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Exact Travelling Wave Solutions for Space-Time Fractional Klein-Gordon Equation and (2+1)-Dimensional Time-Fractional Zoomeron Equation via Auxiliary Equation Method

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Abstract

In this paper, the new exact solutions of nonlinear conformable fractional partial differential equations(CFPDEs) are achieved by using auxiliary equation method for the nonlinear space-time fractional Klein-Gordon equation and the (2+1)-dimensional time-fractional Zoomeron equation. The technique is easily applicable which can be applied successfully to get the solutions for different types of nonlinear CFPDEs. The conformable fractional derivative(CFD) definitions are used to cope with the fractional derivatives.

Keywords: Space-time fractional Klein-Gordon equation; Time-fractional Zoomeron Equation; Conformable fractional derivative; Auxiliary equation method

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1 Introduction

Fractional calculus first started in 1600s, with the question of G.W. Leibnitz, can integer-order derivative be generalized for non-integer derivatives?, to L'Hospital [1]. In the last few decades, fractional differential equation (FDE) has been rediscovered by applied scientists, proving to be very useful in various fields: physics (classic and quantum mechanics, thermodynamics, fluid mechanics, optics, plasma etc.), chemistry, biology, economics, engineering, signal and image processing, and control theory and so on. In recent years, many important definitions of FDEs, Riemann–Liouville, Caputo, Grunwald–Letnikov and conformable derivative etc., are found to solve the nonlinear FDEs [2–5]. Among them, the conformable derivative definition is newly defined by Khalil and et al. [6], which is very closer to definition of general calculus. So it is more applicable and practical than the other definitions. Many effective methods and techniques for obtaining the exact solutions of nonlinear FDEs, such as, Adomian decomposition method [7], the homotopy perturbation method [8], the ansatz method [9,10], the sub-equation method [11, 12], the exp-function method [13, 14], the first integral

method [15, 16], the functional variable method [17, 18], (G'/G) expansion method [19, 20], the extended trial equation method [21], the modified simple equation method [22], the modified Kudryashov method [23], the $\exp(\phi(\epsilon))$ -expansion method [24], the modified trial equation method [25], the finite difference method [26] and so on., have been used and improved various group of mathematicians and scientist.

In this study, first we examine the exact solutions of the space-time nonlinear fractional Klein-Gordon equation, which is well known, linear and nonlinear Klein-Gordon equations model many problems in classical and quantum mechanics, solitons, and condensed matter physics [23, 27, 28]. Then we observe the (2+1) dimensional time-fractional Zoomeron Equation, which is a convenient model to display the novel phenomena associated with boomerons and trappons is studied [29, 30]. These two equations have been solved using different methods before and we have found new exact solutions by using the auxiliary equation method.

The rest of this paper is arranged as follows: In Section 2, we describe the key ideas of the conformable fractional derivative. In Section 3, we consume auxiliary equation method to search the exact solutions of the space-time fractional Klein-Gordon equation and (2+1)-dimensional time-fractional Zoomeron Equation. Finally, Section 4 consists of the conclusion.

2 Preliminaries

In this section, we reviewed some identity and characteristics of the conformable fractional derivative that are exposed further in this article.

Definition 2.1. Given a function $f : [0, \infty) \rightarrow \mathbb{R}$, then the CFD of f of order α is defined by

$$T_{\alpha}(f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \quad (1)$$

for all $t > 0$, $\alpha \in (0, 1)$. If the CFD of f of order α exists, then we simply say f is α -differentiable [31, 32].

Properties: Let $\alpha \in (0, 1)$ and f, g be α -differentiable at a point $t > 0$, then some properties of the CRD are as follows [31, 32]:

1. $T_{\alpha}(af + bg) = aT_{\alpha}(f) + bT_{\alpha}(g)$, $a, b \in \mathbb{R}$,
2. $T_{\alpha}(t^p) = pt^{p-\alpha}$, $p \in \mathbb{R}$,
3. $f(t) = \lambda \in \mathbb{R}$, for all constant functions $T_{\alpha}(\lambda) = 0$,
4. $T_{\alpha}(fg) = fT_{\alpha}(g) + gT_{\alpha}(f)$,
5. $T_{\alpha}\left(\frac{f}{g}\right) = \frac{gT_{\alpha}(f) - fT_{\alpha}(g)}{g^2}$,
6. If, in addition to f differentiable, then $T_{\alpha}(f)(t) = t^{1-\alpha} \frac{df}{dt}(t)$.

Theorem 2.1. (Chain Rule) Assume functions $f, g : [0, \infty) \rightarrow \mathbb{R}$ be α -differentiable, then the following rule is obtained

$$T_{\alpha}(f \circ g)(t) = t^{1-\alpha} g'(t) f'(g(t)), \quad (2)$$

where $0 < \alpha \leq 1$ [31, 32].

Definition 2.2. Let $0 < \alpha \leq 1$ and $0 \leq a < b$. A function $f : [a, b] \rightarrow \mathbb{R}$ is α -fractional integrable on $[a, b]$ if the integral

$$I^{\alpha} f(x) = \int_a^b f(x) d_{\alpha} x = \int_a^b f(x) x^{\alpha-1} dx, \quad (3)$$

exist and is finite [33].

Theorem 2.2. Let $f \in C[a, b]$ and $0 < \alpha \leq 1$. Then [34],

$$\frac{d^{\alpha}}{dx^{\alpha}} I^{\alpha} f(x) = f(x). \quad (4)$$

3 Description of the Method

A brief description of the method is presented in this section. For this purpose, consider the following nonlinear CFPDE,

$$F(u, D_t^\alpha u, u_{x_1}, u_{x_2}, u_{x_3}, \dots, u_{x_m}, D_t^{2\alpha} u, u_{x_1 x_1}, u_{x_2 x_2}, u_{x_3 x_3}, \dots) = 0, \quad 0 < \alpha \leq 1, \quad (5)$$

using the transformation,

$$U(\xi) = u(x_1, x_2, x_3, \dots, x_m, t), \text{ and } \xi = l_1 x_1 + l_2 x_2 + \dots + l_m x_m + k \frac{t^\alpha}{\alpha}, \quad (6)$$

where l_i ($i = 1, 2, 3, \dots, m$) and k are non zero constants, Eq.(5) converted into nonlinear ordinary differential equation as

$$P(U(\xi), U'(\xi), U''(\xi), \dots) = 0. \quad (7)$$

By virtue of the extended tanh-function method we assume that the solution of the Eq. (7) is of the form

$$U(\xi) = \sum_{i=0}^n a_i z^i(\xi), \quad (8)$$

in which a_i ($i = 1, 2, 3, \dots, n$), l_i ($i = 1, 2, 3, \dots, m$) and k are all constant to be determined, the balancing number n is a positive integer which can be determined by balancing the highest order derivative terms with the highest power nonlinear terms in the Eq. (7) and $z(\xi)$ expresses the solutions of the following auxiliary ordinary differential equation

$$\left(\frac{dz}{d\xi}\right)^2 = az^2(\xi) + bz^3(\xi) + cz^4(\xi), \quad (9)$$

where a, b, c are parameters [30, 35]. We substitute the Eqs. (8) and (9) with necessary derivatives into the Eq. (7) with computerized symbolic computation, equating to zero the coefficients of all powers of $z(\xi)$ yields a set of algebraic equations for a, b, c, a_i ($i = 1, 2, 3, \dots, n$), l_i^i ($i = 1, 2, 3, \dots, m$) and k . Finally by inserting each solutions of this set of algebraic equation into the Eqs. (8) and (9) by setting $\xi = l_1 x_1 + l_2 x_2 + \dots + l_m x_m + k \frac{t^\alpha}{\alpha}$, then we obtain the exact travelling wave solutions of the Eq.(5). We observe that some solutions the algebraic equations lead to the trivial solutions or the singular solutions which we do not consider here.

4 Applications

In this section, we consider two nonlinear CFDEs as an application of the auxiliary equation method.

4.1 Space-Time Fractional Klein-Gordon Equation

Let us consider nonlinear space-time fractional Klein-Gordon equation as follow [28]:

$$D_{tt}^{2\alpha} u - D_{xx}^{2\alpha} u + pu - qu^2 = 0, \quad (10)$$

where p and q are nonzero constants. Taking the travelling wave transformation

$$u(x, t) = U(\xi), \quad \xi = l \frac{x^\alpha}{\alpha} - k \frac{t^\alpha}{\alpha}, \quad (11)$$

where k and l are nonzero constants, the Eq. (10) converts into ordinary equation as follow:

$$(k^2 - l^2) U'' + pU - qU^2 = 0. \quad (12)$$

The balancing number is found 2 by balancing the highest order derivative term with the highest power nonlinear term. Then the solution of the Eq. (12) becomes

$$U(\xi) = a_0 + a_1 z(\xi) + a_2 z^2(\xi). \quad (13)$$

Substituting the Eqs. (9) and (13) into the Eq. (12), we find an algebraic equation with powers of $z(\xi)$. Equating the all powers of $z(\xi)$ to zero, we get the following system:

$$z^0(\xi) : qa_0 - ka_0^2 = 0, \quad (14)$$

$$z^1(\xi) : qa_1 - 2ka_0a_1 + p^2aa_1 - l^2a_1a = 0, \quad (15)$$

$$z^2(\xi) : \frac{3}{2}p^2a_1b - \frac{3}{2}l^2a_1b + qa_2 - ka_1^2 - 2ka_0a_2 + 4p^2a_2a - 4l^2a_2a = 0, \quad (16)$$

$$z^3(\xi) : 5p^2a_2b - 5l^2a_2b + 2p^2a_1c - 2l^2a_1c - 2ka_1a_2 = 0. \quad (17)$$

$$z^4(\xi) : 6p^2a_2c - ka_2^2 - 6k^2a_2c = 0. \quad (18)$$

We find the six type of the following coefficients, from the solution of the system (14-18).

Type 1:

$$a_0 = 0, \quad a_1 = -\frac{3b(l^2 - p^2)}{2k}, \quad a_2 = 0 \quad a = \frac{1}{l^2 - p^2}, \quad b = b, \quad c = 0, \quad (19)$$

so we obtain the first and second results in two different cases as follow,

Case 1:

$$(z)_1(\xi) = -\frac{4q \exp(\xi \delta)}{(l^2 - p^2)(2b \exp(\xi \delta) - b^2 \exp(2\xi \delta) - 1)}, \quad (20)$$

$$(U)_1(\xi) = \frac{6bq \exp(\xi \delta)}{k(2b \exp(\xi \delta) - b^2 \exp(2\xi \delta) - 1)}, \quad (21)$$

where $\delta = \sqrt{\frac{1}{l^2 - p^2}}$ and $\xi = l\frac{x^\alpha}{\alpha} - k\frac{t^\alpha}{\alpha}$.

Case 2:

$$(z)_2(\xi) = -\frac{4q \exp(\xi \delta)}{l^2 - p^2(2b \exp(\xi \delta) - \exp(2\xi \delta) - b^2)}, \quad (22)$$

$$(U)_2(\xi) = \frac{6bq \exp(\xi \delta)}{k(2b \exp(\xi \delta) - \exp(2\xi \delta) - b^2)}, \quad (23)$$

where $\delta = \sqrt{\frac{1}{l^2 - p^2}}$ and $\xi = l\frac{x^\alpha}{\alpha} - k\frac{t^\alpha}{\alpha}$.

Type 2:

$$a_0 = \frac{1}{k}, \quad a_1 = -\frac{3b(l^2 - p^2)}{2k}, \quad a_2 = 0 \quad a = \frac{1}{l^2 - p^2}, \quad b = b, \quad c = 0, \quad (24)$$

so we obtain the third and fourth results in two different cases as follow,

Case 1:

$$(z)_3(\xi) = \frac{4q \exp(\xi \rho)}{(l^2 - p^2)(2b \exp(\xi \rho) - b^2 \exp(2\xi \rho) - 1)}, \quad (25)$$

$$(U)_3(\xi) = -\frac{q[1 + 4b \exp(\xi \rho) + b^2 \exp(2\xi \rho)]}{k(2b \exp(\xi \rho) - b^2 \exp(2\xi \rho) - 1)}, \quad (26)$$

where $\rho = \sqrt{\frac{1}{p^2 - l^2}}$ and $\xi = l \frac{x^\alpha}{\alpha} - k \frac{t^\alpha}{\alpha}$.

Case 2:

$$(z)_4(\xi) = -\frac{4q \exp(\xi \rho)}{(l^2 - p^2)(2b \exp(\xi \rho) - \exp(2\xi \rho) - b^2)}, \quad (27)$$

$$(U)_4(\xi) = -\frac{q[b^2 + 4b \exp(\xi \rho) + \exp(2\xi \rho)]}{k(2b \exp(\xi \rho) - \exp(2\xi \rho) - b^2)}, \quad (28)$$

where $\rho = \sqrt{\frac{1}{p^2 - l^2}}$ and $\xi = l \frac{x^\alpha}{\alpha} - k \frac{t^\alpha}{\alpha}$.

Type 3:

$$\begin{aligned} a_0 &= 0, & a_1 &= -\frac{3b(l^2 - p^2)}{k}, & a_2 &= -\frac{3b^2(l^2 - p^2)^2}{2kq}, \\ a &= \frac{1}{l^2 - p^2}, & b &= b, & c &= \frac{3b^2(l^2 - p^2)}{4q}, \end{aligned} \quad (29)$$

so we obtain the fifth and sixth results in two different cases as follow,

Case 1:

$$(z)_5(\xi) = -\frac{4q \exp(\xi \delta)}{(l^2 - p^2)(2b \exp(\xi \delta) - 1)}, \quad (30)$$

$$(U)_5(\xi) = -\frac{12bq \exp(\xi \delta)}{k[2b \exp(\xi \delta) - 1]^2}, \quad (31)$$

where $\delta = \sqrt{\frac{1}{l^2 - p^2}}$ and $\xi = l \frac{x^\alpha}{\alpha} - k \frac{t^\alpha}{\alpha}$.

Case 2:

$$(z)_6(\xi) = -\frac{4q \exp(\xi \delta)}{(l^2 - p^2)(2b \exp(\xi \delta) - \exp(2\xi \delta))}, \quad (32)$$

$$(U)_6(\xi) = -\frac{12bq \exp(\xi \delta)}{k[\exp(\xi \delta) - 2b]^2}, \quad (33)$$

where $\delta = \sqrt{\frac{1}{l^2 - p^2}}$ and $\xi = l \frac{x^\alpha}{\alpha} - k \frac{t^\alpha}{\alpha}$.

Type 4:

$$\begin{aligned} a_0 &= \frac{q}{k}, & a_1 &= -\frac{3b(l^2 - p^2)}{l}, & a_2 &= \frac{3b^2(l^2 - p^2)^2}{2lq}, \\ a &= -\frac{1}{l^2 - p^2}, & b &= b, & c &= -\frac{b^2(l^2 - p^2)}{4q}, \end{aligned} \quad (34)$$

so we obtain the seventh and eighth results in two different cases as follow,

Case 1:

$$(z)_7(\xi) = \frac{4q \exp(\xi \rho)}{(l^2 - p^2)(2b \exp(\xi \rho) - 1)}, \quad (35)$$

$$(U)_7(\xi) = \frac{q[1 + 8b \exp(\xi \rho) + 4b^2 \exp(2\xi \rho)]}{k[2b \exp(\xi \rho) - 1]^2}, \quad (36)$$

where $\rho = \sqrt{\frac{1}{l^2 - p^2}}$ and $\xi = l \frac{x^\alpha}{\alpha} - k \frac{t^\alpha}{\alpha}$.

Case 2:

$$(z)_8(\xi) = \frac{4q \exp(\xi \rho)}{(l^2 - p^2)[2b \exp(\xi \rho) - \exp(2\xi \rho)]}, \quad (37)$$

$$(U)_8(\xi) = \frac{q[4b^2 + 8b \exp(\xi \rho) + \exp(2\xi \rho)]}{k[\exp(\xi \rho) - 2b]^2}, \quad (38)$$

where $\delta = \sqrt{\frac{1}{l^2 - p^2}}$ and $\xi = l \frac{x^\alpha}{\alpha} - k \frac{t^\alpha}{\alpha}$.

Type 5:

$$a_0 = \frac{q}{k}, \quad a_1 = 0, \quad a_2 = -\frac{6c(l^2 - p^2)}{k}, \quad a = -\frac{1}{4(l^2 - p^2)}, \quad b = 0, \quad c = c, \quad (39)$$

so we obtain the ninth and tenth results in two different cases as follow,

Case 1:

$$(z)_9(\xi) = -\frac{q \exp(\frac{1}{2}\xi \rho)}{qc \exp(\xi \rho) + l^2 - p^2}, \quad (40)$$

$$(U)_9(\xi) = \frac{q[q^2 c^2 \exp(2\xi \rho) + 4qc p^2 \exp(\xi \rho) - 4qcl^2 \exp(\xi \rho) + p^4 - 2p^2 l^2 + l^4]}{k[qc \exp(\xi \rho) + l^2 - p^2]^2}, \quad (41)$$

where $\delta = \sqrt{\frac{1}{l^2 - p^2}}$ and $\xi = l \frac{x^\alpha}{\alpha} - k \frac{t^\alpha}{\alpha}$.

Case 2:

$$(z)_{10}(\xi) = -\frac{q \exp(\frac{1}{2}\xi \rho)}{qc + (l^2 - p^2) \exp(\xi \rho)}, \quad (42)$$

$$(U)_{10}(\xi) = -\frac{q[-q^2 c^2 + 4qc(l^2 - p^2) \exp(\xi \rho) + (l^4 - p^4) \exp(2\xi \rho) + 2p^2 l^2 \exp(2\xi \rho)]}{k[qc + (l^2 - p^2) \exp(\xi \rho)]^2}, \quad (43)$$

where $\delta = \sqrt{\frac{1}{l^2 - p^2}}$ and $\xi = l \frac{x^\alpha}{\alpha} - k \frac{t^\alpha}{\alpha}$.

Type 6:

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = -\frac{6c(l^2 - p^2)}{k}, \quad a = \frac{1}{4(l^2 - p^2)}, \quad b = 0, \quad c = c, \quad (44)$$

so we obtain the eleventh and twelfth results in two different cases as follow,

Case 1:

$$(z)_{11}(\xi) = -\frac{q \exp(\xi \delta)}{qc \exp(\xi \delta) + p^2 - l^2}, \quad (45)$$

$$(U)_{11}(\xi) = -\frac{q[-q^2 c^2 \exp(2\xi \rho) + 4qc(l^2 - p^2) \exp(\xi \rho) - l^4 + 2p^2 l^2 - p^4]}{k[qc \exp(\xi \rho) + l^2 - p^2]^2}, \quad (46)$$

where $\delta = \sqrt{\frac{1}{l^2 - p^2}}$, $\rho = \sqrt{\frac{1}{p^2 - l^2}}$ and $\xi = l \frac{x^\alpha}{\alpha} - k \frac{t^\alpha}{\alpha}$.

Case 2:

$$(z)_{12}(\xi) = \frac{q \exp(\frac{1}{2}\xi \delta)}{(l^2 - p^2) \exp(\xi \delta) - qc}, \quad (47)$$

$$(U)_{12}(\xi) = \frac{6(l^2 - p^2) q^2 c \exp(\xi \delta)}{k[(l^2 - p^2) \exp(\xi \delta) - qc]^2}, \quad (48)$$

where $\delta = \sqrt{\frac{1}{l^2 - p^2}}$ and $\xi = l \frac{x^\alpha}{\alpha} - k \frac{t^\alpha}{\alpha}$.

4.2 (2+1)-Dimensional Time-Fractional Zoomeron Equation

Let us consider (2+1)-dimensional time-fractional Zoomeron equation as follow [29, 30]:

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}} \left[\frac{u_{xy}}{u} \right] - \frac{\partial^2}{\partial x^2} \left[\frac{u_{xy}}{u} \right] + 2 \frac{\partial^\alpha}{\partial t^\alpha} [u^2]_x = 0, \quad 0 < \alpha \leq 1, \quad (49)$$

taking the travelling wave transformation,

$$u(x, y, t) = U(\xi), \quad \xi = lx + hy - k \frac{t^\alpha}{\alpha}, \quad (50)$$

where h, k and l are nonzero constants, then the Eq. (49) converts into ordinary equation as follow:

$$hk^2 l \left(\frac{U''}{U} \right)'' - l^3 h \left(\frac{U''}{U} \right)'' - 2kl (U^2)'' = 0. \quad (51)$$

Taking the conformable Integration of the Eq.(51) twice with respect to ξ , then we have,

$$kh(k^2 - l^2)U'' - 2klU^3 - \zeta U = 0, \quad (52)$$

where ζ is a non-zero constant of integration [27, 28]. The balancing number is found 1 by balancing the highest order derivative term with the highest power nonlinear term. Then the solution of the Eq. (52) becomes

$$U(\xi) = a_0 + a_1 z(\xi). \quad (53)$$

Substituting the Eqs. (9) and (53) into the Eq. (52), we find an algebraic equation with powers of $z(\xi)$. Equating the all powers of $z(\xi)$ to zero, we get the following system:

$$z^0(\xi) : -2kla_0^3 - \zeta a_0 = 0, \quad (54)$$

$$z^1(\xi) : alhk^2 a_1^2 - al^3 ha_1 - 6kla_0^2 a_1 - \zeta a_1 = 0, \quad (55)$$

$$z^2(\xi) : \frac{3}{2}blhk^2 a_1 - \frac{3}{2}bl^3 ha_1 - 6kla_0 a_1^2 = 0, \quad (56)$$

$$z^3(\xi) : 2clhk^2 a_1 - 2cl^3 ha_1 - 2kla_1^3 = 0. \quad (57)$$

We find the two type of the following coefficients, from the solution of the system (54-57).

Type 1:

$$\begin{aligned} a_0 &= \pm \frac{\sqrt{-\zeta k}}{\sqrt{2k}}, & a_1 &= \pm \sqrt{-\frac{1}{8\zeta k}}(k^2 - 1)hb, \\ a &= -\frac{2\zeta}{lh(k^2 - l^2)}, & b &= b, & c &= -\frac{(k^2 - l^2)h b^2}{8\zeta}, \end{aligned} \quad (58)$$

so we obtain the first and second results in two different cases as follow,

Case 1:

$$(z)_1(\xi) = \frac{8\zeta \exp(\xi \omega)}{lh(k^2 - l^2)(2b \exp(\xi \omega) - 1)}, \quad (59)$$

$$(U)_1(\xi) = \frac{\sqrt{-\zeta k}}{\sqrt{2k}} + \frac{\sqrt{2}(k^2 - l^2)\zeta \exp(\xi \omega)}{\sqrt{-\zeta k}(k^2 - l^2)l(2 \exp(\xi \omega) - 1)}, \quad (60)$$

where $\omega = \sqrt{-\frac{2\zeta}{lh(k^2-l^2)}}$ and $\xi = lx + hy - k\frac{t^\alpha}{\alpha}$.

Case 2:

$$(z)_2(\xi) = \frac{8\zeta \exp(\xi \omega)}{lh(k^2-l^2)(2b \exp(\xi \omega) - \exp(2\xi \omega))}, \quad (61)$$

$$(U)_2(\xi) = \frac{\sqrt{-\zeta k}}{\sqrt{2k}} + \frac{\sqrt{2}(k^2-l^2)\zeta b \exp(\xi \omega)}{\sqrt{-\zeta k}(k^2-l^2)l(2b \exp(\xi \omega) - \exp(2\xi \omega))}, \quad (62)$$

where $\omega = \sqrt{-\frac{2\zeta}{lh(k^2-l^2)}}$ and $\xi = lx + hy - k\frac{t^\alpha}{\alpha}$.

Type 2:

$$a_0 = 0, \quad a_1 = a_1, \quad a = -\frac{\zeta}{lh(k^2-l^2)}, \quad b = 0, \quad c = \frac{ka_1^2}{h(k^2-l^2)}, \quad (63)$$

so we obtain the third and fourth results in two different cases as follow,

Case 1:

$$(z)_3(\xi) = \frac{4(k^2-l^2)\zeta h \exp(\xi \gamma)}{4\zeta ka_1^2 \exp(2\xi \gamma) - lh^2k^4 + 2l^3h^2k^2 - l^5h^2}, \quad (64)$$

$$(U)_3(\xi) = \frac{4(k^2-l^2)h\zeta a_1 \exp(\xi \gamma)}{4\zeta ka_1^2 \exp(2\xi \gamma) - lh^2k^4 + 2l^3h^2k^2 - l^5h^2}, \quad (65)$$

where $\gamma = \sqrt{\frac{\zeta}{lh(k^2-l^2)}}$ and $\xi = lx + hy - k\frac{t^\alpha}{\alpha}$.

Case 2:

$$(z)_4(\xi) = \frac{4(k^2-l^2)\zeta h \exp(\xi \gamma)}{lh^2k^4 \exp(2\xi \gamma) - 2l^3h^2k^2 \exp(2\xi \gamma) + l^5h^2 \exp(2\xi \gamma) - 4\zeta ka_1^2}, \quad (66)$$

$$(U)_4(\xi) = \frac{4(k^2-l^2)h\zeta a_1 \exp(\xi \gamma)}{lh^2k^4 \exp(2\xi \gamma) - 2l^3h^2k^2 \exp(2\xi \gamma) + l^5h^2 \exp(2\xi \gamma) - 4\zeta ka_1^2}, \quad (67)$$

where $\gamma = \sqrt{\frac{\zeta}{lh(k^2-l^2)}}$ and $\xi = lx + hy - k\frac{t^\alpha}{\alpha}$.

5 Conclusion

In this article, new exact solutions of CFPDEs, namely, the nonlinear space-time fractional Klein-Gordon equation and the (2+1)-dimensional time-fractional Zoomeron equation, have been obtained by using the auxiliary equation method. Conformable fractional derivative definitions were used to cope with the fractional terms in fractional partial differential equation. We have used the new definition of wave transformation for converting the nonlinear CFPDEs into the ordinary differential equation. We have obtained a variety of new solutions of the mentioned equations. Since the technique is efficient and powerful, it can be used to handle a variety of equations which appears in applications in several branches of the nonlinear sciences.

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