for $n = 10; T = 1, 2$, the following symbolic expressions of g's coefficients are

$$g_0 = 0$$
$$g_1 = 1$$
$$g_2 = 0$$
$$g_3 = -\frac{\varepsilon_0}{6}$$
$$g_4 = \frac{\varepsilon_0 \lambda_0}{4}$$

$$g_5 = \frac{1}{120}\left(-\varepsilon_0\right)\left(45\lambda_0^2 + 8\varepsilon_0 - 9\psi_0\right)$$

$$g_6 = \frac{1}{24}\left(\varepsilon_0\lambda_0\right)\left(14\lambda_0^2 + 5\varepsilon_0 - 6\psi_0\right)$$

$$g_7 = -\frac{\varepsilon_0\left(172\varepsilon_0^2 + (36\varepsilon_0)\left(70\lambda_0^2 - 11\psi_0\right) + 225\left(21\lambda_0^4 + \psi_0^2 - \left(14\lambda_0^2\right)\psi_0\right)\right)}{5040}$$

$$g_8 = \frac{1}{320}\left(\varepsilon_0\lambda_0\right)\left(495\lambda_0^4 + (350\varepsilon_0)\lambda_0^2 + 52\varepsilon_0^2 + 75\psi_0^2 - \left(18\left(25\lambda_0^2 + 7\varepsilon_0\right)\right)\psi_0\right)$$

$$g_9 = -\frac{1}{362880}\left(\varepsilon_0\left(7136\varepsilon_0^3 + \left(108\varepsilon_0^2\right)\left(1785\lambda_0^2 - 232\psi_0\right) + (432\varepsilon_0)\left(1925\lambda_0^4\right.\right.\right.$$
$$+ 67\psi_0^2 - \left(1120\lambda_0^2\right)\psi_0\right) + 2205\left(429\lambda_0^6 - 5\psi_0^3 + \left(135\lambda_0^2\right)\psi_0^2\right.$$
$$\left.\left.\left. - \left(495\lambda_0^4\right)\psi_0\right)\right)\right)$$

$$g_{10} = \frac{1}{120960}\left(\left(\varepsilon_0\lambda_0\right)\left(15220\varepsilon_0^3 + \left(12\varepsilon_0^2\right)\left(14938\lambda_0^2 - 4647\psi_0\right) + (81\varepsilon_0)\left(7007\lambda_0^4\right.\right.\right.$$
$$+ 827\psi_0^2 - \left(5698\lambda_0^2\right)\psi_0\right) + 756\left(715\lambda_0^6 - 35\psi_0^3 + \left(385\lambda_0^2\right)\psi_0^2\right.$$
$$\left.\left.\left. - \left(1001\lambda_0^4\right)\psi_0\right)\right)\right)$$

4.4 Applications on orbits

With the application of RVPS algorithm on different kinds of orbits with the initial values components for the position and velocity vectors is given by then, we get the final scalar values of the position and velocity vectors

5 Conclusion

The dynamical system of the two–body problem is addressed, by applying the mathematical modules, which are established to find the scalar values of the position and velocity vectors of the two–body motion. The Application of this module for different kinds of orbits (elliptic, parabolic and hyperbolic) leads to accurate results, that prove module efficiency.

Appendix A: QPS

∗ Purpose
To generate n symbolic expressions of the q's coefficients of the time power series solution $\sum_{i=0}^{\infty} q_i(t - t_0)^i$ of the single harmonic oscillator $\ddot{q} + \varepsilon q = 0$.

∗ Input
$q_0, q_1, \varepsilon_0, \lambda_0, \psi_0, n$

∗ Output
n symbolic expressions of the q's coefficients.

Table 1 The initial values components for the position and velocity vectors

Orbit Type	$x_0(km)$	$y_0(km)$	$z_0(km)$	$\dot{x}_0(km/s)$	$\dot{y}_0(km/s)$	$\dot{z}_0(km/s)$
Elliptic	5096.530625	3997.328251	−1767.35171	4.683016085	0.602386847	4.217758697
	0.462581670	0.063366053	0.94365569	−0.386104670	0.485499770	0.538650150
	0.467585950	0.060430695	0.94778613	−0.343550500	0.464286580	0.582594050
Parabolic	0.000786356	−0.105595880	−1.121732800	0.9715977800	0.4937057600	−0.766396930
	0.201227080	−0.449587910	−0.796666320	1.1039007000	0.4558421100	−0.841963950
	−1616.940994	7756.699643	−7712.188395	−0.6730303137	8.4349309570	0.705548375
Hyperbolic	000010000	0.0000000000	0.00000000000	0.00000000000	0.000000000000	9.20000000
	−1.617401500	−1.0018533	−0.62794583	0.56325092	0.19817751	−1.375402800
	−0.668460760	−2.0580722	−1.96420100	0.78874624	0.74895884	−0.782571152

Table 2 The final scalar values of the position and velocity vectors

Orbit Type	$r(km)$	$v(km/s)$
Elliptic	1.19702000	0.64264300
	1.19989670	0.66492594
	0.42298299	1.87403310
Parabolic	2.1706300	0.95989400
	55.873377	0.18921011
	55.8131610	0.18930981
Hyperbolic	1.6526200	1.1352500
	226.74156	1.1219743
	213.82559	1.0585222

∗ Module list.

$$\text{Module}\Big[\{\}, \text{Do}\Big[\Big\{ q_{n+2} = -\frac{\sum_{i=0}^{n} \varepsilon_i q_{n-i}}{(n+1)(n+2)}, \varepsilon_{n+1} = -\frac{3\sum_{i=0}^{n} \varepsilon_i \lambda_{n-i}}{n+1},$$

$$\lambda_{n+1} = \frac{-2\sum_{i=0}^{n} \lambda_i \lambda_{n-i} + \psi_n - \varepsilon_n}{n+1},$$

$$\psi_{n+1} = -\frac{2\sum_{i=0}^{n} \lambda_i \big(\psi_{n-i} + \varepsilon_{n-i}\big)}{n+1}, \{n, 0, m\}\Big]\Big]$$

Appendix B: FGPS

∗ Purpose
To generate n symbolic expressions of the f's or g's coefficients of the time power series of the Lagrange functions f or g.

∗ Input
T:A positive integer takes the value 1 or 2, such that:

- $T = 1$ if it is required to find the f's coefficients,

- $T = 2$ if it is required to find the *g*'s coefficients,
  $\varepsilon_0, \lambda_0, \psi_0, n$

* Output
  n symbolic expressions of the *f*'s or *g*'s coefficients according to the value of *T*.

* Module list.

$$
\text{Module}\big[\{\}, \text{Which}[T == 1, \text{Goto}[1], T == 2, \text{Goto}[2]];
$$
$$
\text{Label}[1]; q_0 = 1; q_1 = 0; \text{Goto}[3];
$$
$$
\text{Label}[2]; q_0 = 0; q_1 = 1;
$$
$$
\text{Label}[3]; \text{Evaluate}\big[\text{QPS}\big(q_0, q_1, \varepsilon_0, \lambda_0, \psi_0, m\big)\big]\big]
$$

Appendix C: RVPS

* Purpose
  To generate the position and velocity values for any orbit.

* Input
  x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$, μ, *m*, *t* and t_0.

* Output
  The position and velocity values for any orbit.

* Module list.

$$
\text{Module}\Big[\{\}, r_0 = \sqrt{x^2 + y^2 + z^2}; \varepsilon_0 = \frac{\mu}{r_0^3}; \lambda_0 = \frac{x\mathrm{xd} + y\mathrm{yd} + z\mathrm{zd}}{r_0^2};
$$
$$
\psi_0 = \frac{\mathrm{xd}^2 + \mathrm{yd}^2 + \mathrm{zd}^2}{r_0^2}; \tau = t - \mathrm{t0};
$$
$$
F = \sum_{j=0}^{m} q_j \tau^j; \mathrm{Fd} = \sum_{j=0}^{m} j q_j \tau^{j-1};
$$
$$
G = \sum_{j=0}^{m} q_j \tau^j; \mathrm{Gd} = \sum_{j=0}^{m} j q_j \tau^{j-1};
$$
$$
\mathrm{xx} = Fx + G\mathrm{xd}; \mathrm{yy} = Fy + G\mathrm{yd}; \mathrm{zz} = Fz + G\mathrm{zd};
$$
$$
r = \sqrt{\mathrm{xx}^2 + \mathrm{yy}^2 + \mathrm{zz}^2};
$$
$$
\mathrm{xxd} = \mathrm{Fd}x + \mathrm{Gdxd}; \mathrm{yyd} = \mathrm{Fd}y + \mathrm{Gdyd}; \mathrm{zzd} = \mathrm{Fd}z + \mathrm{Gdzd};
$$
$$
v = \sqrt{\mathrm{xxd}^2 + \mathrm{yyd}^2 + \mathrm{zzd}^2}\Big]\Big]
$$

References

[1] Abouelmagd E I, Mortari D, Selim H H (2015). *Analytical study of periodic solutions on perturbed equatorial two-body problem. International Journal of Bifurcation and Chaos.* **25 (14)**: 1540040. https://doi.org/10.1142/S0218127415400404.
[2] Abouelmagd E I, Elshaboury S M, Selim H H (2016). *Numerical integration of a relativistic two-body problem via a multiple scales method.* Astrophys. Space Sci. **361 (1)**: 38. https://doi.org/10.1007/s10509-015-2625-8.

[3] Abouelmagd E I (2018). *Periodic Solution of Two–body problem by KB Averaging Method within Frame of the modified Newtonian Potential* The Journal of the Astronautical Sciences 65(3): 291–306. https://doi.org/10.1007/s40295-018-0128-x.

[4] Abouelmagd E I, Llibre J, Guirao J L G (2017). *Periodic Orbits of the Planar Anisotropic Kepler Problem.* International Journal of Bifurcation and Chaos **27 (3)**: 1750039. https://doi.org/10.1142/S0218127417500390.

[5] Abouelmagd E I,Alhothuali M S, Guirao J L G, Malaikah H M (2015). *On the periodic structure of planar photogravitational Hill problem.* Applied Mathematics & Information Sciences **9 (5)**: 2409–2416. http://dx.doi.org/10.12785/amis/090524.

[6] Elshaboury S M, Mostafa A (2014). *The motion of axisymmetric satellite with drag and radiation pressure.* Astrophys. Space Sci. **352**: 515–519. https://doi.org/10.1007/s10509-014-1975-y.

[7] Engels R C, Junkins J L (1981). *The gravity-perturbed Lambert problem: A KS variation of parameters approach.* Celest. Mech. Dyn. Astron. **24 (1)**: 3–21. https://doi.org/10.1007/BF01228790.

[8] Jezewski D J (1983). *A noncanonical analytic solution to the J_2 perturbed two-body problem.* Celest. Mech. Dyn. Astron. **30 (4)**: 343–361. https://doi.org/10.1007/BF01375505.

[9] Leach P G L (1987). *The first integrals and orbit equation for the Kepler problem with drag.* J. Phys. A **20 (8)**: 1997–2002. https://doi.org/10.1088/0305-4470/20/8/019.

[10] Margheri A, Ortega R, Rebelo C (2012). *Some analytical results about periodic orbits in the restricted three body problem with dissipation.* Celest. Mech. Dyn. Astron. **113**: 279–290. https://doi.org/10.1007/s10569-012-9415-1.

[11] Mavraganis A G, Michalakis D G (1994). *The two-body problem with drag and radiation pressure.* Celest. Mech. Dyn. Astron. **58 (4)**: 393–403. https://doi.org/10.1007/BF00692013.

[12] Mittleman D, Jezewski D (1982). *An analytic solution to the classical two-body problem with drag.* Celest. Mech. Dyn. Astron. **28 (4)**: 401–413. https://doi.org/10.1007/BF01372122.

[13] Russell R P(2006). *Global Search for Planar and Three-Dimensional Periodic Orbits Near Europa.* The Journal of the Astronautical Sciences **54 (2)**: 199–266. https://doi.org/10.1007/BF03256483.

[14] Battin, R.H. (1999), *An Introduction to the Mathematics and Methods of Astrodynamics.* AIAA Education Series, New York. https://doi.org/10.2514/4.861543.

[15] Dyck, J.A., (2015), *Periodic Solutions to the n-Body Problem*, A thesis submitted to the Department of Computer Science, the University of Manitoba Winnipeg, Manitoba, Canada.

[16] Broucke, R.(1971), *Solution of the N-body problem with recurrent power series*, Celest. Mech. Dyn. Astron. 4 110-115. https://doi.org/10.1007/BF01230326.

[17] Benavides J.C. (2010), *Trajectory Design Using Approximate Analytic Solutions of the N-Body Problem.*, A Dissertation in Aerospace Engineering, PSU, USA.

[18] Fejoz, J. (2014), *The N-body problem*, Universite Paris-Dauphine & bservatoire de Paris.

[19] Deprit, A. and Price, J.F.,(1965), *The computation of characteristic exponents in the planar restricted problem of three bodies.*, Astron. J. 70: 836–846. https://doi.org/10.1086/109823

[20] Black, W., (1973), *Practical considerations in the numerical solution of the N-body problem by Taylor series methods*, Celest. Mech. Dyn. Astron. 8 357-370. https://doi.org/10.1007/BF01227806

[21] Sitarski, G., (1984), *Recurrent power series integration of the equations of comet's motion including the non gravitational terms in Marsden's form.*, Acta Astron. 34: 53–63.

[22] Sitarski, G., (1994), *On the perihelion asymmetry for investigations of the nongravitational motion of comets*, Acta Astron. 44 91-98.

[23] Hadjifotinou K.G., (2000), *Numerical integration of satellite orbits around an oblate planet*, Astron. Astrophy. 354: 328–333.

[24] Saad, A.S., Banaszkiewicz, M. and Sitarski, G., (2008), *A new algorithm of the recurrent power series method for the non-gravitational motion of comets*, Applied Mathematica and computation 197: 874–879. https://doi.org/10.1016/j.amc.2007.08.020.

[25] Sharaf, M.A., Ghoneim, R. and Alshaery, A.A., (2011), *Symbolic solution of the three dimensional restricted three-body problem*, Contrib. Astron. Obs. Skalnate'e Pleso 41 1-12.

[26] Alshaery, A.A. and Ebaid, A., (2017), *Accurate analytical periodic solutions of the elliptical Kepler equation using the Adomian decomposition method*, Acta Astronautica 140 27-33. https://doi.org/10.1016/j.actaastro.2017.07.034

[27] Sharaf, M.A., Banajah, M.A. and Al-shaary, A. A., (2005), *Unified Symbolic Algorithmfor Some Expansions of the Two-Body Problem Using Lagranges Fundamental Invariants*, Science, 17(1).

[28] Sharaf, M.A., Saad, A.S. and Alshaery, A.A., (2014), *Universal Symbolic Expression for Radial Distance of Conic Motion*, Serbian Astronomical Journal, 189, 87-92.

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