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(k, m) -type slant helices for partially null and pseudo null curves in Minkowski space $\mathbb{E}_1^4$

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Abstract

In this study we define the notion of (k, m) -type slant helices in Minkowski 4-space and express some characterizations for partially and pseudo null curves in $\mathbb{E}_1^4$.

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1 Introduction

The curve theory has been one of the most studied research area because of having many application area from geometry to the various branch of science. Especially the characterizations on the curvature and torsion play important role to define special curve types such as so-called helices. The curves of this type have drawn great attention in science. Helices appear naturally in structures of DNA, nanosprings. They are also widely used in engineering and architecture. The concept of slant helix defined by Izumiya and Takeuchi [6] based works have been studied in various spaces. For instance in [1] authors extended slant helix concept to E^n and conclude that there are no slant helices with non-zero constant curvatures in the space E^4 . The subject is also considered in 3-, 4-, and n -dimensional Euclidian spaces, respectively in [7, 10, 12]. Moreover different properties of helices are also considered in [8–11, 13, 18]. On the other hand in A.T. Ali, R. Lopez and M. Turgut extended this study to the k -type slant helix in E_1^4 . In this study they called α curve as k -type slant helix if there exists on (non-zero) constant vector field $U \in E_1^4$ such that $\langle V_{k+d}, U \rangle = \text{const}$, for $0 \leq k \leq 3$. Here V_{k+1} shows the Frenet vectors of this curve [2].

One may easily conclude that O-type slant helices are general helices and 1-type slant helices correspond just slant helices. k -type slant helices for partially null and pseudo null curves are also studied. In accordance

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with above studies, the authors introduced (k, m) -type slant helices in E^4 and we show that there do not exist $(1, m)$ type slant helices in E^4 [15].

In the present work, we define the notion of (k, m) -type slant helices in Minkowski 4-space and express some characterizations for partially and pseudo null curves in E_1^4 .

2 Preliminaries

Because of the indefiniteness of the Lorentzian metric g in $\mathbb{E}_1^4$ vector v in this space can have one of three causal characters called spacelike ($g(u, u) > 0$ or $u = 0$), timelike ($g(u, u) < 0$) and lightlike (null) ($g(u, u) = 0, u \neq 0$), respectively. In accordance with the metric, a curve in $\mathbb{E}_1^4$ is called spacelike, timelike or lightlike if its velocity vectors at $\alpha'(s)$ are spacelike, timelike and lightlike, respectively. In $\mathbb{E}_1^4$, if u is a unit vector, we know that $g(u, u) = \pm 1$ and the norm of a vector u is given by $\|u\| = \sqrt{|g(u, u)|}$. In addition a spacelike or timelike curve is said to be arclength parametrized if a $\alpha'(s)$ is a unit vector for any s [2].

Suppose that $\alpha = \alpha(s)$ is a spacelike curve with its arclength parameter. The Frenet frame along the curve α can be denoted by $\{T(s), N(s), B_1(s), B_2(s)\}$. Here T, N, B_1, B_2 are called tangent, principal normal, the first binormal and the second binormal vector fields of the curve α , respectively. Because of the indefiniteness of the metric g , the vectors N, B_1 and B_2 have different causal characters. In this study we will assume that T is spacelike, since α is a spacelike curve.

Definition 2.1 A spacelike curve called partially null curve if N is spacelike and B_1 is lightlike [16, 17].

For partially null curves the second binormal B_2 is the only lightlike vector orthogonal to T and N such that $g(B_1, B_2) = 1$. The Frenet equations are given as follows

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & -\tau & 0 & -\sigma \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \quad [16, 17] \quad (2.1)$$

where κ, τ and σ are first, second and third curvature of the curve α , respectively. Note that after a null rotation of the ambient space the third curvature σ can be chosen as zero, and τ is determined up to a constant which means that any partially null curve lies in a three dimensional lightlike subspace orthogonal to B_1 .

Definition 2.2. A spacelike curve called pseudo null curve if $\alpha''(s)$ is a lightlike vector for all s where the normal vector is $N = T'$ [16, 17].

For the case N' is lightlike the curve α lies in the lightlike plane which we omit this trivial case. For the other cases, B_1 is a unit spacelike vector orthogonal to $\{T, N\}$ and B_2 is the only lightlike vector orthogonal to T and B_1 such that $g(N, B_2) = 1$. The Frenet equations are given as follows

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -0 & 0 & \tau & 0 \\ 0 & \sigma & 0 & -\tau \\ -\kappa & -0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \quad [16, 17] \quad (2.2)$$

In this case the first curvature κ , can take only 0 and 1 values. As is well known the curve is a straight line if curvature vanishes. We will focus on the cases $\kappa = 1$ and $\sigma\tau \neq 0$ [2].

3 (k, m) -type slant helices for partially null curves in $\mathbb{E}_1^4$

In this section, we will define (k, m) type partially null slant helices in $\mathbb{E}_1^4$.

From (2.1) we have that $\sigma = 0$. We also suppose that $\kappa, \tau \neq 0$.

Definition 3.1. Let α be a regular unit speed curve in $\mathbb{E}_1^4$ with Frenet frame $\{V_1, V_2, V_3, V_4\}$. We call α is a (k, m) type slant helix if there exists a non-zero constant vector field $U \in \mathbb{E}_1^4$ satisfies $\langle V_k, U \rangle = a$ and $\langle V_m, U \rangle = b$ (a, b constant) for $1 \leq k, m \leq 4, k \neq m$. The constant vector U is on axis of (k, m) -type slant helix. We decompose U with respect to Frenet frame $\{T, N, B_1, B_2\}$ as $U = u_1T + u_2N + u_3B_1 + u_4B_2$, where $u_i = u_i(s)$ are differentiable functions of s . Here we denote $V_1 = T, V_2 = N, V_3 = B_1, V_4 = B_2$. From now on, in sake of easinesss, we will use these notations and assume that $k_i \neq O, (1 \leq i \leq 3)$.

Theorem 3.1. There are no $(1, 2)$ type partially null slant helix in $\mathbb{E}_1^4$.

Proof. Assume that α is a $(1, 2)$ type slant helix. Then for a constant vector field U . $g(T, U) = a$ is constant and $g(N, U) = b$ is constant. Differentiating this equation and using Frenet equations, we obtain $\kappa g(N, U) = 0$ means that U is orthogonal to N . Hence there are no $(1, 2)$ type partially null slant helices in $\mathbb{E}_1^4$.

Theorem 3.2. Let α be a $(1, 3)$ partially null slant helix in $\mathbb{E}_1^4$. Then α is a general helix.

Proof. Assume that α is a $(1, 3)$ partially null slant helix in $\mathbb{E}_1^4$. Then we may write $\langle T, U \rangle = \text{const} = a, \langle B, U \rangle = \text{const} = b$. Also taking account Theorem 3.1 we decompose U as follows

$$U = \alpha T + bB_1 + u_1B_2 \quad (3.1)$$

$$0 = U' = a(\kappa N) + b(\sigma B_1) + u_1' B_2 + u_1(-\tau N - \sigma B_2) \quad (3.2)$$

$$0 = (a\kappa - u_1\tau)N + (b\sigma)B_1 + (u_1' - u_1\sigma)B_2 \quad (3.3)$$

$$a\kappa - u_1\tau = 0 \quad (3.4)$$

$$-(u_1' - u_1\sigma) = 0 \quad (3.5)$$

From Definition 2.1 we have chosen $\sigma = O$ which means that $u_1' = 0$ hence $u_1 = \text{constant}$. From (3.3) we get $\frac{\kappa}{\tau} = \text{constant}$. Then this completes the proof.

Theorem 3.3. Let α be a $(1, 4)$ partially null slant helix in $\mathbb{E}_1^4$. Then α is a general helix.

Proof. Assume that α is a $(1, 4)$ type slant helix. Then we may write

$$U = aT + u_1B_1 + bB_2 \quad (3.6)$$

$$0 = U' = a(\kappa N) + u_1' B_1 + u_1(\sigma B_1) + b(-\tau N - \sigma B_2) \quad (3.7)$$

$$0 = (a\kappa - b\tau)N + (u_1' + u_1\sigma)B_1 + b\sigma B_2 \quad (3.8)$$

From Definition 2.1 we have chosen $\sigma = O$ which means that $u_1' = 0$ hence $u_1 = \text{constant}$. From (3.3) we get $\frac{\kappa}{\tau} = \text{constant}$. Then this completes the proof.

Theorem 3.4. There are no $(2, 3)$ partially null type slant helix in $\mathbb{E}_1^4$.

Proof. Assume that α is a $(2, 3)$ partially null type slant helix in $\mathbb{E}_1^4$. Then we may write

$$U = u_1T + aN + bB_1 + u_2B_2 \quad (3.9)$$

$$0 = U' = u_1' T + u_1(\kappa N) + a(-\kappa T + \tau B_1) + b(\sigma B_1) + u_2' B_2 + u_2(-\tau N - \sigma B_2) \quad (3.10)$$

$$0 = U' = (u_1' - a\kappa)T + (u_1\kappa - u_2\tau)N + (a\tau + b\sigma)B_1 + (u_2' - u_2\sigma)B_2 \quad (3.11)$$

Taking into account of $\sigma = O$ we get $a = O$ which is a contradiction. Hence there are no $(2, 3)$ type slant helix in $\mathbb{E}_1^4$.

Theorem 3.5. There are no $(2, 4)$ type partially null slant helix in $\mathbb{E}_1^4$.

Proof. Assume that α is a $(2, 4)$ partially null type slant helix in $\mathbb{E}_1^4$. Then we may write

$$U = u_1T + aN + u_2B_1 + bB_2 \quad (3.12)$$

$$\begin{aligned} 0 = U' &= u_1' T + u_1(\kappa N) + a(-\kappa T + \tau B_1) + u_2' B_1 + u_2(\sigma B_1) + b(-\tau N - \sigma B_2) \\ &= (u_1' - a\kappa)T + (u_1\kappa - b\tau)N + (a\tau + u_2' + u_2\sigma)B_1 + (-b\sigma)B_2 \end{aligned} \quad (3.13)$$

Noting that $\sigma = O$ we get $b = O$ which means that there are no $(2, 4)$ type partially null type slant helix in $\mathbb{E}_1^4$.

Theorem 3.6. There are no (3, 4) type partially null slant helix in $\mathbb{E}_1^4$.

Proof. Assume that α is a (3, 4) type partially null slant helix in $\mathbb{E}_1^4$. Then we may write

$$U = u_1T + u_2N + aB_1 + bB_2 \quad (3.14)$$

$$0 = U' = u'_1T + u_1(\kappa N) + u'_2N + u_2(-\kappa T + \tau B_1) + a(\sigma B_1) + b(-\tau N - \sigma B_2) \quad (3.15)$$

$$0 = (u'_1 - u_2\kappa)T + (u_1\kappa + u'_2 - b\tau)N + (u_2\tau + a\sigma)B_1 + (u_2\tau + a\sigma)B_1 + (-b\sigma)B_2 \quad (3.16)$$

In virtue of $\sigma = 0$ we get $b = 0$ which means that there are no (3, 4) type partially null slant helix in $\mathbb{E}_1^4$.

4 (k, m) -type slant helices for pseudo null curves in $\mathbb{E}_1^4$

In this part, we will focus on the (k, m) -type pseudo null slant helices. Recall that we assume $\kappa = 1$ and $\sigma, \tau \neq 0$.

Theorem 4.1. There are no (1, 2) type pseudo null slant helix in $\mathbb{E}_1^4$.

Proof. Assume that α is a (1, 2) type pseudo null slant helix. Then for a constant vector field U , $g(T, U) = a$ is constant and $g(N, U) = b$ is constant. Differentiating this equation and using Frenet equations, we obtain $\kappa g(N, U) = 0$ means that U is orthogonal to N . Hence there are no (1, 2) type pseudo null slant helices in $\mathbb{E}_1^4$.

Theorem 4.2. There are no (1, 3) type pseudo null slant helix in $\mathbb{E}_1^4$.

Proof. Assume that α is a (1, 3) type pseudo null slant helix in $\mathbb{E}_1^4$. Then we may write $\langle T, U \rangle = \text{const} = a$, $\langle B, U \rangle = \text{const} = b$. Also taking account Theorem 3.1 we decompose U as follows

$$U = aT + bB_1 + u_1B_2 \quad (4.1)$$

$$0 = U' = a(N) + b(\sigma N - \tau B_2) + u'_1B_2 + u_1(-T - \sigma B_1) \quad (4.2)$$

$$0 = (-u_1)T + (a + b\sigma)N + (-u_1\sigma)B_1 + (u'_1 - b\tau)B_2 \quad (4.3)$$

$$u_1 = 0 \quad (4.4)$$

$$(a + b\sigma) = 0 \quad (4.5)$$

$$u'_1 + b\tau = 0 \quad (4.6)$$

From (3.25) and (3.27) we conclude $b = 0$ which is a contradiction. Hence there do not exist (1, 3) type pseudo null slant helix in $\mathbb{E}_1^4$.

Theorem 4.3. There are no (1, 4) type pseudo null slant helix in $\mathbb{E}_1^4$.

Proof. Assume that α is a (1, 4) type pseudo null slant helix in $\mathbb{E}_1^4$. Then we decompose U as follows

$$U = aT + u_1B_1 + bB_2 \quad (4.7)$$

$$0 = U' = a(N) + b(-T - \sigma B_1) + u'_1B_1 + u_1(\sigma N - \tau B_2) \quad (4.8)$$

$$0 = (-b)T + (a + u_1\sigma)N + (u'_1 - b\sigma)B_1 + (-u_1\tau)B_2 \quad (4.9)$$

Using (3.25) we get $b = 0$ which is a contradiction. Hence there do not exist (1, 4) type pseudo null slant helix in $\mathbb{E}_1^4$.

From the theorem proved above we conclude the following corollary.

Corollary 4.1. There do not exist (1, k) type pseudo null slant helix in $\mathbb{E}_1^4$ for $2 \leq k \leq 4$.

Theorem 4.4. Let α be a (2, 3) type pseudo null slant helix in $\mathbb{E}_1^4$ if and only

$$-\left(a\sigma \frac{\int \tau ds}{\tau}\right) - b \int \tau ds = 0$$

Proof. Assume that α is a (2, 3) type pseudo null slant helix in $\mathbb{E}_1^4$. Then we decompose U as follows

$$U = u_1T + aN + bB_1 + u_2B_2 \quad (4.10)$$

$$0 = U' = u_1'T + u_1(N) + a(\tau B_1) + b(\sigma N - \tau B_2) + u_2'B_2 + u_2(-T - \sigma B_1) \quad (4.11)$$

$$0 = (u_1' - u_2)T + (u_1 + b\sigma)N + (a\tau - u_2\sigma)B_1 + (u_2' - b\tau)B_2 \quad (4.12)$$

Hence we get

$$u_1' - u_2 = 0 \quad (4.13)$$

$$u_1 + b\sigma = 0 \quad (4.14)$$

$$a\tau - u_2\sigma = 0 \quad (4.15)$$

$$(u_2' - b\tau) = 0 \quad (4.16)$$

Using (4.17) we get $u_2 = b \int \tau ds$ and taking into account of (4.15) with (4.14) we get $u_1 = -a\sigma \frac{\int \tau ds}{\tau}$. Considering these facts in (4.13) we conclude the desired proof.

Corollary 4.2 An axis of a (2, 3) type pseudo null slant helix is the vector given by

$$D = \left[\sigma \frac{\int \tau ds}{\tau} \right] T + N + \frac{\int \tau ds}{\tau} B_1 + \left[b \int \tau ds \right] B_2$$

Theorem 4.5. There are no (2, 4) type pseudo null slant helix in $\mathbb{E}_1^4$.

Proof. Assume that α is a (2, 4) type pseudo null slant helix in $\mathbb{E}_1^4$. Then we decompose U as follows

$$U = u_1T + aN + u_2B_1 + bB_2 \quad (4.17)$$

$$0 = U' = u_1'T + u_1(N) + a(\tau B_1) + b(-T - \sigma B_1) + u_2'B_2 + u_2(\sigma N - \tau B_2) \quad (4.18)$$

$$0 = (u_1' - b)T + (u_1 + u_2\sigma)N + (u_2' + a\tau - b\sigma)B_1 + (-u_2\tau)B_2 \quad (4.19)$$

$$u_1' - b = 0 \quad (4.20)$$

$$u_1 + u_2\sigma = 0 \quad (4.21)$$

$$u_2' + a\tau - b\sigma = 0 \quad (4.22)$$

$$-u_2\tau = 0 \quad (4.23)$$

From (4.23) we conclude that $u_2 = 0$ and considering this fact in (4.20) we also see that $u_1 = 0$ which means a contradiction. Hence there do not exist (2, 4) type pseudo null slant helix in $\mathbb{E}_1^4$.

Theorem 4.6. There are no (3, 4) type pseudo null slant helix in $\mathbb{E}_1^4$.

Proof. Assume that α is a (3, 4) type pseudo null slant helix in $\mathbb{E}_1^4$. Then we decompose U as follows

$$U = u_1T + u_2N + aB_1 + bB_2 \quad (4.24)$$

$$0 = U' = u_1'T + u_1(N) + u_2'N + u_2(\tau B_1) + a(\sigma N - \tau B_2) + b(-T - \sigma B_1) \quad (4.25)$$

$$0 = (u_1' - b)T + (u_1 + u_2' + a\sigma)N + (u_2\tau - b\sigma)B_1 + (-a\tau)B_2 \quad (4.26)$$

$$u_1' - b = 0 \quad (4.27)$$

$$u_1 + u_2' + a\sigma = 0 \quad (4.28)$$

$$u_2\tau - b\sigma = 0 \quad (4.29)$$

$$-a\tau = 0 \quad (4.30)$$

Taking into account of (3.55) we see that $a = 0$ and this is a contradiction.

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