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On the existence of periodic solution and the transition to chaos of Rayleigh-Duffing equation with application of gyro dynamic

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Abstract

In this article, the study of qualitative properties of a special type of non-autonomous nonlinear second order ordinary differential equations containing Rayleigh damping and generalized Duffing functions is considered. General conditions for the stability and periodicity of solutions are deduced via fixed point theorems and the Lyapunov function method. A gyro dynamic application represented by the motion of axi-symmetric gyro mounted on a sinusoidal vibrating base is analyzed by the interpretation of its dynamical motion in terms of Euler's angles via the deduced theoretical results. Moreover, the existence of homoclinic bifurcation and the transition to chaotic behaviour of the gyro motion in terms of main gyro parameters are proved. Numerical verifications of theoretical results are also considered.

Keywords: Nonlinear Ordinary Differential Equations; Stability Theory; Periodic Solutions; Bifurcation; Chaotic Dynamic; Gyroscope.
AMS 2010 codes: 34L30; 37C75; 34C25; 34F10; 37D45; 70E05.

1 Introduction

Many applied problems represented by dynamical systems have many difficulties to obtain their solutions in explicit forms definitely when these problems are modelled by (non-autonomous)nonlinear systems, cf. [5,7,12]. Therefore, the directions of progress to analyze qualitatively for obtaining the main mathematical properties now become must. Most of these properties are stuck in the presence of periodic solutions, the bifurcations and the route to the chaotic behaviours or more less based on their dynamics as well. One of these celebrated applied problems is the general planar motion of a particle exerted by conservative and nonconservative fields that can be represented by the following system:

$$\dot{x} = X(x,y,t), \qquad \dot{y} = Y(x,y,t). \tag{1}$$

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A special case of this system (Eq.1) is the general second order (non-autonomous)nonlinear ordinary differential equation of the form

$$\dot{x} = y, \qquad \dot{y} + f(t,x,y) = 0, \tag{2}$$

where f is a nonlinear function satisfying certain conditions specified in the tackled problem.
The nonlinear system represented by Eq.2 is mostly used for mathematical modelling of various engineering phenomena and it is attracting engineers and mathematical researchers due to its rich applications. A great deal of papers and books published during the last decades have extensively studied the qualitative properties of Eq.2 in directions of existence and uniqueness, stability, periodicity(quasi-periodicity), bifurcation, chaos and some numerous particular properties, cf. [2, 3, 15, 21, 25, 27].
In particular, one of the fruitful special cases of Eq.2 tackled in numerous applications for their qualitative constant interests is the following non-autonomous undamped duffing equation

$$\ddot{x} + f(x,t) = 0. \tag{3}$$

Ortega in [22] initiated the study of Eq.3 for the stability of periodic solutions of using the relation between topological degree and stability. In the other hand, regarding the driven damped duffing equation

$$\ddot{x} + c\dot{x} + f(x,t) = r(t), \tag{4}$$

where c is the coefficient of linear damping and $r(t)$ is the driven function. Chen et al. [11] studied it to obtain the exact multiplicity of periodic solutions with $f(x,t)$ and $r(t)$ as continuous and 2π periodic functions of time. In [29], it is studied the existence, uniqueness and asymptotic stability of periodic solutions of some special type of Van Der Pol oscillators under periodic time excitation is studied. However, so many literatures have been discussed on the qualitative analysis of Eq.2 and its periodic solutions by different techniques, cf. [1, 8–10, 14, 23, 24, 26].

Motivated by this argument, in this work, we discuss the stability and the existence of periodic solutions as well as the transition to chaos for very specific type of non-autonomous nonlinear ODEs, so-called the Rayleigh–Duffing equation

$$\ddot{x} + h(\dot{x}) + f(x,t) = r(t), \tag{5}$$

where $h(\dot{x})$ and $f(x,t)$ are nonlinear continuous functions and $r(t)$ is a continuous periodically driven function.
In general, the most general engineering applications of the Rayleigh–Duffing equation (Eq.5) are modelling of RLC circuits with actual voltage, the planar motion of a particle under the exciting and dissipative forces and some of gyro motions under external torques, cf. [19].
Here, the function $h(\dot{x})$ represents the dissipative forces that are considered as a general bounded function or have the generalized form of Rayleigh damping and $f(x,t)$ represents the conservative and nonconservative forces of duffing type.
Using a dimensionless small parameter, denoted by ε, the Rayleigh–Duffing equation can be transformed to the following perturbed form:

$$\ddot{x} + \omega_n^2 x = \varepsilon \mathbb{F}(x,\dot{x},t), \tag{6}$$

where ω_n represents the natural frequency of the system and $\mathbb{F}$ represents a perturbed force on the system.
The concept of chaotic behaviour is important due to its existence profusely in complex systems; therefore, the dynamic application of Eq.5 will concern some what a part of this study. In this work, an interesting application of Eq.5 being the motion of axi-symmetric gyroscope mounted on a vibrating base with periodic excitation is considered, cf. [28]. The general treatment of the governing equation is handled by sides of stability analysis and existence of periodic solutions. Moreover, the theoretical results are applied to the tackled application besides the study of bifurcation and transition to chaos using the perturbed form of the differential equation. Melnikov's method is used to clarify the range of chaotic behaviour affected by the change in main parameters of the system. Numerical verification by using numerical solution diagrams and phase plane trajectories for proving the

deduced theoretical results of the gyro motion are considered.

This work is organized as follows: In section 2, some theoretical results for the Rayleigh–Duffing equation are presented. Section 3 is concerned with a dynamic application using an example of gyro motion governed by the Rayleigh–Duffing equation to verify the theoretical results. In the last section, the conclusion is given.

2 Theoretical results

Let us assume that, the Rayleigh–Duffing equation possesses a T-periodic solution in the presence of continuous functions $h(\dot{x})$, $f(x,t)$ and $r(t)$, with the later two T-periodic with respect to time. In addition to, it is assumed that the following identity at the equilibrium point $x^* = 0$

$$h(0) + f(0,t) = 0, \tag{7}$$

holds and the only solution of $h(0) + f(x,t) = 0$ is $x = x^*$. Hence, rewriting Eq.5

$$\ddot{x} + f(x,t) = r(t) - h(\dot{x}), \tag{8}$$

where the RHS represents the absorption and the supply of system energy. Thus, the total energy($\mathbb{E}$) of the system is non-conservative and reads

$$\mathbb{E} = \frac{1}{2}y^2 + F(x,t), \quad \text{where} \quad F(x,t) = \int_0^x f(u,t)du. \tag{9}$$

Then, the energy rate is

$$\frac{d\mathbb{E}}{dt} = \frac{\partial F}{\partial t} - h(y)y. \tag{10}$$

From Eq.10, it is easy to obtain the general conditions for energy decaying, but on the other hand one can obtain a general representation of energy function (i.e. Lyapunov's function) as

$$\mathbb{V}(x,y,t) = e^{\sqrt{y^2+F(x,t)+k} - \int_0^t |r(t)| - kt}, \tag{11}$$

on the domain $|x| < \infty$, $|y| < \infty$, $0 \le t \le T$ and $x^2 + y^2 \ge k^2$ where $k \in \mathbb{R}^+$.
Thus, the following theorem gives general imposed conditions based on Eq.5 to obtain asymptotically stable solutions.

Theorem 1. *The Rayleigh-Duffing equation has an asymptotically continuable stable solution if the following conditions are satisfied:*
i) $h(y)$ *and* $f(x,t)$ *are locally Lipschitzian in* y *and* x *respectively, and* $r(t)$ *is continuous on* $\mathbb{R}$*.*
ii) $f(x,t)$ *and* $r(t)$ *are periodic in* t *of period* T*.*
iii) $F(x,t) = \int_0^x f(u,t)du > -k$ *for all* t *and* x *and* $\frac{F_t}{\sqrt{F(x,t)+k}}$ *is bounded.*

Proof. For the domain $t \in [0,T]$, $|x| < \infty$, $|y| < \infty$ and $x^2 + y^2 \ge k^2$, and by considering the function $\mathbb{V}$ using Eq.11, we have

$$\dot{V} \le 0, \tag{12}$$

only if the three conditions *i*, *ii* and *iii* are satisfied. Then, the conclusion holds. □

The existence of periodic solutions of the Rayleigh–Duffing equation can be proved on the basis of Schauder's fixed-point theorem. General conditions to obtain at least one limit cycle are stated and proved in the following theorem.

Theorem 2. *The Rayleigh–Duffing equation has at least one T-periodic solution if the following conditions are satisfied,*
i) $f(x,t)$ is locally Lipschitzian in x and $r(t)$ is a continuous on $\mathbb{R}$.
ii) $f(x,t)$ and $r(t)$ are periodic in t of period T and $\int_0^T r(t)dt = 0$.
iii) $|f(x,t)| \leq |r(t)|$ uniformly in t.
iv) the function $h(y)$ is bounded, i.e $|h(y)| < L$.
v) $f(x,t)sgnx \geq 0$.

Proof. Let us rewrite Eq.5 in its perturbed form as follows,

$$\ddot{x} + \omega_n^2 x = \varepsilon(\tilde{r}(t) - \tilde{h}(\dot{x}) - \tilde{f}(t,x) + (1-\varepsilon)\omega_n^2 x), \tag{13}$$

where ω_n here represents an arbitrary positive constant. Hence, if $\varepsilon \approx 0$ then we obtain a homogenous equation with ω_n periodic solution but for $0 < \varepsilon < 1$ all periodic solutions and their first derivative are uniformly bounded. So that, let $x(t) = x(t+T)$ be a solution of Eq.13 and

$$\max_{0 \leq t < \infty} |x(t)| = R, \ \max_{0 \leq t < \infty} |\dot{x}(t)| = \tilde{R}, \ \max_{0 \leq t < \infty} |r(t)| = \Gamma. \tag{14}$$

Hence, this equation can be written as

$$\ddot{x} + \omega_n^2 x = q(t), \tag{15}$$

with the following periodic boundary conditions

$$x(0) = x(T), \qquad y(0) = y(T). \tag{16}$$

Green function($\mathbb{G}$) of the boundary value problem of Eq.15 with periodic boundary conditions of Eq.16 reads

$$\mathbb{G}(s;t) = \frac{1}{2\omega_n \sin\frac{\omega_n T}{2}} \begin{cases} \cos\omega_n(s-t+\frac{T}{2}) \text{ if} & 0 \leq s \leq t \leq T, \\ \cos\omega_n(t-s+\frac{T}{2}) \text{ if} & 0 \leq t \leq s \leq T, \end{cases} \tag{17}$$

with the identity $\mathbb{G}_s(t+0,t) - \mathbb{G}_s(t-0,t) = 1$. This implies that $q(t+T) = q(t)$, then we obtain the following representation of the solution $x(t)$

$$x(t) = \int_0^T G(s;t)q(s)ds. \tag{18}$$

Hence, by inserting the explicit expression for $q(t)$ to derive the following estimates

$$|x(t)| \leq \rho(2\Gamma + L + \omega_n^2 R), \quad \rho = \max(1, \frac{1}{\omega_n^2}) \tag{19}$$

Now, let us integrate term by term of Eq.13 to yield

$$[y - P(t)]_0^T + \int_0^T [\omega_n^2(1-\varepsilon)x(t) + f(x,t) + h(y)]dt = 0, \tag{20}$$

or

$$\int_0^T [\omega_n^2(1-\varepsilon)x(t) + f(x,t) + h(y)]dt = 0, \tag{21}$$

then, we have

$$\omega_n^2(1-\varepsilon)|x(t)| + f(x,t)sgnx + |h(y)| > 0. \tag{22}$$

This leads to

$$f(x,t)sgnx > 0. \tag{23}$$

For $|x| > x_o$, $t \in [0,T]$ it follows that $|x(t)| \geq x_o$ for $0 \leq t \leq T$ is excluded, therefore there exits τ such that $0 < \tau < T$ then $x(\tau) < x_o$. Applying the mean value theorem to an arbitrary interval $[\tau,t] \subset [\tau,T]$, then we have

$$|y(\tau+\theta(t-\tau)| = \frac{|x(t)-x(\tau)|}{|t-\tau|}, \tag{24}$$

then

$$|y(\tau+\theta(t-\tau)| \leq \frac{(2\Gamma+L+\omega_n R)}{T\rho}, \tag{25}$$

so we have

$$|y(t)| \leq \frac{\rho(2\Gamma+L+\omega_n R)}{T\rho}, \tag{26}$$

it is easily to get

$$\tilde{R} \leq \frac{2\Gamma+L+\omega_n R}{T}, \tag{27}$$

for $R \to 0$, then we have the following priori estimates

$$|x(t)| \leq x_o + T\rho(2\Gamma+L), \quad |y(t)| \leq \frac{2\Gamma+L}{T}, \tag{28}$$

These estimates ensure at least the existence of one periodic solution in future, then the conclusion holds. □

The following theorem ensures the existence of periodic solutions if the damping function is a Rayleigh damping term $h(y) = cy + ey^3$ where c and e are real constants under the following stated conditions.

Theorem 3. *The Rayleigh-Duffing equation under the Rayleigh damping term for $t \in \mathbb{R}^+$ and $x \in [a,b]$ has at least one periodic solution of period T if the following condition are satisfied,*
i) $h(y)$ and $f(x,t)$ are locally Lipschitzian in y and x respectively and $r(t)$ is a continuous on $\mathbb{R}$.
ii) $f(x,t)$ and $r(t)$ are periodic in t of period T,
iii) $F(x,t) = \int_0^x f(u,t)du > -k$ for all t and $\frac{F_t}{\sqrt{F(x,t)+k}}$ is bounded.
iv) $h(y)y > 0$ for $c,\ e \in \mathbb{R}$.
v) There exists $a,b,\ a < b,\ f(a,t) - r(t) \geq 0$ and $f(b,t) - r(t) \leq 0$.
vi) $|f(t,x)| + |r(t)| \leq \ell$.

Proof. For the domain $t \in [0,T]$, $|x| < \infty$, $|y| < \infty$ and $x^2+y^2 \geq k^2$, and by considering the following two function $\mathbb{V}_1$ and $\mathbb{V}_2$ as follow

$$\mathbb{V}_1 = e^{x+\int_k^y \frac{u}{h(u)+\ell}du}, \qquad \mathbb{V}_2 = e^{x+\int_{-k}^y \frac{u}{h(u)+\ell}du}. \tag{29}$$

where

$$h(u) = cu + eu^3. \tag{30}$$

The two Liapunov's functions satisfy the following conditions

$$\mathbb{V}_1 \leq \mathbb{B}(|y|), \quad \mathbb{V}_2 \leq \mathbb{B}(|y|), \quad \mathbb{B}(|y|) > 0. \tag{31}$$

and $\mathbb{V}_1 \to \infty$ and $\mathbb{V}_2 \to \infty$ uniformly for (x,t) as $|y| \to \infty$.
Thus, there exist in the interior of two domains D_1 and D_2 such that

$$\dot{\mathbb{V}}_1 \geq 0, \qquad \dot{\mathbb{V}}_2 \leq 0. \tag{32}$$

Then, the equation has a solution $x(t)$ such that $|x(t)| + |y(t)|$ is bounded for all $t \geq 0$, so that all solutions exist in future and one of them is bounded then there exists at least a periodic solution of period T in future. □

3 Gyro dynamic application

The Rayleigh–Duffing equation has rich applications in the gyro dynamics, cf. [6, 13, 19]. In our case, the application represents the motion of axi-symmetric gyroscope(Lagrange's gyroscope) mounted on a vibrating base exerted by a sinusoidal periodic force along the vertical fixed axis(OZ) as shown in Fig.1.

The motion can be described by the Routhian's function($\Re$) as a function of the Euler's angles θ(nutation),

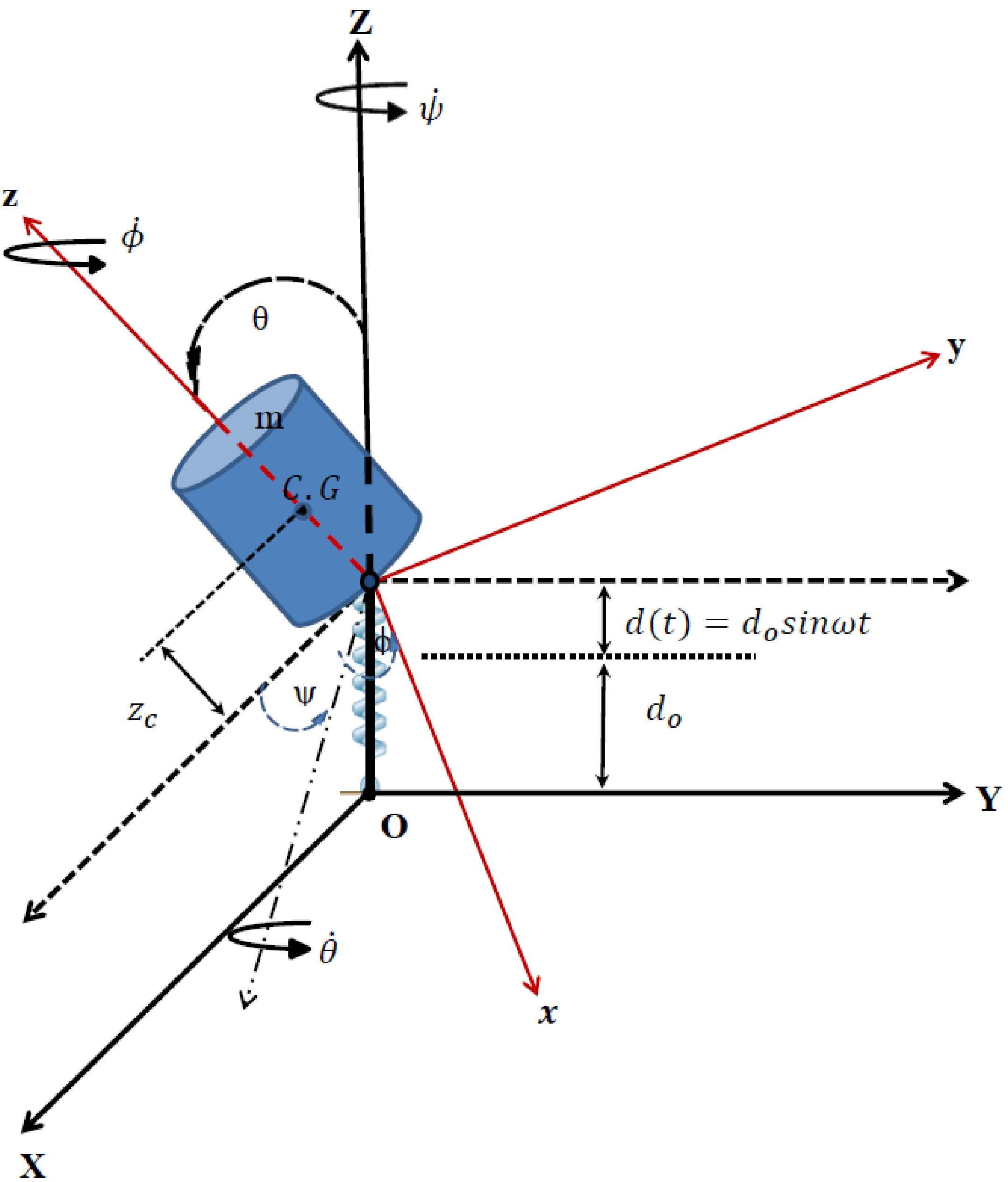

Fig. 1 Motion of Lagrange's gyroscope represented by a cylindrical body mounted on a vibrating base with periodic excitation($d(t) = d_o \sin \omega t$).

ψ(precession), and ϕ(spin), cf. [16–18, 28],

$$\mathfrak{R} = \mathbb{L} - p_\psi \dot{\psi}(p_\psi, p_\phi, \theta) - p_\phi \dot{\phi}(p_\psi, p_\phi, \theta), \tag{33}$$

where $\mathbb{L}$ is the Lagrangian function

$$\mathbb{L} = \frac{1}{2} I(\dot{\theta} + \dot{\psi}^2 \sin^2 \theta) + \frac{1}{2} I_3(\dot{\phi} + \dot{\psi} \cos \theta)^2 - mr\dot{d}\dot{\theta} \sin \theta - mgr \cos \theta, \tag{34}$$

p_ϕ and p_ψ are the corresponding conserved angular momenta to the cyclic coordinates ϕ and ψ, respectively.

$$p_\phi = I_3(\dot{\phi} + \dot{\psi} \cos \theta) = const., \qquad p_\psi = I\dot{\psi} \sin^2 \theta + p_\phi \cos \theta = const. \tag{35}$$

Let us define m as the mass of the gyroscope, $I = A + mz_c^2$ is the principle equatorial moment of inertia, A, B and C are the principal moments of inertia along the moving axes Ox, Oy and Oz respectively, z_c is the distance along the polar axis(Oz) of the centre of gravity(C.G.) of the gyro from its point of support(O) and $d(t) = d_0 \sin \omega t$ is time varying amplitude of the vertical support motion that has constant amplitude d_o and forced frequency ω along the vertical fixed axis (OZ).
Then, the governing equation reads

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{R}}{\partial \dot{\theta}}\right) - \frac{\partial \mathfrak{R}}{\partial \theta} = -\mathfrak{D}(\dot{\theta}), \tag{36}$$

where $-\mathfrak{D}(\dot{\theta})$ is the nonconservative force of damping, which has the following generalized Rayleigh damping form

$$\mathfrak{D}(\dot{\theta}) = c_1 \dot{\theta} + c_2 \dot{\theta}^3, \qquad c_1, c_2 \in \mathbb{R}. \tag{37}$$

If $p_\psi = p_\phi = p_o$, then the governing equation reads

$$\ddot{\theta} + \frac{1}{I} D(\dot{\theta}) + \frac{p_o^2 (1 - \cos \theta)^2}{I^2 \sin^3 \theta} - \frac{mgz_c}{I} \sin \theta - \frac{mz_c}{I} \sin \theta \ddot{d}(t) = 0. \tag{38}$$

If $\theta = x$, $\dot{\theta} = y$, then we get the following system

$$\dot{x} = y, \tag{39a}$$

$$\dot{y} = -\frac{1}{I} D(y) - \frac{p_o^2 (1 - \cos x)^2}{I^2 \sin^3 x} + \frac{mgz_c}{I} \sin x + \frac{mz_c}{I} \sin x \ddot{d}(t). \tag{39b}$$

The vibrating axi-symmetric gyro equation(Eq.38) can be drawn into the following general form:

$$\ddot{\theta} + h(\dot{\theta}) + f(\theta, t) = 0, \tag{40}$$

where,

$$f(\theta, t) = \frac{p_o^2}{I^2} \frac{(1 - \cos \theta)^2}{\sin^3 \theta} - \frac{mgz_c}{I} \sin \theta + \frac{1}{I} \omega^2 mz_c d_o \sin \omega t \sin \theta, \tag{41}$$

and

$$h(\dot{\theta}) = c\dot{\theta} + e\dot{\theta}^3, \qquad c = \frac{c_1}{I}, \quad e = \frac{c_2}{I}. \tag{42}$$

The approximate form of Eq.38 under the following approximations

$$\sin \theta = \theta - \frac{\theta^3}{6}, \quad \frac{(1 - \cos \theta)^2}{\sin^3 \theta} = \frac{\theta}{4} + \frac{\theta^3}{12}, \quad \omega_n^2 = \frac{\alpha}{4} - \beta,$$

$$\alpha = \frac{p_o^2}{I^2}, \quad \beta = \frac{mgz_c}{I}, \quad \Gamma = \frac{1}{I} \omega^2 mz_c d_o, \quad \Omega = \frac{\alpha}{12} + \frac{\beta}{6}, \tag{43}$$

reads

$$\ddot{\theta}+c\dot{\theta}+e\dot{\theta}^3+(\omega_n^2+\Gamma\sin\omega t)\theta+(\Omega-\frac{\Gamma}{6}\sin\omega t)\theta^3=0. \tag{44}$$

With the following choices,

$$\Gamma=\varepsilon\bar{\Gamma},\quad c=\varepsilon\bar{c},\quad e=\varepsilon\bar{e},\quad \Omega=\varepsilon\bar{\Omega}, \tag{45}$$

Eq.44 can be put in the following perturbed form

$$\ddot{\theta}+\omega_n^2\theta=\varepsilon(-\bar{c}\dot{\theta}-\bar{e}\dot{\theta}^3-\bar{\Gamma}\sin\omega t\theta+(-\bar{\Omega}+\frac{\bar{\Gamma}}{6}\sin\omega t)\theta^3). \tag{46}$$

The theoretical results can be applied to the equation of the vibrating axi-symmetric gyro model (Eq.38) to obtain the following theorems

Theorem 4. *According to the conditions of* theorem 1*, the solution of the vibrating axi-symmetric gyro equation is globally asymptotically stable.*

Theorem 5. *According to the conditions of* theorem 3*, the vibrating axi-symmetric gyro equation has at least one periodic solution.*

3.1 Approximate form of the periodic solution

In [20], it is found that the straightforward expansion methods fails sometime to obtain a form of periodic solution for the linear problem with periodic coefficient, for instance Mathieu's equation. Therefore, it is suggested to use the method of strained parameters by expanding the solution(θ) and the natural frequency(ω_n) in powers of ε by using the perturbed form of vibrating axi-symmetric gyro equation(Eq.46). Thus, we seek the periodic solution by using the following uniform expansions:

$$\omega_n^2=n^2+\varepsilon\omega_1+\varepsilon^2\omega_2+\ldots \tag{47a}$$
$$\theta=\theta_0+\varepsilon\theta_1+\varepsilon^2\theta_2+\ldots. \tag{47b}$$

Comparing coefficients of $\varepsilon^0,\varepsilon^1,\varepsilon^2$, we get the following differential equations,

$$\frac{d^2\theta_0}{dt^2}+n^2\theta_0=0, \tag{48}$$
$$\frac{d^2\theta_1}{dt^2}+n^2\theta_1=-\omega_1\theta_0-\bar{c}\dot{\theta}_0-\bar{e}\dot{\theta}_0^3-\bar{\Gamma}\sin\omega t\theta_0+(-\bar{\Omega}+\frac{1}{6}\bar{\Gamma}\sin\omega t)\theta_0^3, \tag{49}$$
$$\frac{d^2\theta_2}{dt^2}+n^2\theta_2=-\omega_1\theta_1-\omega_2\theta_0-\bar{c}\dot{\theta}_1-2\bar{e}\dot{\theta}_0^2\dot{\theta}_1-\bar{\Gamma}\sin\omega t\theta_1+2(-\Omega+\frac{1}{6}\bar{\Gamma}\sin\omega t)\theta_0^2\theta_1. \tag{50}$$

By solving these equations, we obtain,

$$\theta_0=a\cos(nt+\phi), \tag{51}$$

$$\theta_1=-\frac{-1}{\omega^2-2\omega n}\frac{5}{12}\bar{\Gamma}a\sin((\omega-n)t+\phi)+\frac{1}{\omega^2+2\omega n+2n^2}\frac{5}{12}\bar{\Gamma}a\sin((\omega+n)t+\phi)$$
$$-\frac{1}{8n^2}\bar{e}a^3n^3\sin(3nt+3\phi), \tag{52}$$

$$
\begin{aligned}
\theta_2 = & \frac{-1}{\omega^2+2\omega n-n^2} D(\omega_1+\bar{\Omega})\sin((\omega-n)t+\phi) \\
& -\frac{1}{\omega^2+2\omega n} M(\omega_1+\bar{\Omega})\sin((\omega+n)t+\phi) \\
& -\frac{1}{64n^2}(\omega_1+\bar{\Omega})\sin(3nt+3\phi) \\
& +\frac{1}{4\omega(n-\omega)}\cos((2\omega-n)t+\phi) \\
& +\frac{1}{n^2-(\omega-3n^2)^2}\frac{\bar{\Gamma}}{16n^2}\cos((\omega-3n)t-3\phi)-\frac{3\bar{c}}{64n^3}\cos(3nt+3\phi) \\
& -\frac{1}{n^2-(\omega+3n^2)^2}\frac{\bar{\Gamma}}{16n^2}\cos((\omega+3n)t+3\phi) \\
& -\frac{1}{\omega(2n-\omega)}\bar{c}D\cos((\omega-n)t+\phi) \\
& -\frac{1}{\omega(\omega+2n)}\bar{c}M(\omega+n)\cos((\omega-n)t+\phi) \\
& +\frac{1}{\omega-3n}\frac{3}{2}\bar{e}aD\sin(2n-\omega)t \\
& -\frac{1}{\omega+n}\frac{3}{2}\bar{e}aD\sin(\omega t+2\phi)-\frac{1}{\omega-n}\frac{3}{2}\bar{e}aM\sin\omega t \\
& +\frac{1}{3n+\omega}\frac{3}{2}\bar{e}aD\sin((\omega+2n)t+2\phi)-\frac{3}{16n^3}\bar{e}a\sin(2nt+2\phi) \\
& -\frac{80}{3n^3}\bar{e}\sin(4nt+\phi)-\frac{1}{16\omega(\omega+n)}\bar{\Gamma}M\cos((2\omega+n)t+\phi) \\
& -\frac{1}{(4n-\omega)(2n-\omega)}\frac{3}{16n}\bar{\Gamma}\sin((\omega-3n)t-3\phi) \\
& -\frac{1}{(4n+\omega)(2n+\omega)}\frac{3}{16n}\bar{\Gamma}\sin((\omega+3n)t+3\phi),
\end{aligned} \tag{53}
$$

where a and ϕ are arbitrary constants, $D=-\frac{5}{12}\frac{1}{\omega(\omega-2n)}\bar{\Gamma}a$ and $M=\frac{5}{12}\frac{1}{(\omega+n)^2}\bar{\Gamma}a$. Then, the general approximate solution reads

$$\theta=\theta_0+\varepsilon\theta_1+\varepsilon^2\theta_2+O(\varepsilon^3). \tag{54}$$

The gyro system generates a periodic solution under the following conditions which come from deleting the secular terms

$$a=\frac{1}{n}\sqrt{-\frac{\bar{c}}{\bar{e}}},\quad \omega_1=-\bar{\Omega},\quad \omega_2=\frac{-7}{12}\frac{\bar{\Gamma}}{aD}-\frac{1}{4}\bar{\Gamma}M. \tag{55}$$

3.2 Stability of the approximate periodic solution

By rewriting the perturbed form of vibrating axi-symmetric gyro equation(Eq.46) to have the form

$$\ddot{\theta}+\omega_n^2\theta=\varepsilon\mathbb{F}(\theta,\dot{\theta},t), \tag{56}$$

where

$$\mathbb{F}=-\bar{c}\dot{\theta}-\bar{e}\dot{\theta}^3-\bar{\Gamma}\sin\omega t\,\theta+(-\bar{\Omega}+\frac{\bar{\Gamma}}{6}\sin\omega t)\theta^3. \tag{57}$$

Hence, the solution of the unperturbed problem has the form

$$\theta=a(t)\sin(\omega_n t+\phi(t)) \tag{58}$$

Following Krylov and Bogoliubov approximations in [20], a and ϕ are replaced with their average values over the period $T = \frac{2\pi}{\omega_n}$. By considering that, a and ϕ are constants in taking the values of average and apply the method of averaging, we obtain

$$\dot{a} = -\varepsilon \frac{1}{2\pi} \int_0^T \cos(\psi)\mathbb{F}(a\sin(\psi), a\omega_n \cos\psi, t)dt, \tag{59a}$$

$$\dot{\phi} = \varepsilon \frac{1}{2\pi a} \int_0^T \sin(\psi)\mathbb{F}(a\sin(\psi), a\omega_n \cos\psi, t)dt. \tag{59b}$$

where $\psi = \omega_n t + \phi$.

Once the integrals have been evaluated, then we have first order differential equations to obtain the amplitude and the phase angle and an approximate solution is obtained. In the case of existence of limit cycles, then the amplitudes of possible ones can be obtained from

$$\mathbb{G}(a) = \int_0^T \cos(\psi)\mathbb{F}(a\sin(\psi), a\omega_n \cos\psi, t)dt = 0. \tag{60}$$

By considering the existence of roots(a=a_1,a_2,a_3,...), then we obtain the amplitudes of the periodic solutions, consequently to get the condition of stability of the limit cycle, the following condition must be satisfied

$$\frac{d\mathbb{G}(a)}{da}|_{a=a_i} < 0, \qquad i = 1,2,3,.... \tag{61}$$

By applying, we have

$$\mathbb{G}(a) = \rho_1 a^4 - \rho_2 a^2, \tag{62}$$

then we obtain

$$a_1 = 0, \quad a_2 = \sqrt{\frac{\rho_2}{\rho_1}}, \tag{63}$$

where,

$$\begin{aligned}\rho_1 = &-\frac{\bar{\Gamma}\omega_n}{96}[-\frac{1}{(2\omega_n-\omega)}\sin\frac{2\pi(2\omega_n-\omega)}{\omega_n} + \frac{1}{(2\omega_n+\omega)}\sin\frac{2\pi(2\omega_n+\omega)}{\omega_n} \\ &-\frac{1}{(4\omega_n+\omega)}\sin\frac{2\pi(4\omega_n+\omega)}{\omega_n} + \frac{1}{(4\omega_n-\omega)}\sin\frac{2\pi(4\omega_n-\omega)}{\omega_n}] - \frac{3\pi}{4}\bar{e}\omega_n^3\end{aligned} \tag{64a}$$

$$\rho_2 = \bar{c}\pi\omega_n - \frac{\bar{\Gamma}\omega_n}{4}[\frac{1}{2\omega_n-\omega}\sin(2\omega_n-\omega)\frac{2\pi}{\omega_n} + \frac{1}{2\omega_n+\omega}\sin\frac{2\pi(2\omega_n+\omega)}{\omega_n}]. \tag{64b}$$

By considering a special case to verify the theoretical result of Eq.63, we compare it with the following numerical results by plotting the solution and the phase plane of the continuous governing equation of gyro(Eq.39). As shown in Fig.2 for $\alpha = 10$, $\beta = 1$, $c = 0.1$, $e = 0.05$, $\Gamma = 1$ and $\omega = 3$, a stable limit cycle is obtained with radius around one, which roughly fits the theoretical result.

3.3 Stability analysis of gyro relative equilibria

From Eq.38

$$\ddot{\theta} = P(t)\theta + Q(t)\theta^3 - c\dot{\theta} - e\dot{\theta}^3, \tag{65}$$

where

$$P(t) = -\omega_n^2 - \Gamma\sin\omega t, \qquad Q(t) = -\Omega + \frac{\Gamma\sin\omega t}{6}, \tag{66}$$

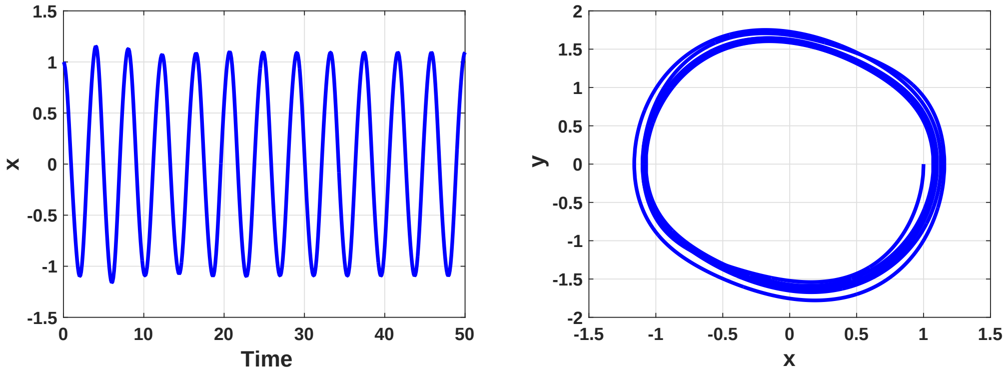

Fig. 2 Existence of stable limit cycle with $\alpha = 10$, $\beta = 1$, $c = 0.1$, $e = 0.05$, $\Gamma = 1$ and $\omega = 3$.

By using the following assumptions

$$\theta = x, \quad \dot{\theta} = y. \tag{67}$$

Then, Eq.65 reads

$$\dot{x} = y, \quad \dot{y} = P(t)x + Q(t)x^3 - cy - ey^3. \tag{68}$$

Following the Lyapunov stability analysis in [4] at the equilibrium points $(0,0)$ and $(0,\pm\pi)$, let the disturbance of motion be at $(0,0)$ and using Lyapunov function,

$$V(t,x,y) = \frac{1}{2}P(t)x^2 + 0.5y^2 + xy. \tag{69}$$

Then, this function is positive definite under condition of $V > 0$ $P(t) > 1$ and $1 + \Gamma + \omega_n^2 < 0$. To obtain asymptotic stability conditions where $-\dot{V} > 0$,

$$\dot{V}(t,x,y) = (P(t) - \frac{\Gamma\omega\cos\omega t}{2})x^2 + (2P(t) - c)xy + (1-c)y^2 - ey^4. \tag{70}$$

Thus, the requirements to obtain asymptotically stable node at the zero position are

$$c > 1, \quad c > 2P(t) \tag{71a}$$

$$\omega_n^2 + \Gamma\sin\omega t + \frac{1}{2}\Gamma\omega\cos\omega t > 0 \tag{71b}$$

$$\frac{(c - 2P(t))^2}{4(1-c)} > \omega_n^2 + f\sqrt{1 + \frac{\omega^2}{4}}. \tag{71c}$$

The inequalities in Eq.71 are conditions for the asymptotic stable solution of the vibrating axi-symmetric gyro equation at the fixed point $(0,0)$. We can verify this condition by the following numerical case: $\alpha = 10$, $\beta = 1$, $c = 0.5$, $e = 0.2$, $\omega = 3$ and $\Gamma = 1$ as shown in Fig.3 to assure the theoretical conditions at $(0,0)$.
The second relative equilibrium points are at $(\theta,\dot{\theta}) = (\pm\pi,0)$ where the gyro is inverted to down or up respectively. Considering the downward position,

$$\theta = x + \pi, \quad \dot{\theta} = y. \tag{72}$$

Then, the system reads

$$\dot{x} = y, \quad \dot{y} = P(t)(x+\pi) - cy - ey^3 + Q(t)(x+\pi)^3. \tag{73}$$

Similarly, we use Lyapunov function

$$V(t,x,y) = \frac{1}{2}(P(t) - 3\pi^2 Q(t))x^2 + \frac{1}{2}y^2 + xy. \tag{74}$$

This equation verifies Lyapunov function condition where $V(t,x,y) > 0$ under the condition $P(t) - 3\pi^2 Q(t) > 0$. If we test the conditions for stability where $-\dot{V} > 0$, then we obtain

$$c > 1, \quad c > 2P(t) - 3\pi^2 Q(t) \tag{75a}$$

$$P(t) < \frac{1}{2}(1 + \frac{\pi^2}{2})\Gamma\omega\cos\omega t \tag{75b}$$

$$\frac{\alpha}{4}(1-\pi^2) + \beta(1+\frac{\pi^2}{2}) - \frac{(c - 2P(t) + 3\pi^2 Q(t))^2}{4(c-1)} > (1+\frac{\pi^2}{2})\Gamma\sqrt{1+\frac{\omega^2}{4}}. \tag{75c}$$

By using the Lyapunov theorem, the inequalities in Eq.75 are sufficient conditions for asymptotic stability at the fixed points $(-\pi, 0)$.

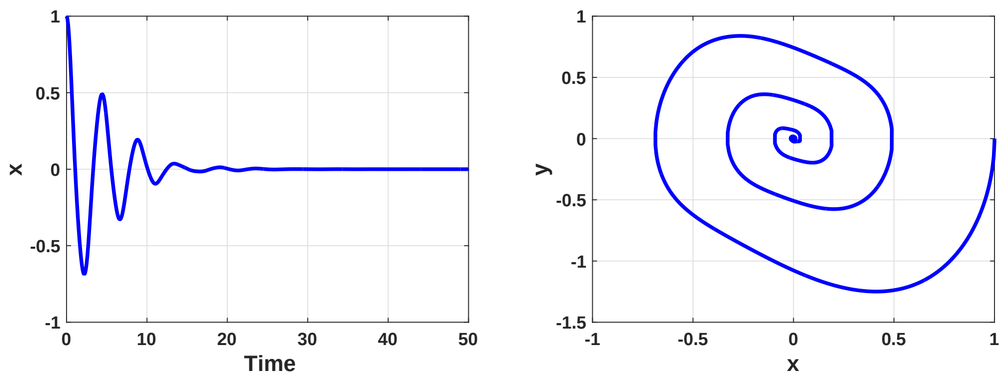

Fig. 3 Solution and phase plane in case of a stable motion at $\alpha = 10$, $\beta = 1$, $c = 0.5$, $e = 0.2$, $\Gamma = 1$ and $\omega = 3$.

3.4 Homoclinic bifurcation and transition to chaos

To investigate the existence of the homoclinic bifurcation, we use firstly the following approximate or perturbed equation of the gyro system to deduce such critical values using Melnikov's function,

$$\ddot{\theta} + \omega_n^2\theta + \Omega\theta^3 = \varepsilon\mathbb{H}(\theta, \dot{\theta}, t), \tag{76}$$

where

$$\mathbb{H} = -\bar{c}\dot{\theta} - \bar{e}\dot{\theta}^3 - \bar{\Gamma}\sin\omega t\,\theta + \frac{\bar{\Gamma}}{6}\sin(\omega t)\theta^3, \tag{77}$$

Theorem 6. *Eq.76 admits a homoclinic bifurcation if* $\omega_n^2 < 0$ *or* $\beta > 1 + \frac{\alpha}{4}$.

Proof. Under the condition of $\beta = 1 + \frac{\alpha}{4}$, the unperturbed equation can be represented as

$$\ddot{\theta} + \omega_n^2\theta + \Omega\theta^3 = 0, \tag{78}$$

then, at the point $(0,0)$ for $\omega_n^2 < 0$ and $\Omega > 0$, the unperturbed system has homoclinic paths meeting at the origin. Consequently, the perturbed system has a homoclinic bifurcation at the origin. □

When we take the assumed values of $\omega_n = -1$, $\Omega = 1$ as a special case to deduce the critical values of the system that generates the homoclinic bifurcation of vibrating axi-symmetric gyro equation, then the following unperturbed equation of the system($\varepsilon \approx 0$) reads

$$\dot{x}_o = y_o, \quad \dot{y}_o = x_o - x_o^3. \tag{79}$$

Then, the solution reads

$$x_o(t) = \sqrt{2}sech(t - t_o), \qquad y_o t = -\sqrt{2}sech(t - t_o)\tanh(t - t_o). \tag{80}$$

Melnikov's function can be expressed as

$$M(t_o) = \int_{-\infty}^{\infty} y_o(t - t_o)\mathbb{H}(x_o, y_o, t)dt. \tag{81}$$

Using the theory of residues, we obtain

$$M(t_o) = \bar{\Gamma}\pi(\omega^4 + \frac{5}{6}\omega^2 - \frac{35}{36})cosech\frac{\omega\pi}{2} - \frac{4}{3}\bar{c} + \frac{16}{63}\bar{e}. \tag{82}$$

To calculate the critical value of $\bar{\Gamma}$ as $M(t_o) = 0$, we have

$$\bar{\Gamma}_c = \frac{\frac{4}{3}\bar{c} - \frac{16}{36}\bar{e}}{\pi(\omega^4 + \frac{5}{6}\omega^2 - \frac{35}{36})}\sinh\frac{\omega\pi}{2}. \tag{83}$$

The critical values of the amplitude can be verified numerically when we take $c = 0.5$, $\omega = 1$, $e = 0$ and $e = 0.1$ then it is around $\bar{\Gamma}_c = 0.522$ and $\bar{\Gamma}_c = 0.529$ respectively. These results can be shown in Fig.4 of bifurcation diagrams. It is obviously from the pre-analysis of bifurcation diagrams for varying values of revealed parameter Γ that, there is one chaotic region found without windows of periodicity in the region of $\Gamma \geq 0.522$ and $\Gamma \geq 0.529$ corresponding to $e = 0$ and $e = 0.1$ respectively.

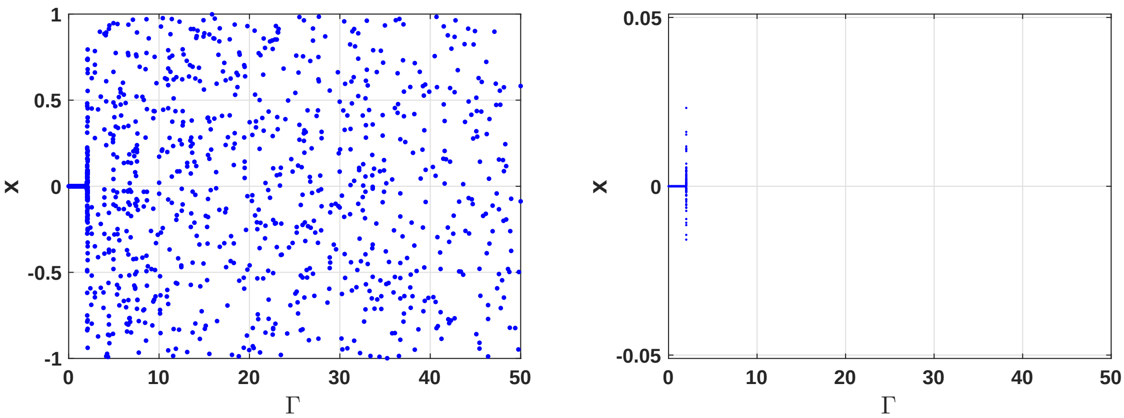

Fig. 4 Bifurcation diagrams in case of $c = 0.5$, $\omega = 1$, $e = 0$ (left) and $e = 0.1$ (right).

To obtain a large insight of the chaotic behaviour, we analyze the nonlinear behavior of the vibrating axi-symmetric gyro equation numerically by using the Range–kutta fourth-order method solver to determine the transition to chaotic stages using the phase plane trajectories. In Fig.5 and Fig.6, it can be easily noticed that the output of the system transit from double route bifurcation to complex chaotic behaviour when the normalized amplitude Γ is increased further. By the way, one can also notice that the damping coefficient e has an effect to delay or to slow down the transition to the chaos region.

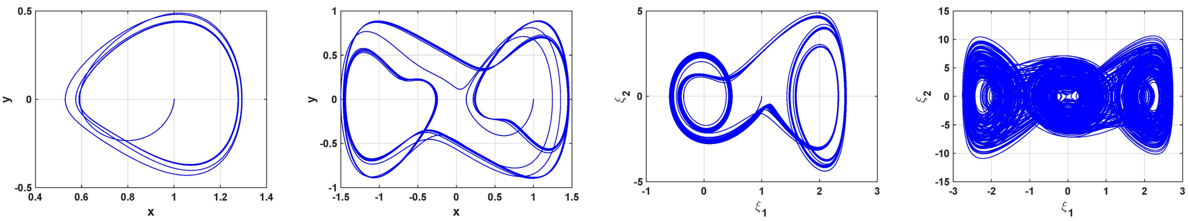

Fig. 5 Chaos diagrams in case of $c = 0.5$, $\omega = 1$, $e = 0$ at Γ=0.522,1,25 and 50 from to left to right.

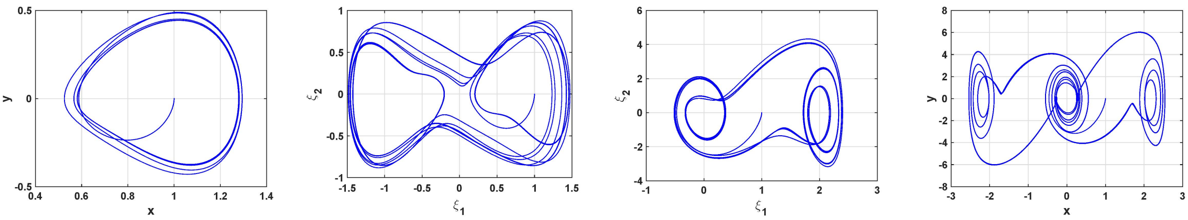

Fig. 6 Chaos diagrams in case of $c = 0.5$, $\omega = 1$, $e = 0.1$ at Γ=0.529,1,25 and 50 from to left to right.

4 Conclusion

In this work, special type of nonlinear ordinary differential equations called the Rayleigh–Duffing equation is qualitatively studied by seeking the stability of solutions and existence of periodic solutions via the fixed point method and the second method of Lyapunov. An engineering application governed by the Rayleigh–Duffing equation represented by a motion of vibrating axi-symmetric gyro is investigated based on the deduced theoretical results. In general, it is concluded that the gyro motion under a driven periodic force is affected by the variation of the normalized excitation amplitude from stable or periodic to a complex chaotic dynamic. An approximate measure of the periodic solution and its amplitude using the perturbed forms is deduced. The stability of equilibria of the gyro motion have been explicated using Lyapunov stability analysis. The existence of the homoclinic bifurcation and the transition to chaos by obtaining the range of chaotic behaviour with respect to the normalized value of excitation amplitude are shown. Lastly, all theoretical results are verified and fit well with the numerical ones.

Conflict of Interest: The authors declare that they have no conflicts of interest.

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