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A Numerical Scheme For Semilinear Singularly Perturbed Reaction-Diffusion Problems

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Abstract

In this study we investigated the singularly perturbed boundary value problems for semilinear reaction-difussion equations. We have introduced a basic and computational approach scheme based on Numerov's type on uniform mesh. We indicated that the method is uniformly convergence, according to the discrete maximum norm, independently of the parameter of ε . The proposed method was supported by numerical example.

Keywords: Singular perturbations, reaction-diffusion problems, numerov method **AMS 2010 codes:** 65L10,65L11,65L12.

1 Introduction

Let us take into consideration of the following singularly perturbed semilinear reaction-diffusion boundary value problem:

$$Lu(x) \equiv -\varepsilon u''(x) + f(x, u(x)) = 0, \quad 0 < x < l,$$

$$\tag{1}$$

$$u(0) = A, \quad u(l) = B.$$
 (2)

where ε , $0 < \varepsilon \le 1$ is the perturbation parameter, f is given sufficiently smooth functions that $f(x, u(x)) \in C([0, l]x\mathbb{R}), \quad \frac{\partial f(x, u)}{\partial u} \ge \alpha > 0$. The problem (1)-(2) has boundary layers at the boundary points.

In a differential equation, if a small parameter is multiplied by the highest- order derivative term in the differential equation, generally it is called the singularly perturbed problem (denoted here by ε).[1-3]

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Second-order reaction-diffusion type boundary value problems with singularly perturbed occur frequently in fluid mechanics and other fields of applied mathematics. Examples of such studied problems can be seen in [4-6].

Since the continuous solutions of singularly perturbed problems change very quickly in certain layers, it's numerical analysis is always important. It is well known that when the parameter of perturbation is small enough, conventional numerical methods to solve the problem do not work well. Therefore should be develop develop appropriate numerical methods for such problems, whose convergence does not depend on the perturbation parameter. It can be found in the literature that there are many numerical finite difference schemes that are stable for all values parameter of perturbation [2-9]. One of the most important ways to easily find the methods that give such results is use of finite difference schemes with exponentially fitted [11-13].

As a numerical study, several examples of the second order singularly perturbed convection-diffusion problems can also be seen in [5],[21].

In [13-14,16,19] introduced numeric methods and special mesh methods for various reaction-diffusion type problem. In [20] semi-linear reaction diffusion equations are discussed. The discrete and upper of solutions were investigated for the asymptotic properties. And also it is numerical solutions are investigated on pisewise mesh.

The Numerov method is undoubtedly one of the most well known methods for reaction-diffusion type equations since it has fourth-order approach and it has been widely used in practical computational methods. Recently much fitted dinite difference scheme has been studied based on Numerov's method in [11-18]. In [11], Phaneendra et al gave a finite difference Numerov scheme with a fitted multiplier three bands for solving singularly perturbed boundary value problem. Based on Numerov method, singularly perturbed nonlinear reaction-diffusion problem were investigated in [16-19].

In this study, we present finite difference scheme based on Numerov method for (1)-(2) problem on an uniform mesh. The some properties of the exact solution is given in section 2. In section 3, a finite difference scheme on a uniform mesh is introduced which is based on Numerov's method. In section 4, the convergence of the approximate solution was presented and it was shown that uniform convergence was achieved at the discrete maximum norm. A numerical example and its results are given in section 5.

Notation. The C symbol in the throughout the article indicates a positive constants and does not have to be the same in each occurence which is independent of ε and of the mesh.

2 Properties of the exact solution

The semilinear equation (1) can be written in the take form;

$$Lu(x) \equiv -\varepsilon u''(x) + a(x)u(x) = F(x), 0 < x < l,$$
(3)

$$u(0) = A, \ u(l) = B,$$
 (4)

where $a(x) = \frac{\partial f(x,\tilde{u})}{\partial u}$, $\tilde{u} = \gamma u$, $0 < \gamma < 1$, F(x) = -f(x,0). Here we will give some important properties of the solution of (3)-(4) problem, which are required in later sections for the analysis of the numerical solution. We will indicate the maximum norm of any continuous g(x) function on the interval with $||g||_{\infty}$.

Definition 1. (Maximum Principle). Assume that $v(0) \ge 0$ and $v(l) \ge 0$. Then $Lv(x) \ge 0$, 0 < x < l, implies that $v(x) \ge 0$, for all $0 \le x \le l$.

The following two lemma and its solutions are given in [22].

Lemma 1. For any v(x) function, let $v(x) \in C[0,l] \cap C^2(0,l)$. Then the following estimate is true.

$$|v(x)| \le |v(0)| + |v(l)| + \alpha^{-1} \max_{1 \le i \le N} |Lv(x)|, \quad 0 \le x \le l.$$
(5)

Proof. Let us define the $\Psi(x)$ function as follows:

$$\Psi(x) = \pm v(x) + |v(0)| + |v(l)| + \alpha^{-1} \max_{1 \le i \le N} |Lv(x)|, \qquad 0 \le x \le l.$$
(6)

Then the following inequalities are satisfied

$$\Psi(0) \ge 0, \ \Psi(l) \ge 0 \quad and \ L\Psi(x) \ge 0. \tag{7}$$

The maximum principle gives $\Psi(x) \ge 0$ *, for all* $0 \le x \le l$ *, and so the inequality (5) holds.*

Lemma 2. Let a(x), F(x) are given sufficiently smooth functions and u(x) be the solution of the problem (3)-(4). Then the following estimates hold.

$$|||u(x)||_{\infty} \le C, \qquad 0 \le x \le l.$$
(8)

$$\left|u'(x)\right| \le C\left\{1 + \frac{1}{\sqrt{\varepsilon}}\left(e^{-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}} + e^{-\frac{\sqrt{\alpha}(l-x)}{\sqrt{\varepsilon}}}\right)\right\}.$$
(9)

Proof. Appliying Lemma1.to (3)-(4) we have (8).

$$Lv(x) = \varphi(x), \tag{10}$$

$$v(0) = O\left(\frac{1}{\sqrt{\varepsilon}}\right), \qquad v(l) = O\left(\frac{1}{\sqrt{\varepsilon}}\right),$$
(11)

where

$$v(x) = u'(x), \quad \varphi(x) = F'(x) - a'(x)u(x).$$
 (12)

The solution of the problem (10)-(11) *has the following form:*

$$v(x) = v_0(x) + v_1(x).$$
(13)

Respectively, in here, the functions $v_0(x)$ *and* $v_1(x)$ *are the solutions of the following problems:*

$$\begin{cases} Lv_0(x) = \varphi(x), 0 < x < l, \\ v_0(0) = v_0(l) = 0, \end{cases}$$
(14)

$$\begin{cases} Lv_1(x) = 0, 0 < x < l, \\ v_1(0) = v_1(l) = 0, \end{cases}$$
(15)

from Lemma1., for the solution of the problem (14), we have

$$|v_0(x)| \leq \alpha^{-1} \max_{1 \leq s \leq l} |\varphi(s)|.$$

Thus, we obtain

$$|v_0(x)| \le C, \ \ 0 \le x \le l.$$
(16)

Applying maximum principle to the problem (15), we get

$$|v_1(x)| \le w(x), \tag{17}$$

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where w(x) is the solution of the following problem:

$$\begin{cases} -\varepsilon w'' + \alpha w' = 0, \ 0 < x < l, \\ w(0) = |v_1(0)|, \ w(l) = |v_1(l)|. \end{cases}$$
(18)

The solution of this problem is has the form

$$w(x) = \frac{1}{\sinh\left(\frac{\sqrt{\alpha}l}{\sqrt{\varepsilon}}\right)} \left\{ |v_1(0)| \sinh\left(\frac{\sqrt{\alpha}(l-x)}{\sqrt{\varepsilon}}\right) + |v_1(l)| \sinh\left(\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}\right) \right\},\tag{19}$$

and it is from that

$$w(x) \le \frac{C}{\sqrt{\varepsilon}} \left(e^{-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}} + e^{-\frac{\sqrt{\alpha}(l-x)}{\sqrt{\varepsilon}}} \right),\tag{20}$$

is hold. Then combining (16),(17) and (20) in the following inequality, it can be easely obtained:

$$|u'(x)| \le |v_0(x)| + |v_1(x)|.$$

Thus the proof is completed.

3 Discretization and Mesh

In this section, we construct a numerical scheme for solving the problem (1)-(2) on a uniform mesh. Let w_h denote the uniform mesh on [0, l].

$$w_h = \{x_i = ih, i = 1, 2, \dots, N-1, h = 1/N\}, \bar{w}_h = w_h \cup \{x = 0, l\}.$$

Let us show $w_i = w(x_i)$ for any function w(x), and moreover any approximation of the function u(x) at point x_i with y_i . We will use the following notations for any mesh function $\{w_i\}$ defined on \bar{w}_N :

$$w_{\bar{x},i} = \frac{w_i - w_{i-1}}{h}, \quad w_{x,i} = \frac{w_{i+1} - w_i}{h}, \quad w_{\bar{x}x,i} = \frac{w_{x,i} - w_{\bar{x},i}}{h} = \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2},$$
$$\|w\|_{C(\bar{w}_h)} := \max_{1 \le i \le N} |w_i|.$$

and

To find the difference approach that corresponds to (1), let us use the following identity and use the interpolating quadrature formulas in [8] on each intervals (x_{i-1}, x_i) and (x_i, x_{i+1}) ,

$$\chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_{i+1}} Lu(x) \varphi_i(x) dx = 0, \ 1 \le i \le N - 1,$$
(21)

then we obtain the foolowing relation:

$$lw_{i} \equiv -\varepsilon \theta_{i} w_{\bar{x}x,i} + f(x_{i}, w_{i}) = R_{i}, \quad i = 1, 2, \dots, N-1,$$
(22)

where

$$R_{i} = \chi_{i}^{-1} h^{-1} \left\{ \int_{x_{i-1}}^{x_{i+1}} (f(x, w) - f(x_{i}, w_{i})\varphi_{i}(x) dx \right\}.$$
(23)

In here θ_i is called fitting factor and after a simple calculation, the value of $\theta_i = \chi_i^{-1} = 1$. if R_i omited in equation (22) then we have numerical scheme for (1)-(2)

$$\begin{cases} lw_i \equiv -\varepsilon w_{\bar{x}x,i} + f(x_i, w_i) = 0, \ i = 1, 2, \dots, N-1, \\ w_0 = A, \ w_N = B. \end{cases}$$
(24)

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We note that $\varphi_i^{(1)}$ and $\varphi_i^{(2)}$ are basis functions such that they are solutions of the following problems respectively,

$$-\varepsilon\varphi_{i}^{(1)''}=0, \ \varphi_{i}^{(1)}(x_{i})=1, \ \varphi_{i}^{(1)}(x_{i-1})=0, -\varepsilon\varphi_{i}^{(2)''}=0, \ \varphi_{i}^{(2)}(x_{i})=1, \ \varphi_{i}^{(2)}(x_{i+1})=0,$$

and which are

$$\varphi_{i}(x) = \begin{cases} \varphi_{i}^{(1)} \equiv \frac{(x-x_{i-1})}{h}; & x_{i-1} < x < x_{i}, \\ \varphi_{i}^{(2)} \equiv \frac{(x_{i+1}-x)}{h}; & x_{i} < x < x_{i+1}, \\ 0 & , & x \notin (x_{i-1}, x_{i+1}), \end{cases}$$

and also $\chi_i = h^{-1} \int_{x_{i-1}}^{x_{i+1}} \varphi_i(x) dx = 1.$

3.1 Description of the Numerov's method

For convenience, let $y_i = y(x_i)$, $y^{(n)}(x_i) = y_i^{(n)}$ at $x = x_i$, for any function y(x), using the Taylor series expansion, we obtain:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = y_i'' + \frac{1}{12}h^2 y_i'''' + O(h^4).$$

If the error term is omited, and instead of y_i'' and $y_i^{(4)}$ the corresponding terms are written, then we have the following a Numerov finite difference scheme:

$$\left(\frac{\varepsilon}{h^2} - \frac{a_{i-1}}{12}\right)y_{i-1} - \left(\frac{2\varepsilon}{h^2} + \frac{10a_i}{12}\right)y_i + \left(\frac{\varepsilon}{h^2} - \frac{a_{i+1}}{12}\right)y_{i+1} = -\frac{1}{12}\left(F_{i-1} + 10F_i + F_{i+1}\right) \cdot i = 1, 2, \dots, N-1.$$
(25)

Let us

$$A_{i} = \frac{\varepsilon}{h^{2}} - \frac{a_{i-1}}{12} , \quad C_{i} = \frac{2\varepsilon}{h^{2}} + \frac{10a_{i}}{12} , \qquad B_{i} = \frac{\varepsilon}{h^{2}} - \frac{a_{i+1}}{12} , \qquad G_{i} = \frac{1}{12} \left(F_{i-1} + 10F_{i} + F_{i+1} \right) ,$$

then we obtain the followig equation:

$$A_{i}y_{i-1} - C_{i}y_{i} + B_{i}y_{i+1} = -G_{i}, \quad i = 1, 2, \dots, N-1.$$
(26)

From [10],

$$A_i > 0, \ B_i > 0, \ C_i - A_i - B_i > 0,$$
 (27)

are valid and the siystem has only one solution. This system can be solved by Thomas algorithm.

3.2 An Algorithm for Numerov Type Scheme

If Numerov method is applied to problem (1)-(2) we obtain following equation:

$$\begin{cases} -\varepsilon w_{\bar{x}x,i} + \left[1 + \frac{1}{12}h^2 w_{\bar{x}x,i}\right] f(x_j, w_j) = 0, \quad j = 1, 2, \dots, N; \\ w_0 = A, \quad w_{N+1} = B \end{cases}$$
(28)

With the help of the Newton-Raphson-Kantorovich approach, we'll get a new Numerov type difference scheme. Instead of $f(x_i, u_i)$ function, in the equation for (28), let's write the following equivalent,

$$f\left(x_{j}, w_{j}^{(k)}\right) = f\left(x_{j}, w_{j}^{(k-1)}\right) + \frac{\partial f}{\partial w}\left(x_{j}, w_{j}^{(k-1)}\right) \cdot \left(w_{j}^{(k)} - w_{j}^{(k-1)}\right)$$

then we obtain the following relation.

$$-\varepsilon w_{\bar{x}x} w_j^{(k)} + \left[1 + \frac{1}{12} h^2 w_{\bar{x}x}\right] \left\{ f\left(x_j, w_j^{(k-1)}\right) + \frac{\partial f}{\partial w}\left(x_j, w_j^{(k-1)}\right) \cdot \left(w_j^{(k)} - w_j^{(k-1)}\right) \right\} = 0, \ j = 1, 2, \dots, N; \ k = 1, 2, \dots$$

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If necessary arrangements are made we have the following difference equation.

$$A_{j}^{(k-1)}w_{j-1}^{(k)} - C_{j}^{(k-1)}w_{j}^{(k)} + B_{j}^{(k-1)}w_{j+1}^{(k)} = -G_{j}^{(k-1)}, \ j = 1, 2, \dots, N-1; \ k = 1, 2, \dots$$
(29)

where

$$\begin{split} A_{j}^{(k-1)} &= \left(\frac{\varepsilon}{h^{2}} - \frac{1}{12} f_{w_{j}}^{(k-1)}\right), C_{j}^{(k-1)} = \left[\frac{2\varepsilon}{h^{2}} + \frac{1}{12} \left(f_{w_{j+1}}^{(k-1)} + 8f_{w_{j}}^{(k-1)} + f_{w_{j-1}}^{(k-1)}\right)\right], B_{j}^{(k-1)} = \left(\frac{\varepsilon}{h^{2}} - \frac{1}{12} f_{w_{j}}^{(k-1)}\right), \\ G_{j}^{(k-1)} &= \frac{1}{12} \left(f_{w_{j+1}}^{(k-1)} + 10f_{w_{j}}^{(k-1)} + f_{w_{j-1}}^{(k-1)}\right) \cdot w_{j}^{(k-1)} - \left(f_{j+1}^{(k-1)} + 10f_{j}^{(k-1)} + f_{j-1}^{(k-1)}\right) \\ &\quad + f_{w_{j}}^{(k-1)} \cdot (w_{j+1}^{(k-1)} - 2w_{j}^{(k-1)} + w_{j-1}^{(k-1)}) \end{split}$$

and

$$w_{j+1}^{(0)}, w_j^{(0)}, w_{j-1}^{(0)}$$

are given. This scheme has (27) properties and therefore it is stable and has a one solution.

4 Convergence Analysis

Let $z_i = y_i - u_i, 0 \le i \le N$, where y_i the solution of (22) and u_i the solution of (1)-(2) at mesh point x_i . We now estimate the approximate error z_i , which satisfies the following discrete problem

$$lz_i = R_i \; ; \; z_0 = z_N = 0 \tag{30}$$

where the truncation error R_i is in the equation (23).

Definition 2. (Discrete Maximum Principle).Suppose that a mesh function v_i satisfies $v_0 \ge 0$ and $v_N \ge 0$. Then $\ell v_i \ge 0$, for all $0 \le i \le N - 1$ implies that $v_i \ge 0$ for all $0 \le i \le N$.

Theorem 3. Let $f(x, u) \in C^2[0, l]$. Then the following estimate holds.

$$\|y - u\|_{C(\bar{w}_h)} \le Ch^2 \tag{31}$$

Proof. If we find an estimate for R_i the proof is complete. For function f(x, u), using the Taylor series expansion, we obtain:

$$f(x,u) - f(x_i,u_i) = (x - x_i) \left\{ \frac{\partial f(x_i,u_i)}{\partial x} + \frac{\partial f(x_i,u_i)}{\partial u} \frac{\partial u(x_i)}{\partial x} \right\} + \frac{(x - x_i)^2}{2!} \left\{ \frac{\partial^2 f(\xi_i, u(\xi_i))}{\partial x^2} + \frac{\partial^2 f(\xi_i, u(\xi_i))}{\partial x \partial u} \frac{\partial u(\xi_i)}{\partial x} + \frac{\partial^2 f(\xi_i, u(\xi_i))}{\partial u^2} \left(\frac{\partial u(\xi_i)}{\partial x}\right)^2 \right\}$$

If we write this expression instead of R_i we obtain the following relation:

$$(x-x_i)\left\{\frac{\partial f(x_i,u_i)}{\partial x} + \frac{\partial f(x_i,u_i)}{\partial u}\frac{\partial u(x_i)}{\partial x}\right\} + \frac{(x-x_i)^2}{2!}\left(\frac{\partial^2 f(\xi_i,u(\xi_i))}{\partial x^2} + \frac{\partial^2 f(\xi_i,u(\xi_i))}{\partial x\partial u}\frac{\partial u(\xi_i)}{\partial x} + \frac{\partial^2 f(\xi_i,u(\xi_i))}{\partial u^2}\left(\frac{\partial u(\xi_i)}{\partial x}\right)^2\right)\varphi_i(x)\,dx$$

Considering the equivalents of u' and u'', using of discrete maximum principle we have

$$\|R\|_{C(\bar{w}_h)} \le Ch^2$$

This estimate conclude the proof of Theorem 4.

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ε	N = 32	N = 64	N = 128	N = 256	N = 512	R_{ϵ}^{N}	
2^{-5}	2.3e-4	5.9e-5	1.5e-5	3.7e-6	9.2e-7	2.009	
2^{-6}	3.8e-4	9.5e-5	2.4e-5	6.2e-6	1.5e-6	2.014	
2^{-7}	6.4e-4	1.6e-4	4.0e-5	1.0e-5	2.5e-6	2.019	
2^{-8}	1.1e-3	2.4e-4	6.9e-5	1.7e-5	4.3e-6	2.020	
2^{-9}	1.9e-3	4.9e-4	1.2e-4	3.1e-5	7.6e-6	2.015	
R^N	2.013	2.003	2.002	1.998	2.020		

Table 1 Maximum errors and rates of convergence w_h

5 Numerical example

To illustrate the applicability of the method proposed in this article, we applied it to an example. Connsider the following semilinear problem:

Example 4. $-\varepsilon u'' - exp(-(x^2 + u)) = 0, x \in [0, 1], u(0) = 0, u(1) = 1$

For numerical approximation of solution, we have shown that the method uniform convergens according to the ε -parameter. Since the exact solution to this problem could not be found, we used the following the double mesh prensiple for calculate of the maximum absolute errors.

$$E_{\varepsilon}^{N} = \max_{0 \le i \le N} \left| u_{i}^{N} - u_{2i}^{2N} \right|$$

For any N , the ε -uniform maximum absolute error is calculated by

$$E^N = \max_{\varepsilon} E_{\varepsilon}.$$

The numerical rate of convergence and ε -uniform convergence rate for example has been calculated by the following formulas;

$$R_{\varepsilon}^{N} = \frac{\log \left| E_{\varepsilon}^{N} / E_{\varepsilon}^{2N} \right|}{log2}, R^{N} = \frac{\log \left| E^{N} / E^{2N} \right|}{log2}$$

The maximum point wise errors and the rates of convergence of the problem in example is presented in Table 1. $(u_i^0 = x^2, 1 \le i \le N - 1)$, for arbitrary initial function)

6 Conclusion

In this paper we have presented a Numerov's scheme to solve a class of singularly perturbed semilinear reaction-diffusion problem. We have introduced computational technique based Numerov's scheme on a uniform mesh. Uniform convergence of the method is demonstrated with respect to the parameter of perturbation. The accuracy of the uniform convergence was supported by a numerical example.

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