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Existence of solution for Mean-field Reflected Discontinuous Backward Doubly Stochastic Differential Equation

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Abstract

In this paper we prove the existence of a solution for mean-field reflected backward doubly stochastic differential equations (MF-RBDSDEs) with one continuous barrier and discontinuous generator (left-continuous). By a comparison theorem establish here for MF-RBDSDEs, we provide a minimal or a maximal solution to MF-RBDSDEs.

Keywords: Reflected Backward doubly stochastic differential equations; Mean-field; Minimal solution; Comparison theorem; Discontinuous generator

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1 Introduction

The theory of nonlinear backward stochastic differential equations (**BSDEs** in short) have been first introduced by Pardoux and Peng [6] (1990). They proved the existence and uniqueness of the adapted processes (Y, Z) , solution of the following equation:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T,$$

where the terminal value ξ is square integrable and the coefficient f is uniformly Lipschitz in (y, z) , several authors interested in weakening this assumption; In [5] (1997), the authors prove the existence of a solution for one dimensional backward stochastic differential equations where the coefficient is continuous and it has a linear growth, they also obtain the existence of a minimal solution. In [3] (2008) the author prove the existence of the solution to BSDEs whose coefficient may be discontinuous in y and continuous in z .

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A new kind of backward stochastic differential equations was introduced by Pardoux and Peng [7] in (1994) which is a class of backward doubly stochastic differential equation (BDSDE for short) of the form:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) d\overleftarrow{B}_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T,$$

with two different directions of stochastic integrals, i.e., the equation involves both a standard (forward) stochastic integral dW_t and a backward stochastic integral $d\overleftarrow{B}_t$ and ξ is a random variable termed the terminal condition.

After the authors have proved an existence and unique solution when f and g are uniform Lipschitz, several authors interested to weakening this assumption, see [4]. In [9](2005) the authors obtained the existence of the solution of BDSDE under continuous assumption and gave the comparison theorem for one dimensional BDSDE.

On the other hand Bahlali et al [1] (2009) introduced a special class of reflected BDSDEs (RBDSDEs in short) which is a BDSDE but the solution is forced to stay above a lower barrier. In particular, a solution of such equation is a triplet of processes (Y, Z, K) satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) dB_s + \int_t^T dK_s - \int_t^T Z_s dW_s, \quad t \in [0, T], \quad (1)$$

and $Y_t \geq S_t$ a.s. for any $t \in [0, T]$. The role of the nondecreasing continuous process $(K_t)_{t \in [0, T]}$ is to push upward the process Y in order to keep it above S , it satisfies the skorohod condition

$$\int_0^T (Y_s - S_s) dK_s = 0.$$

In this paper, motivated by the above results and by the result introduced by Xu, R. (2012) [10], we establish the existence of the a minimal solution to the following reflected MF-BDSDE,

$$\begin{aligned} Y_t = & \xi + \int_t^T E' \left(f(s, \omega, \omega', Y_s, (Y_s)', Z_s, (Z_s)') \right) ds + \int_t^T dK_s \\ & + \int_t^T E' \left(g(s, \omega, \omega', Y_s, (Y_s)', Z_s, (Z_s)') \right) d\overleftarrow{B}_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \end{aligned} \quad (2)$$

whose coefficient may be discontinuous in y and continuous in z .

- **In Section 2**, we give some preliminaries about MF-BDSDE with one continuous barrier.
- **In Section 3**, under certain assumptions, we obtain the existence for a minimal solution to the Mean-field backward doubly stochastic differential equation with one continuous barrier and discontinuous generator (left-continuous).

2 Framework

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. For $T > 0$, let $\{W_t, 0 \leq t \leq T\}$ and $\{B_t, 0 \leq t \leq T\}$ be two independent standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^d$ and $\mathbb{R}$, respectively.

Let $\mathcal{F}_t^W := \sigma(W_s; 0 \leq s \leq t)$, and $\mathcal{F}_{t,T}^B := \sigma(B_s - B_t; t \leq s \leq T)$, completed with P -null sets. We put,

$$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B.$$

It should be noted that $(\mathcal{F}_t)$ is not an increasing family of sub σ -fields, and hence it is not a filtration.

Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) = (\Omega \times \Omega, \mathcal{F}_t \otimes \mathcal{F}_t, P \otimes P)$ be the (non-completed) product of $(\Omega, \mathcal{F}, P)$ with itself. We denote the filtration of this product space by $\tilde{\mathcal{F}} = \{\tilde{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{F}_t, 0 \leq t \leq T\}$.

A random variable $\xi \in L^0(\Omega, \mathcal{F}, P; \mathbb{R}^n)$ originally defined on Ω is extended canonically to $\bar{\Omega}$: $\bar{\xi}(\bar{\omega}, \omega) = \xi(\bar{\omega}), (\bar{\omega}, \omega) \in \bar{\Omega} = \Omega \times \Omega$.

For every $\theta \in L^1(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$, the variable $\theta(\cdot, \omega) : \Omega \rightarrow \mathbb{R}$ belongs to $L^1(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}), P(d\omega) - a.s.$. We denote its expectation by $\bar{E}(\theta(\cdot, \omega)) = \int_{\bar{\Omega}} \theta(\bar{\omega}, \omega) P(d\bar{\omega})$.

Notice that

$$\begin{cases} \bar{E}(\theta) = \bar{E}(\theta(\cdot, \omega)) \in L^1(\Omega, \mathcal{F}, P) \\ \text{and} \\ \bar{E}(\theta) = \int_{\bar{\Omega}} \theta d\bar{P} = \int_{\Omega} \bar{E}(\theta(\cdot, \omega)) P(d\omega) = E(\bar{E}(\theta)). \end{cases}$$

We consider the following spaces of processus:

- Let $\mathcal{M}^2(0, T, \mathbb{R}^d)$ denote the set of d -dimensional, $\mathcal{F}_t$ -progressively measurable processes $\{\varphi_t; t \in [0, T]\}$, such that $\mathbb{E} \int_0^T |\varphi_t|^2 dt < \infty$.
- We denote by $\mathcal{S}^2(0, T, \mathbb{R}^d)$, the set of $\mathcal{F}_t$ -adapted càdlàg processes $\{\varphi_t; t \in [0, T]\}$, which satisfy $\mathbb{E}(\sup_{0 \leq t \leq T} |\varphi_t|^2) < \infty$.
- $\mathcal{A}^2$ set of continuous, increasing, $\mathcal{F}_t$ -adapted process $K : [0, T] \times \Omega \rightarrow [0, +\infty)$ with $K_0 = 0$ and $\mathbb{E}(K_T)^2 < +\infty$.
- $\mathbb{L}^2$ set of $\mathcal{F}_T$ -measurable random variables $\xi : \Omega \rightarrow \mathbb{R}$ with $\mathbb{E}|\xi|^2 < +\infty$.

Definition 1. A solution of equation (2) is a triple (Y, Z, K) which belongs to the space $\mathcal{S}^2(0, T, \mathbb{R}^d) \times \mathcal{M}^2(0, T, \mathbb{R}^d) \times \mathcal{A}^2$ and satisfies (2) such that:

$$\begin{cases} S_t \leq Y_t, 0 \leq t \leq T, \\ \int_0^T (Y_s - L_s) dK_s = 0. \end{cases}$$

Remark 1. In the case where $S = -\infty$ (i.e., MF-BDSDEs without lower barrier), the process K has no effect i.e., $K \equiv 0$.

Remark 2. In the setup of system (2) the process $S(\cdot)$ play the role of reflecting barrier.

Remark 3. The state process $Y(\cdot)$ is forced to stay above the lower barrier $S(\cdot)$, thanks to the action of the increasing reflection process $K(\cdot)$.

The coefficient of mean-field Reflected BDSDE is a function. We assume that f and g satisfy the following assumptions on the data (ξ, f, g, S) :

- **(H.1)** The terminal value ξ be a given random variable in $\mathbb{L}^2$.
- **(H.2)** $(S_t)_{t \geq 0}$, is a continuous progressively measurable real valued process satisfying

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} (S_t^+)^2 \right) < +\infty, \quad \text{where} \quad S_t^+ := \max(S_t, 0).$$

- **(H.3)** For $t \in [0, T]$, $S_T \leq \xi$, $\mathbb{P}$ -almost surely.
- **(H.4)** $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$; $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^k$ be jointly measurable such that for any $(y, y', z, z') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{cases} f(\cdot, \omega, y, y', z, z') \in \mathcal{M}^2(0, T, \mathbb{R}^d), \\ \text{and} \\ g(\cdot, \omega, y, y', z, z') \in \mathcal{M}^2(0, T, \mathbb{R}^d). \end{cases}$$

- **(H.5)** There exist constant $C \geq 0$ and a constant $0 \leq \alpha \leq \frac{1}{2}$ such that for every $(\omega, t) \in \Omega \times [0, T]$ and $(y, y') \in \mathbb{R}^2, (z, z') \in \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{cases} (i) & \left| f(t, y_1, y'_1, z_1, z'_1) - f(t, y_2, y'_2, z_2, z'_2) \right|^2 \leq C \left\{ |y_1 - y_2|^2 + |y'_1 - y'_2|^2 + |z_1 - z_2|^2 + |z'_1 - z'_2|^2 \right\}, \\ (ii) & \left| g(t, y_1, y'_1, z_1, z'_1) - g(t, y_2, y'_2, z_2, z'_2) \right|^2 \leq C \left\{ |y_1 - y_2|^2 + |y'_1 - y'_2|^2 \right\} + \alpha \left\{ |z_1 - z_2|^2 + |z'_1 - z'_2|^2 \right\}. \end{cases}$$

- **(H.6)** (i) For a.e. (t, ω) the mapping $(y, y', z, z') \rightarrow f(t, y, y', z, z')$ is a continuous.
- (ii) There exist constant $C \geq 0$ and a constant $0 \leq \alpha \leq \frac{1}{2}$ such that for every $(\omega, t) \in \Omega \times [0, T]$ and $(y, y') \in \mathbb{R}^2, (z, z') \in \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{cases} \left| f(t, y, y', z, z') \right| \leq C (1 + |y| + |y'| + |z| + |z'|), \\ g \text{ satisfies (H.2) (ii)}. \end{cases}$$

We recall the following existence results.

Proposition 1. [2] (2014). Under the assumptions **(H.1)-(H.5)** the reflected BDSDE (2) has a unique solution $(Y, Z, K) \in \mathcal{S}^2(0, T, \mathbb{R}^d) \times \mathcal{M}^2(0, T, \mathbb{R}^d) \times \mathcal{A}^2$.

3 Existence result

In this section we are interested in weakening the conditions on f . We assume that f and g satisfy the following assumptions:

- **(H.7)** Linear growth: There exists a nonnegative process $f_t \in \mathbb{M}^2(0, T, \mathbb{R}^d)$ such that

$$\forall (t, y, y', z) \in [0, T] \times \mathbb{R}^2 \times \mathbb{R}^d, \quad \left| f(t, y, y', z) \right| \leq f_t(\omega) + C (|y| + |y'| + |z|).$$

- **(H.8)** $f(t, \cdot, y', z) : \mathbb{R} \rightarrow \mathbb{R}$ is a left continuous and $f(t, y, \cdot, \cdot)$ is a continuous.
- **(H.9)** There exists a continuous function $\pi : [0, T] \times (\mathbb{R})^2 \times \mathbb{R}^d$ satisfying for $y_1 \geq y_2, (y'_1, y'_2) \in (\mathbb{R})^2, (z_1, z_2) \in (\mathbb{R}^d)^2$

$$\begin{cases} \left| \pi(t, y, y', z) \right| \leq C (|y| + |y'| + |z|), \\ f(t, \omega, y_1, y'_1, z_1) - f(t, \omega, y_2, y'_2, z_2) \geq \pi(t, y_1 - y_2, y'_1 - y'_2, z_1 - z_2). \end{cases}$$

- **(H.10)** Monotonicity in y' : $\forall (y, y', z), f(t, y, y', z)$ is increasing in y' .
- **(H.11)** g satisfies (H.5) (ii) and $g(t, 0, 0, 0) \equiv 0$.

Hence, we only consider the following type of Mean-field reflected BDSDE:

$$\begin{aligned} Y_t = & \xi + \int_t^T E' \left(f(s, \omega, \omega', Y_s, (\tilde{Y}_s)', Z_s) \right) ds + \int_t^T dK_s \\ & + \int_t^T E' \left(g(s, \omega, \omega', Y_s, (\tilde{Y}_s)', Z_s) \right) d\overleftarrow{B}_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \end{aligned} \quad (3)$$

Proposition 2. [2] (2014). Under the assumption (H.1)-(H.4) and (H.6), and for any random variable $\xi \in \mathbb{L}^2$ the mean-field RBDSDE (3) has an adapted solution $(Y, Z, K) \in \mathcal{S}^2(0, T, \mathbb{R}^d) \times \mathcal{M}^2(0, T, \mathbb{R}^d) \times \mathcal{A}^2$, which is a minimal one, in the sense that, if (Y^*, Z^*, K^*) is any other solution we $Y \leq Y^*$, $\mathbb{P} - a.s.$

Now we prove a technical Lemma before we introduce the main theorem.

Lemma 3. Let $\pi(t, y, y', z)$ satisfies (H.9), g satisfies (H.11) and h belongs in $\mathcal{M}^2(0, T, \mathbb{R}^d)$. For a continuous function of finite variation $\tilde{K}$ belong in $\mathcal{A}^2$ we consider the processes $(\tilde{Y}, \tilde{Z}) \in \mathcal{S}^2(0, T, \mathbb{R}) \times \mathcal{M}^2(0, T, \mathbb{R}^d)$ such that:

$$\begin{cases} (i) \quad \tilde{Y}_t = \xi + \int_t^T E' \left(\pi(s, \omega, \omega', \tilde{Y}_s, (\tilde{Y}_s)', \tilde{Z}_s) + h(s) \right) ds + \int_t^T d\tilde{K}_s \\ \quad + \int_t^T E' \left(g(s, \omega, \omega', \tilde{Y}_s, (\tilde{Y}_s)', \tilde{Z}_s) \right) d\overleftarrow{B}_s - \int_t^T \tilde{Z}_s dW_s, \quad 0 \leq t \leq T, \\ (ii) \quad \int_0^T \tilde{Y}_s^- d\tilde{K}_s \geq 0. \end{cases} \quad (4)$$

Then we have

- (i) The MF-RBDSDE (4) has a least one solution $(\tilde{Y}, \tilde{Z}, \tilde{K}) \in \mathcal{S}^2(0, T, \mathbb{R}^d) \times \mathcal{M}^2(0, T, \mathbb{R}^d) \times \mathcal{A}^2$
- (ii) if $h(t) \geq 0$ and $\xi \geq 0$, we have $\tilde{Y}_t \geq 0$, $d\mathbb{P} \times dt - a.s.$

Proof. (i) See [2], (2014). (ii) Applying Tanaka's formula to $|\tilde{Y}_t^-|^2$, we have

$$\begin{aligned} \mathbb{E} |\tilde{Y}_t^-|^2 + \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s < 0\}} |\tilde{Z}_s|^2 ds = & \mathbb{E} |\xi^-|^2 - 2\mathbb{E} \int_t^T \tilde{Y}_s^- E' \left(\pi(s, \tilde{Y}_s, (\tilde{Y}_s)', \tilde{Z}_s) + h(s) \right) ds \\ & - 2\mathbb{E} \int_t^T \tilde{Y}_s^- d\tilde{K}_s + \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s < 0\}} \left\| E' \left(g(s, \tilde{Y}_s, (\tilde{Y}_s)', \tilde{Z}_s) \right) \right\|^2 ds. \end{aligned}$$

Since $-2\mathbb{E} \int_t^T \tilde{Y}_s^- d\tilde{K}_s \leq 0$, $h(s) \geq 0$ and $\xi \geq 0$, we get

$$\begin{aligned} \mathbb{E} |\tilde{Y}_t^-|^2 + \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s < 0\}} |\tilde{Z}_s|^2 ds \leq & -2\mathbb{E} \int_t^T \tilde{Y}_s^- E' \left(\pi(s, \tilde{Y}_s, (\tilde{Y}_s)', \tilde{Z}_s) \right) ds \\ & + \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s < 0\}} \left\| E' \left(g(s, \tilde{Y}_s, (\tilde{Y}_s)', \tilde{Z}_s) \right) \right\|^2 ds \end{aligned}$$

By (H.9), we get $\left| \pi(s, \tilde{Y}_s, (\tilde{Y}_s)', \tilde{Z}_s) \right| \leq C \left(|\tilde{Y}_s| + |(\tilde{Y}_s)'| + |\tilde{Z}_s| \right)$ and by assumption (H.11) for g , we have

$$\begin{aligned} & \mathbb{E} |\tilde{Y}_t^-|^2 + \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s < 0\}} |\tilde{Z}_s|^2 ds \\ & \leq \left(4C^2 + \frac{C^2}{\beta} + 2C \right) \mathbb{E} \int_t^T |\tilde{Y}_s^-|^2 ds + (\alpha + \beta) \mathbb{E} \int_t^T 1_{\{\tilde{Y}_s < 0\}} |\tilde{Z}_s|^2 ds. \end{aligned}$$

Therefore, choosing $0 \leq \beta \leq 1 - \alpha$ and using Gronwall inequality, we have $\tilde{Y}_t^- = 0$, $\mathbb{P} - a.s.$, $\forall t \in [0, T]$, which implies that $\tilde{Y}_t \geq 0$ $\mathbb{P} - a.s.$, $\forall t \in [0, T]$. $\square$

Before we prove the main result, we construct a sequence of MF-RBDSDEs as follows:

$$\left\{ \begin{array}{l} \bar{Y}_t^0 = \xi + \int_t^T E' \left(-C \left(|\bar{Y}_s^0| + (\bar{Y}_s^0)' + |\bar{Z}_s^0| \right) - f_s \right) ds \\ \quad + \int_t^T E' \left(g \left(s, \bar{Y}_s^0 + (\bar{Y}_s^0)' + \bar{Z}_s^0 \right) \right) d\bar{B}_s + \int_t^T d\bar{K}_s^0 - \int_t^T \bar{Z}_s^0 dW_s, \quad 0 \leq t \leq T, \\ (ii) \quad \bar{Y}_t^0 \geq S_t, \\ (iii) \quad \int_0^T (\bar{Y}_s^0 - S_s) d\bar{K}_s^0 = 0. \end{array} \right. \quad (5)$$

$$\left\{ \begin{array}{l} (i) \quad \bar{Y}_t^n = \xi + \int_t^T E' \left(f \left(s, \bar{Y}_s^{n-1}, (\bar{Y}_s^{n-1})', \bar{Z}_s^{n-1} \right) + \pi \left(s, \delta \bar{Y}_s^n, \delta (\bar{Y}_s^n)', \delta \bar{Z}_t^n \right) \right) ds \\ \quad + \int_t^T E' \left(g \left(s, \bar{Y}_s^n, (\bar{Y}_s^n)', \bar{Z}_s^n \right) \right) d\bar{B}_s + \int_t^T d\bar{K}_s^n - \int_t^T \bar{Z}_s^n dW_s, \quad 0 \leq t \leq T, \\ (ii) \quad \bar{Y}_t^n \geq S_t, \\ (iii) \quad \int_0^T (\bar{Y}_s^n - S_s) d\bar{K}_s^n = 0. \end{array} \right. \quad (6)$$

$$\left\{ \begin{array}{l} (i) \quad Y_t^0 = \xi + \int_t^T E' \left(C \left(|Y_s^0| + (Y_s^0)' + |Z_s^0| \right) + f_s \right) ds + \int_t^T dK_s^0 \\ \quad + \int_t^T E' \left(g \left(s, Y_s^0, (Y_s^0)', Z_s^0 \right) \right) d\bar{B}_s - \int_t^T Z_s^0 dW_s, \quad 0 \leq t \leq T, \\ (ii) \quad Y_t^0 \geq S_t, \\ (iii) \quad \int_0^T (Y_s^0 - S_s) dK_s^0 = 0. \end{array} \right. \quad (7)$$

For these solutions above, we get some properties as follows:

Lemma 4. *Under the assumptions (H.1) – (H.4) and (H.7) – (H.11), we have for any $n \geq 1$ and $t \in [0, T]$*

$$\bar{Y}_t^0 \leq \bar{Y}_t^n \leq \bar{Y}_t^{n+1} \leq Y_t^0.$$

Proof. We will prove $\bar{Y}_t^0 \leq \bar{Y}_t^n$ at first. By Eqs. (5), and (6), we have

$$\begin{aligned} \bar{Y}_t^1 - \bar{Y}_t^0 &= \int_t^T E' \left(\pi \left(s, \delta \bar{Y}_s^1, \delta (\bar{Y}_s^1)', \delta \bar{Z}_s^1 \right) + \Lambda_s^1 \right) ds \\ &\quad + \int_t^T E' \left(g \left(s, \bar{Y}_s^1, (\bar{Y}_s^1)', \bar{Z}_s^1 \right) - g \left(s, \bar{Y}_s^0 + (\bar{Y}_s^0)' + \bar{Z}_s^0 \right) \right) d\bar{B}_s \\ &\quad + \int_t^T (d\bar{K}_s^1 - d\bar{K}_s^0) - \int_t^T \delta \bar{Z}_s^1 dW_s, \end{aligned}$$

where $\Lambda_s^1 = f \left(s, \bar{Y}_s^0, (\bar{Y}_s^0)', \bar{Z}_s^0 \right) + C \left(|\bar{Y}_s^0| + (\bar{Y}_s^0)' + |\bar{Z}_s^0| \right) + f_s$. By hypothesis (H.7) we have $\Lambda_s^1 \geq 0$, because $(\bar{Y}_t^0, \bar{Z}_t^0)$ is the solution of Eq. (5), we get $\Lambda_s^1 \in \mathcal{M}^2(0, T, \mathbb{R}^d)$. Therefore, from Lemma 3 we get $\bar{Y}_t^1 \geq \bar{Y}_t^0$. Now we want to prove $\bar{Y}_t^n \leq \bar{Y}_t^{n+1}$, for any $n \geq 0$. We set

$$\left\{ \begin{array}{l} \delta \rho_s^{n+1} = \rho_s^{n+1} - \rho_s^n, \\ \Delta \psi^{n+1} \left(s, \delta \bar{Y}_s^{n+1}, \delta (\bar{Y}_s^{n+1})', \delta \bar{Z}_s^{n+1} \right) \\ = \psi \left(s, \delta \bar{Y}_s^{n+1} + \bar{Y}_s^n, \delta (\bar{Y}_s^{n+1})' + (\bar{Y}_s^n)', \delta \bar{Z}_s^{n+1} + \bar{Z}_s^n \right) - \psi \left(s, \bar{Y}_s^n, (\bar{Y}_s^n)', \bar{Z}_s^n \right). \end{array} \right.$$

Using Eq. (6), we have

$$\begin{aligned}\delta\bar{Y}_t^{n+1} &= \int_t^T E' \left(\pi \left(s, \delta\bar{Y}_s^{n+1}, \delta \left(\bar{Y}_s^{n+1} \right)', \delta\bar{Z}_s^{n+1} \right) + \theta_s^{n+1} \right) ds - \int_t^T \delta\bar{Z}_s^{n+1} dW_s \\ &\quad + \int_t^T E' \left(\Delta g^{n+1}(s, \delta\bar{Y}_s^{n+1}, \delta \left(\bar{Y}_s^{n+1} \right)', \delta\bar{Z}_s^{n+1}) \right) d\bar{B}_s + \int_t^T d(\delta\bar{K}_s^{n+1}),\end{aligned}$$

where $\theta_s^{n+1} = \Delta f^n \left(s, \delta\bar{Y}_s^n, \delta \left(\bar{Y}_s^n \right)', \delta\bar{Z}_s^n \right) - \pi \left(s, \delta\bar{Y}_s^n, \delta \left(\bar{Y}_s^n \right)', \delta\bar{Z}_s^n \right)$ and $\theta_s^0 = \Lambda_s^1, \forall n \geq 0$. According to its definition, one can show that θ_s^0 and $\Delta g^{n+1}, \forall n \geq 0$ satisfy all assumptions of Lemma 3. Moreover, since $\bar{K}_t^n$ is a continuous and increasing process, for all $n \geq 0$, $\delta\bar{K}_s^{n+1}$ is a continuous process of finite variation and, using the same argument as one appears in [2], one can show that

$$\begin{aligned}\int_0^T (\bar{Y}_s^{n+1} - \bar{Y}_s^n)^- d(\delta\bar{K}_s^{n+1}) &= \int_0^T (\bar{Y}_s^{n+1} - \bar{Y}_s^n)^- d\bar{K}_s^{n+1} - \int_0^T (\bar{Y}_s^{n+1} - \bar{Y}_s^n)^- d\bar{K}_s^n \\ &= \int_0^T (\bar{Y}_s^{n+1} - \bar{Y}_s^n)^- d\bar{K}_s^{n+1} \geq 0,\end{aligned}$$

by Lemma 3, we deduce that $\delta\bar{Y}_t^{n+1} \geq 0$, i.e. $\bar{Y}_t^{n+1} \geq \bar{Y}_t^n \forall t \in [0, T]$, we have

$$\bar{Y}_t^{n+1} \geq \bar{Y}_t^n \geq \bar{Y}_t^0.$$

Now we shall prove that $\bar{Y}_t^{n+1} \leq Y_t^0 \forall n \geq 0$, by Eqs.(3) and (7)

$$\begin{aligned}Y_t^0 - \bar{Y}_t^{n+1} &= \int_t^T E' \left(-C \left(|Y_s^0 - \bar{Y}_s^{n+1}| + \left| (Y_s^0)' - (\bar{Y}_s^{n+1})' \right| + |Z_s^0 - \bar{Z}_s^{n+1}| \right) + \Lambda_s^{n+1} \right) ds \\ &\quad + \int_t^T E' \left(g(s, Y_s^0, (Y_s^0)', Z_s^0) - g(s, \bar{Y}_s^n, (\bar{Y}_s^n)', \bar{Z}_s^n) \right) d\bar{B}_s \\ &\quad + \int_t^T (dK_s^0 - d\bar{K}_s^{n+1}) + \int_t^T (Z_s^0 - \bar{Z}_s^{n+1}) dW_s,\end{aligned}$$

where

$$\begin{aligned}\Lambda_s^{n+1} &= C \left(|Y_s^0 - \bar{Y}_s^{n+1}| + \left| (Y_s^0)' - (\bar{Y}_s^{n+1})' \right| + |Z_s^0 - \bar{Z}_s^{n+1}| + |Y_s^0| + (Y_s^0)' + |Z_s^0| \right) \\ &\quad + f_s - f(s, \bar{Y}_s^n, (\bar{Y}_s^n)', \bar{Z}_s^n) + \pi \left(s, \delta\bar{Y}_s^{n+1}, \delta \left(\bar{Y}_s^{n+1} \right)', \delta\bar{Z}_s^{n+1} \right).\end{aligned}$$

By Lemma 3, we deduce that $Y_t^0 - \bar{Y}_t^{n+1} \geq 0$, i.e. $Y_t^0 \geq \bar{Y}_t^{n+1}$, for all $t \in [0, T]$. Thus we have for all $n \geq 0$

$$Y_t^0 \geq \bar{Y}_t^{n+1} \geq \bar{Y}_t^n \geq \bar{Y}_t^0, \quad d\bar{\mathbb{P}} \times dt - a.s. \quad \forall t \in [0, T].$$

The proof of Lemma 4 is complete. □

Theorem 5. Let $\xi \in \mathbb{L}^2(\mathcal{F}_T, \mathbb{R})$ and $t \in [0, T]$. Under assumption (H.1) – (H.4) and (H.7) – (H.11), the reflected MF-BDSDEs (2) has a minimal solution

$$(Y_t, Z_t, K_t)_{0 \leq t \leq T} \in \mathcal{S}^2(0, T, \mathbb{R}) \times \mathcal{M}^2(0, T, \mathbb{R}^d) \times \mathcal{A}^2.$$

Proof. From Lemma 4, we know $(\bar{Y}_t^n)_{n \geq 0}$ is increasing and bounded in $\mathcal{M}^2(0, T, \mathbb{R}^d)$. Since $|\bar{Y}_t^n| \leq \max(\bar{Y}_t^0, Y_t^0) \leq |\bar{Y}_t^0| + |Y_t^0|$ for all $t \in [0, T]$, we have

$$\sup_n \mathbb{E} \left(\sup_{0 \leq t \leq T} |\bar{Y}_t^n|^2 \right) \leq \mathbb{E} \left(\sup_{0 \leq t \leq T} |\bar{Y}_t^0|^2 \right) + \mathbb{E} \left(\sup_{0 \leq t \leq T} |Y_t^0|^2 \right) < \infty,$$

then according to the Lebesgue's dominated convergence theorem, we deduce that $(\bar{Y}_t^n)_{n \geq 0}$ converges in $\mathcal{S}^2(0, T, \mathbb{R})$. We denote by $\bar{Y}$ the limit of $(\bar{Y}_t^n)_{n \geq 0}$.

On the other hand from Eq. (6), we deduce that

$$\begin{aligned} \bar{Y}_0^{n+1} &= \bar{Y}_T^{n+1} + \int_0^T E' \left(f \left(s, \bar{Y}_s^n, (\bar{Y}_s^n)' , \bar{Z}_s^n \right) + \pi \left(s, \delta \bar{Y}_s^{n+1}, \delta (\bar{Y}_s^{n+1})' , \delta \bar{Z}_s^{n+1} \right) \right) ds \\ &\quad + \int_t^T E' \left(g \left(s, \bar{Y}_s^{n+1}, (\bar{Y}_s^{n+1})' , \bar{Z}_s^{n+1} \right) \right) d\bar{B}_s + \int_t^T d\bar{K}_s^{n+1} - \int_t^T \bar{Z}_s^{n+1} dW_s. \end{aligned}$$

Applying Itô's formula, we obtain

$$\begin{aligned} \mathbb{E} |\bar{Y}_0^{n+1}|^2 + \mathbb{E} \int_0^T |\bar{Z}_s^{n+1}|^2 ds &\leq \mathbb{E} |\bar{Y}_T^{n+1}|^2 + 2\mathbb{E} \int_0^T \bar{Y}_s^{n+1} d\bar{K}_s^{n+1} + \mathbb{E} \int_0^T \left\| E' \left(g \left(s, \bar{Y}_s^{n+1}, (\bar{Y}_s^{n+1})' , \bar{Z}_s^{n+1} \right) \right) \right\|^2 ds \\ &\quad + 2\mathbb{E} \int_0^T \bar{Y}_s^{n+1} E' \left(f \left(s, \bar{Y}_s^n, (\bar{Y}_s^n)' , \bar{Z}_s^n \right) + \pi \left(s, \delta \bar{Y}_s^{n+1}, \delta (\bar{Y}_s^{n+1})' , \delta \bar{Z}_s^{n+1} \right) \right) ds. \end{aligned}$$

From assumption (H.7) and Yöung's inequality, we get

$$\begin{aligned} &2\mathbb{E} \int_0^T \bar{Y}_t^{n+1} E' \left[f \left(s, \bar{Y}_s^n, (\bar{Y}_s^n)' , \bar{Z}_s^n \right) \right] ds \\ &\leq 2\mathbb{E} \int_0^T \bar{Y}_s^{n+1} E' \left[f_s(\omega) + C \left(1 + |\bar{Y}_s^n| + |(\bar{Y}_s^n)'| + |\bar{Z}_s^n| \right) \right] ds, \\ &\leq 4C^2 \mathbb{E} \int_0^T |\bar{Y}_s^{n+1}|^2 ds + (4C^2 + 1) \mathbb{E} \int_0^T |\bar{Y}_s^{n+1}|^2 ds + \mathbb{E} \int_0^T |\bar{Y}_s^n|^2 ds \\ &\quad + 16C^2 \mathbb{E} \int_0^T |\bar{Y}_s^{n+1}|^2 ds + \frac{1}{16} \mathbb{E} \int_0^T |\bar{Z}_s^n|^2 ds + \mathbb{E} \int_0^T |f_s(\omega)|^2 ds, \\ &= (24C^2 + 1) \mathbb{E} \int_0^T |\bar{Y}_s^{n+1}|^2 ds + \mathbb{E} \int_0^T |\bar{Y}_s^n|^2 ds + \frac{1}{16} \mathbb{E} \int_0^T |\bar{Z}_s^n|^2 ds + \mathbb{E} \int_0^T |f_s(\omega)|^2 ds, \end{aligned} \tag{8}$$

and from hypothesis (H.9) we get

$$\begin{aligned} &2\mathbb{E} \int_0^T \bar{Y}_s^{n+1} E' \left(\pi \left(s, \delta \bar{Y}_s^{n+1}, \delta (\bar{Y}_s^{n+1})' , \delta \bar{Z}_s^{n+1} \right) \right) ds \\ &\leq 2\mathbb{E} \int_0^T \bar{Y}_s^{n+1} E' \left(C \left(|\delta \bar{Y}_s^{n+1}| + |(\delta \bar{Y}_s^{n+1})'| + |\delta \bar{Z}_s^{n+1}| \right) \right) ds, \\ &\leq 4C \mathbb{E} \int_0^T |\bar{Y}_s^{n+1}|^2 ds + 4C^2 \mathbb{E} \int_0^T |\bar{Y}_s^{n+1}|^2 ds + \mathbb{E} \int_0^T |\bar{Y}_s^n|^2 ds + 8C^2 \mathbb{E} \int_0^T |\bar{Y}_s^{n+1}|^2 ds \\ &\quad + \frac{1}{8} \mathbb{E} \int_0^T |\bar{Z}_s^{n+1}|^2 ds + 16C^2 \mathbb{E} \int_0^T |\bar{Y}_s^{n+1}|^2 ds + \frac{1}{16} \mathbb{E} \int_0^T |\bar{Z}_s^n|^2 ds, \\ &= (4C + 28C^2) \mathbb{E} \int_0^T |\bar{Y}_s^{n+1}|^2 ds + \mathbb{E} \int_0^T |\bar{Y}_s^n|^2 ds + \frac{1}{8} \mathbb{E} \int_0^T |\bar{Z}_s^{n+1}|^2 ds + \frac{1}{16} \mathbb{E} \int_0^T |\bar{Z}_s^n|^2 ds. \end{aligned} \tag{9}$$

Using the two inequalities (8) and (9), we obtain

$$\begin{aligned} &2\mathbb{E} \int_0^T \bar{Y}_s^{n+1} E' \left(f \left(s, \bar{Y}_s^n, (\bar{Y}_s^n)' , \bar{Z}_s^n \right) + \pi \left(s, \delta \bar{Y}_s^{n+1}, \delta (\bar{Y}_s^{n+1})' , \delta \bar{Z}_s^{n+1} \right) \right) ds \\ &\leq (52C^2 + 4C + 1) \mathbb{E} \int_0^T |\bar{Y}_s^{n+1}|^2 ds + 2\mathbb{E} \int_0^T |\bar{Y}_s^n|^2 ds \\ &\quad + \frac{1}{8} \mathbb{E} \int_0^T \left(|\bar{Z}_s^{n+1}|^2 + |\bar{Z}_s^n|^2 \right) ds + \mathbb{E} \int_0^T |f_s(\omega)|^2 ds. \end{aligned}$$

Then, we get

$$\begin{aligned} \mathbb{E} \int_0^T |\bar{Z}_s^{n+1}|^2 ds &\leq \mathbb{E} |\xi|^2 + \mathbb{E} \int_0^T \left\| E' \left(g(s, \bar{Y}_s^{n+1}, (\bar{Y}_s^{n+1})', \bar{Z}_s^{n+1}) \right) \right\|^2 ds \\ &\quad + C + 2\mathbb{E} \int_0^T \langle \bar{Y}_s^{n+1}, d\bar{K}_s^{n+1} \rangle + \frac{1}{8} \mathbb{E} \int_0^T \left(|\bar{Z}_s^{n+1}|^2 + |\bar{Z}_s^n|^2 \right) ds, \end{aligned}$$

where $C = 2\mathbb{E} \int_0^T |\bar{Y}_s^n|^2 ds + (52C + 4C + 1) \int_0^T |\bar{Y}_s^{n+1}|^2 ds + \mathbb{E} \int_0^T |f_s(\omega)|^2 ds$.

Applying hypothesis **(H.11)**, we have

$$\begin{aligned} &\mathbb{E} \int_0^T \left\| E' \left(g(s, \bar{Y}_s^{n+1}, (\bar{Y}_s^{n+1})', \bar{Z}_s^{n+1}) \right) \right\|^2 ds \\ &\leq 4C\mathbb{E} \int_0^T |\bar{Y}_s^{n+1}|^2 ds + 2\alpha\mathbb{E} \int_0^T |\bar{Z}_s^{n+1}|^2 ds + 2\mathbb{E} \int_0^T \|g(s, 0, 0, 0)\|^2 ds. \end{aligned}$$

Using Yöung's inequality, we obtain

$$2\mathbb{E} \int_0^T \bar{Y}_s^{n+1} d\bar{K}_s^{n+1} \leq 2\mathbb{E} \int_0^T S_s d\bar{K}_s^{n+1} \leq \frac{1}{\theta} \mathbb{E} \left(\sup_{0 \leq t \leq T} |S_t|^2 \right) + \theta \mathbb{E} |\bar{K}_T^{n+1}|^2.$$

Therefore, there exists a constant C^θ depending on α , ξ , C and θ , we derive

$$\mathbb{E} \int_0^T |\bar{Z}_s^{n+1}|^2 ds \leq C^\theta + \left(\frac{1}{8} + 2\alpha \right) \mathbb{E} \int_0^T |\bar{Z}_s^{n+1}|^2 ds + \frac{1}{8} \mathbb{E} \int_0^T |\bar{Z}_s^n|^2 ds + \theta \mathbb{E} |\bar{K}_T^{n+1}|^2, \quad (10)$$

where $C^\theta = C + \mathbb{E} |\xi|^2 + 4C \int_0^T |\bar{Y}_s^{n+1}|^2 ds + \frac{1}{\theta} \mathbb{E} \left(\sup_{0 \leq t \leq T} |S_t|^2 \right) + 2\mathbb{E} \int_0^T \|g(s, 0, 0, 0)\|^2 ds$.

Chossing α such that $0 < \frac{1}{8} + 2\alpha < 1$, we obtain

$$\mathbb{E} \int_0^T |\bar{Z}_s^{n+1}|^2 ds \leq C^\theta + \frac{1}{8} \mathbb{E} \int_0^T |\bar{Z}_s^n|^2 ds + \theta \mathbb{E} |\bar{K}_T^{n+1}|^2.$$

Moreover, since

$$\begin{aligned} \bar{K}_T^{n+1} &= \bar{Y}_0^{n+1} - \xi - \int_0^T E' \left(f \left(s, \bar{Y}_s^n, (\bar{Y}_s^n)', \bar{Z}_s^n \right) + \pi \left(s, \delta \bar{Y}_s^{n+1}, \delta (\bar{Y}_s^{n+1})', \delta \bar{Z}_s^{n+1} \right) \right) ds \\ &\quad - \int_0^T E' \left(g \left(s, \bar{Y}_s^{n+1}, (\bar{Y}_s^{n+1})', \bar{Z}_s^{n+1} \right) \right) d\bar{B}_s + \int_t^T \bar{Z}_s^{n+1} dW_s, \end{aligned}$$

by the Hölder inequality and **B-D-G** inequality, $\mathbb{E}(X)^2 \leq \mathbb{E}(X^2)$ and the properties on f , g , π that there exists two constants C_1 and C_2 depending on α , ξ and C of n such that

$$\mathbb{E} |\bar{K}_T^{n+1}|^2 \leq C_1 + C_2 \left(\mathbb{E} \int_0^T |\bar{Z}_s^{n+1}|^2 + |\bar{Z}_s^n|^2 ds \right).$$

Return to inequality (10), we get

$$\mathbb{E} \int_0^T |\bar{Z}_s^{n+1}|^2 ds \leq C^\theta + \theta C_1 + \left(\frac{1}{8} + \theta C_2 \right) \mathbb{E} \int_0^T |\bar{Z}_s^n|^2 ds + \theta C_2 \mathbb{E} \int_0^T |\bar{Z}_s^{n+1}|^2 ds,$$

we chosing θ , such that $\theta C_2 \leq 1$, we have

$$\begin{aligned} \mathbb{E} \int_0^T |\bar{Z}_s^{n+1}|^2 ds &\leq C^\theta + \theta C_1 + \left(\frac{1}{8} + \theta C_2 \right) \mathbb{E} \int_0^T |\bar{Z}_s^n|^2 ds \\ &\leq \left(C^\theta + \theta C_1 \right) \sum_{i=0}^{i=n-1} \left(\frac{1}{8} + \theta C_2 \right)^i + \left(\frac{1}{8} + \theta C_2 \right)^n \mathbb{E} \int_0^T |\bar{Z}_s^0|^2 ds. \end{aligned}$$

Now choosing θ such that $\frac{1}{8} + \theta C_2 < 1$ and noting $\mathbb{E} \int_0^T |\bar{Z}_s^0|^2 ds < \infty$. Obtain

$$\sup_{n \in \mathbb{N}} \mathbb{E} \int_0^T |\bar{Z}_s^{n+1}|^2 ds < \infty,$$

consequently, we deduce

$$\mathbb{E} |\bar{K}_T^{n+1}|^2 < \infty.$$

Now we shall prove that $(\bar{Z}^n, \bar{K}^n)$ is a Cauchy sequence in $\mathcal{M}^2(0, T, \mathbb{R}^d) \times \mathcal{A}^2$.

Applying Itô's formula to $|\delta \bar{Y}_s^{n,m}|^2 = |\bar{Y}_s^n - \bar{Y}_s^m|^2$, we have

$$\begin{aligned} \mathbb{E} |\bar{Y}_t^n - \bar{Y}_t^m|^2 + \mathbb{E} \int_0^T |\bar{Z}_s^n - \bar{Z}_s^m|^2 ds &= 2\mathbb{E} \int_0^T (\bar{Y}_s^n - \bar{Y}_s^m) E'(\Gamma_s^n - \Gamma_s^m) ds + 2 \int_0^T \bar{Y}_s^{n+1} (d\bar{K}_s^n - d\bar{K}_s^m) \\ &\quad + \int_0^T \left\| E' \left(g(s, \bar{Y}_s^n, (\bar{Y}_s^n)', \bar{Z}_s^n) - g(s, \bar{Y}_s^m, (\bar{Y}_s^m)', \bar{Z}_s^m) \right) \right\|^2 ds. \end{aligned}$$

where $\Gamma_s^n = f(s, \bar{Y}_s^{n-1}, (\bar{Y}_s^{n-1})', \bar{Z}_s^{n-1}) + \pi(s, \delta \bar{Y}_s^n, \delta (\bar{Y}_s^n)', \delta \bar{Z}_s^n)$. Since $\int_0^T \bar{Y}_s^{n+1} (d\bar{K}_s^n - d\bar{K}_s^m) \leq 0$, we obtain

$$\begin{aligned} \mathbb{E} \int_0^T |\bar{Z}_s^n - \bar{Z}_s^m|^2 ds &\leq 2\mathbb{E} \int_0^T (\bar{Y}_s^n - \bar{Y}_s^m) E'(\Gamma_s^n - \Gamma_s^m) ds \\ &\quad + \int_0^T \left\| E' \left(g(s, \bar{Y}_s^n, (\bar{Y}_s^n)', \bar{Z}_s^n) - g(s, \bar{Y}_s^m, (\bar{Y}_s^m)', \bar{Z}_s^m) \right) \right\|^2 ds. \end{aligned}$$

By the Hölder inequality and hypothesis (H.11), we deduce that

$$\begin{aligned} (1 - \alpha) \mathbb{E} \int_0^T |\bar{Z}_s^n - \bar{Z}_s^m|^2 ds &\leq 2\mathbb{E} \left(\int_0^T |\bar{Y}_s^n - \bar{Y}_s^m|^2 ds \right)^{\frac{1}{2}} \mathbb{E} \left(\int_0^T |E'(\Gamma_s^n - \Gamma_s^m)|^2 ds \right)^{\frac{1}{2}} \\ &\quad + 2C \mathbb{E} \int_0^T |\bar{Y}_s^n - \bar{Y}_s^m|^2 ds. \end{aligned}$$

The boundedness of the sequence $(\bar{Y}^n, \bar{Z}^n, \bar{K}^n)$, we deduce that the $\Lambda = \sup_{n \in \mathbb{N}} \left[\mathbb{E} \int_0^T |E'(\Gamma_s^n)|^2 ds \right] < \infty$, this yields that

$$(1 - \alpha) \mathbb{E} \int_0^T |\bar{Z}_s^n - \bar{Z}_s^m|^2 ds \leq 4\Lambda \mathbb{E} \left(\int_0^T |\bar{Y}_s^n - \bar{Y}_s^m|^2 ds \right)^{\frac{1}{2}} + 2C \mathbb{E} \int_0^T |\bar{Y}_s^n - \bar{Y}_s^m|^2 ds,$$

which yields that $(\bar{Z}^n)_{n \geq 0}$ is a Cauchy sequence in $\mathcal{M}^2(0, T, \mathbb{R}^d)$. Then there exists $Z \in \mathcal{M}^2(\mathbb{R}^d)$ such that

$$\mathbb{E} \int_0^T |\bar{Z}_s^n - Z_s|^2 ds \rightarrow 0 \text{ as } n \rightarrow \infty.$$

On the other hand, by Burkholder-Davis-Gundy inequality, we get

$$\begin{cases} \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_t^T \bar{Z}_s^n dW_s - \int_t^T Z_s dW_s \right|^2 \leq \mathbb{E} \int_t^T |\bar{Z}_s^n - Z_s|^2 ds \rightarrow 0, \text{ as } n \rightarrow \infty, \\ \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_t^T E' \left(g(s, \bar{Y}_s^n, (\bar{Y}_s^n)', \bar{Z}_s^n) \right) - E' \left(g(s, Y_s, (Y_s)', Z_s) \right) \right|^2 \\ \leq 2C \mathbb{E} \int_0^T |\bar{Y}_s^n - Y_s|^2 ds + \alpha \mathbb{E} \int_0^T |\bar{Z}_s^n - Z_s|^2 ds \rightarrow 0, \text{ as } n \rightarrow \infty. \end{cases}$$

Therefore, from the properties of f and π

$$\Gamma_s^n = f(s, \bar{Y}_s^{n-1}, (\bar{Y}_s^{n-1})', \bar{Z}_s^{n-1}) + \pi(s, \delta \bar{Y}_s^n, \delta (\bar{Y}_s^n)', \delta \bar{Z}_s^n) \rightarrow f(s, Y_s, (Y_s)', Z_s),$$

$\bar{\mathbb{P}} - a.s.$, for all $t \in [0, T]$ as $n \rightarrow \infty$. Then follows by Lebesgue's dominated convergence theorem that

$$\mathbb{E} \int_0^T \left| E' \left(\Gamma_s^n - f(s, Y_s, (Y_s)', Z_s) \right) \right|^2 ds \rightarrow 0, \quad n \rightarrow \infty$$

Since $(\tilde{Y}_s, \tilde{Z}_s, \Gamma_s^n)$ converges in $\mathcal{S}^2(0, T, \mathbb{R}) \times \mathcal{M}^2(0, T, \mathbb{R}^d) \times \mathcal{M}^2(0, T, \mathbb{R}^d)$ and

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq t \leq T} |\bar{K}_t^n - \bar{K}_t^m|^2 \right) &\leq \mathbb{E} |\bar{Y}_0^n - \bar{Y}_0^m|^2 + \mathbb{E} \sup_{0 \leq t \leq T} |\bar{Y}_t^n - \bar{Y}_t^m|^2 + \mathbb{E} \int_0^T \left| E' (\Gamma_s^n - \Gamma_s^m) \right|^2 ds \\ &\quad + \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t E' \left(g(s, \bar{Y}_s^n, (\bar{Y}_s^n)', \bar{Z}_s^n) - g(s, \bar{Y}_s^m, (\bar{Y}_s^m)', \bar{Z}_s^m) \right) d\bar{B}_s \right|^2 \\ &\quad + \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t (\bar{Z}_s^n - \bar{Z}_s^m) dW_s \right|^2 \end{aligned}$$

for any $n \geq 0$, we deduce from Bukhlder-Davis-Gundy inequality that

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |\bar{K}_t^n - \bar{K}_t^m|^2 \right) \rightarrow 0,$$

as $n \rightarrow \infty$. Consequently, there exists a $\mathcal{F}_t$ -measurable process K with value in $\mathbb{R}$ such that

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |\bar{K}_t^n - K_t|^2 \right) \rightarrow 0,$$

as $n \rightarrow \infty$. Obviously, $K_0 = 0$ and $\{K_t; 0 \leq t \leq T\}$ is a increasing and continuous process. From Eq. (6), we have for all $n \geq 0$, $\bar{Y}_t^n \geq S_t$, $\forall t \in [0, T]$, then $Y_t \geq S_t$, $\forall t \in [0, T]$. On the other hand, from the result of Saisho [8] (in 1987, p. 465), we have

$$\int_0^T (\bar{Y}_s^n - S_s) d\bar{K}_s^n \rightarrow \int_0^T (Y_s - S_s) dK_s,$$

$\bar{\mathbb{P}} - a.s.$, as $n \rightarrow \infty$. Using the identite $\int_0^T (\bar{Y}_s^n - S_s) d\bar{K}_s^n = 0$, for all $n \geq 0$ we conclude that $\int_0^T (Y_s - S_s) dK_s \geq 0$. Letting $n \rightarrow +\infty$ in Eq. (3), we prove that (Y, Z, K) is solution to Eq. (3). Let (Y^*, Z^*, K^*) be any solution of the MF-RBDSDE (3), we have $\bar{Y}_t^n \leq Y_t^*$, for all $n \geq 0$ and therefore, $Y_t \leq Y_t^*$ i.e., Y is the minimal solution. $\square$

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