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A new approach for nuclear family model with fractional order Caputo derivative

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Abstract

A work on a mathematical modeling is very popular in applied sciences. Nowadays many mathematical models have been considered and new methods have been used for approaching of these models. In this paper we are considering mathematical modeling of nuclear family model with fractional order Caputo derivative. Also the existence and uniqueness results and numerical scheme are given with Adams-Bashforth scheme via fractional order Caputo derivative.

Keywords: Fractional order derivative, existence and uniqueness, Adams-Bashforth numerical scheme.
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1 Introduction

Recently, new efficient numerical methods have been developed for solutions of differential equations with different definitions of derivatives. For example the kernels including the power law for the Riemann-Liouville and Caputo type, the exponential decay law for the Caputo-Fabrizio case and the Mittag-Leffler law for the Atangana-Baleanu derivative [2-6, 11-14]. So these kernels history are beginning from the Leibniz's letter to L'Hospital to Atangana-Baleanu derivative. In this work we are interesting in mathematical modeling of nuclear family. Model was introduced by Koca in 2015 with Caputo type fractional derivative [1]. In addition to previous paper, we reconsider model with searching the existence and uniqueness results of solutions and we give numerical approach for solutions of model with Caputo derivative. We believe that classical (ordinary) derivative is weak to explain the memory effect of the family dynamics. Because of this, we considered numerical solutions via fractional order Caputo derivative. Also the aim of the choose of the Caputo derivative is to give better meaning for modeling.

Adams-Bashforth is a powerful numerical method to solve linear and non-linear ordinary differential equations. Method was used only for ordinary differential equations generally with integer order. After that Atangana

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and Batogna have extended this method for partial differential equation with Caputo-Fabrizio derivative [10] in their thesis. Also Owolabi and Atangana formulated a new three-step fractional Adams-Bashforth scheme with Caputo-Fabrizio derivative [7-9]. Method has been used for the solution of linear and nonlinear fractional differential equations .

In this paper we extend the applicability of the proposed scheme to solve system that is modeled by the Caputo derivative. The remainder of this paper is follows that in section one; some useful definitions of fractional order differentiation are given, in section two; we present in detail the existence and uniqueness results of solutions of our system. Finally in numerical part; we consider the solutions of system with two-step Adams-Bashforth scheme via fractional order Caputo derivative.

2 Preliminaries

Definition 1 : Caputo fractional derivative of order $\alpha > 0$ of a function $f : (0,\infty) \to R$, according to Caputo, the fractional derivative of a continuous and differentiable function f is given as :

$$ {}^{C}D_t^{\alpha}(f(t)) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-x)^{-\alpha} \frac{d}{dx} f(x)dx, \quad 0 < \alpha \leq 1. \tag{1} $$

Definition 2 : The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to R$, according to Riemann-Liouville, the fractional integral that is considered as anti-fractional derivative of a function f is :

$$ I_t^{\alpha}(f(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x)dx, \quad x > a. \tag{2} $$

Now we give two important properties for Caputo and Riemann-Liouville derivatives.
Property 1 : If $f(t)$ is defined in the interval $[a,b]$ and

$$ \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x)dx = 0 \tag{3} $$

for $\alpha > 0$ and for all $t \in [a,b]$, then

$$ f(t) \equiv 0. \tag{4} $$

Property 2 : The following equation

$$ \begin{aligned} {}^{C}D_t^{\alpha}(f(t)) &= g(x), \quad \alpha \in (0,1), \quad x \in R, \\ f(0) &= f_{0,} \end{aligned} \tag{5} $$

doesn't have a periodic solution if f_0 does not solve $g(x) = 0$, where $g(x)$ is continuous.

3 Model derivation and existence and uniqueness of solutions for the nuclear family model

In this section, first we give integer order nuclear family model that is introduced by Koca in 2015 with four state variables [1]. The model describes baby's emotions, in which baby (B) is involved in emotions with mother (M) and father (F). In model, the following notations for variables were used:

$B(t)$: Baby's love for the baby's father,

$F(t)$: Father's love for the baby and his wife,

$M(t)$: Mother's love for the baby and her husband,
$B_1(t)$: Baby's love for the baby's mother.
The integer order nuclear family model is given as

$$\begin{aligned}\frac{dB}{dt} &= aB+b(F-M)(c-(F-M))+\gamma_1,\\ \frac{dF}{dt} &= eF+gB(h-B)+jM+\gamma_2,\\ \frac{dM}{dt} &= kM+mB_1(n-B_1)+pF+\gamma_3,\\ \frac{dB_1}{dt} &= aB_1+b(M-F)(d-(M-F))+\gamma_4,\end{aligned} \tag{6}$$

with initial conditions

$$B(0)=B_0,\ F(0)=F_0,\ M(0)=M_0,\ B_1(0)=B_{10}, \tag{7}$$

where e,g,h,j are specify father's emotional style, k,m,n,p are specify mother's emotional style and $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ are attraction constants.

3.1 Existence of solution for the nuclear family model

In this part, we will present in detail the existence of the solutions of our system. The fixed-point theorem will help achieve this. Let $P=K(q)\times K(q)$ and $K(q)$ be the Banach space of continuous $R\to R$ valued function defined on the interval q with the norm

$$\|B,F,M,B_1\| = \|B\|+\|F\|+\|M\|+\|B_1\|. \tag{8}$$

Here

$$\begin{aligned}\|B\| &= \sup\{|B(t)| : t\in q\},\\ \|F\| &= \sup\{|F(t)| : t\in q\},\\ \|M\| &= \sup\{|M(t)| : t\in q\},\\ \|B_1\| &= \sup\{|B_1(t)| : t\in q\}.\end{aligned}$$

Let us redefine the nuclear family model spread by replacing the time derivative by Caputo fractional derivative:

$$\begin{aligned}{}^C_aD^\alpha_t B(t) &= F_1(t,B(t)),\\ {}^C_aD^\alpha_t F(t) &= F_2(t,F(t)),\\ {}^C_aD^\alpha_t M(t) &= F_3(t,M(t)),\\ {}^C_aD^\alpha_t B_1(t) &= F_4(t,B_1(t)),\end{aligned} \tag{9}$$

with initial conditions $B(t_0)=B_0, F(t_0)=F_0, M(t_0)=M_0$ and $B_1(t_0)=B_{10}$.

Here,

$$\begin{aligned}F_1(t,B(t)) &= aB(t)+b(F(t)-M(t))(c-(F(t)-M(t))+\gamma_1,\\ F_2(t,F(t)) &= eF(t)+gB(t)(h-B(t))+jM(t)+\gamma_2,\\ F_3(t,M(t)) &= kM(t)+mB_1(t)(n-B_1(t))+pF(t)+\gamma_3,\\ F_4(t,B_1(t)) &= aB_1(t)+b(M(t)-F(t))(d-(M(t)-F(t))+\gamma_4.\end{aligned} \tag{10}$$

The above system (10) can be converted to the Caputo fractional integral. By definition (2), the model can be written as

$$B(t) = B_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_1(\tau, B(\tau)) d\tau, \tag{11}$$

$$F(t) = F_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_2(\tau, F(\tau)) d\tau,$$

$$M(t) = M_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_3(\tau, M(\tau)) d\tau,$$

$$B_1(t) = B_{10} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_4(\tau, B_1(\tau)) d\tau.$$

Theorem 1 : The kernels F_1, F_2, F_3 and F_4 satisfy the Lipschitz condition if the following inequalities can be obtained :

$$0 \leq L_i < 1, \text{ for } i = 1,2,3,4. \tag{12}$$

Proof : Let us start the kernel F_1. Let B and B^1 be two function, so we have following:

$$\begin{aligned} & \left\| F_1(t, B(t)) - F_1(t, B^1(t)) \right\| \\ = & \left\| \begin{array}{c} aB(t) + b(F(t) - M(t))(c - (F(t) - M(t)) + \gamma_1 \\ -aB^1(t) - b(F(t) - M(t))(c - (F(t) - M(t)) - \gamma_1 \end{array} \right\| \\ \leq & \, a \left\| B(t) - B^1(t) \right\| \end{aligned} \tag{13}$$

Taking as $L_1 = a$, then we get

$$\left\| F_1(t, B(t)) - F_1(t, B^1(t)) \right\| \leq L_1 \left\| B(t) - B^1(t) \right\|. \tag{14}$$

Hence, the Lipschitz condition is satisfied for F_1, and if $0 \leq L_1 < 1$, then it is also a contraction for F_1. Similarly the other kernels have the Lipschitz condition as follows:

$$\begin{aligned} \left\| F_2(t, F(t)) - F_2(t, F^1(t)) \right\| &\leq L_2 \left\| F(t) - F^1(t) \right\|, \\ \left\| F_3(t, M(t)) - F_3(t, M^1(t)) \right\| &\leq L_3 \left\| M(t) - M^1(t) \right\|, \\ \left\| F_4(t, B_1(t)) - F_4(t, B_1^1(t)) \right\| &\leq L_4 \left\| B_1(t) - B_1^1(t) \right\|. \end{aligned} \tag{15}$$

When considering the kernels for the model, eq. (9) can be rewritten as follows:

$$B(t) = B_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_1(\tau, B(\tau)) d\tau, \tag{16}$$

$$F(t) = F_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_2(\tau, F(\tau)) d\tau,$$

$$M(t) = M_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_3(\tau, M(\tau)) d\tau,$$

$$B_1(t) = B_{10} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_4(\tau, B_1(\tau)) d\tau.$$

Now we can present the following recursive formula:

$$B_n(t) = B_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_1(\tau, B_{n-1}(\tau)) d\tau, \tag{17}$$

$$F_n(t) = F_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_2(\tau, F_{n-1}(\tau)) d\tau,$$

$$M_n(t) = M_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_3(\tau, M_{n-1}(\tau)) d\tau,$$

$$B_{1n}(t) = B_{10} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_4(\tau, B_{1(n-1)}(\tau)) d\tau.$$

Also the initial conditions are given as $B(t_0) = B_0$, $F(t_0) = F_0$, $M(t_0) = M_0$ and $B_1(t_0) = B_{10}$. Now, we obtain the difference between the successive terms in the expression.

$$\begin{aligned} \phi_n(t) &= B_n(t) - B_{n-1}(t) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(F_1(\tau, B_{n-1}(\tau)) - F_1(\tau, B_{n-2}(\tau))\right) d\tau, \\ \psi_n(t) &= F_n(t) - F_{n-1}(t) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(F_2(\tau, F_{n-1}(\tau)) - F_2(\tau, F_{n-2}(\tau))\right) d\tau, \\ \mu_n(t) &= M_n(t) - M_{n-1}(t) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(F_3(\tau, M_{n-1}(\tau)) - F_3(\tau, M_{n-2}(\tau))\right) d\tau, \\ \varepsilon_n(t) &= B_{1n}(t) - B_{1(n-1)}(t) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(F_4(\tau, B_{1(n-1)}(\tau)) - F_4(\tau, B_{1(n-2)}(\tau))\right) d\tau. \end{aligned} \tag{18}$$

It is worth noticing that

$$B_n(t) = \sum_{i=1}^{n} \phi_n(t), \tag{19}$$

$$F_n(t) = \sum_{i=1}^{n} \psi_n(t),$$

$$M_n(t) = \sum_{i=1}^{n} \mu_n(t),$$

$$B_{1n}(t) = \sum_{i=1}^{n} \varepsilon_n(t).$$

Let us consider equality (18), applying the norm on both sides of the equation and considering triangular inequality and then the equation reduces to (20),

$$\begin{aligned}\|\phi_n(t)\| &= \|B_n(t) - B_{n-1}(t)\| \\ &\leq \frac{1}{\Gamma(\alpha)} \left\| \int_0^t (t-\tau)^{\alpha-1} \left(F_1(\tau, B_{n-1}(\tau)) - F_1(\tau, B_{n-2}(\tau))\right) d\tau \right\|.\end{aligned} \tag{20}$$

As the kernel satisfies the Lipschitz condition, we have

$$\|B_n(t) - B_{n-1}(t)\| \leq \frac{L_1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|B_{n-1}(\tau) - B_{n-2}(\tau)\| d\tau, \tag{21}$$

then we get

$$\|\phi_n(t)\| \leq \frac{L_1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|\phi_{n-1}(t)\| d\tau. \tag{22}$$

Similarly, we get the following results:

$$\begin{aligned}\|\psi_n(t)\| &\leq \frac{L_2}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|\psi_{n-1}(t)\| d\tau, \\ \|\mu_n(t)\| &\leq \frac{L_3}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|\mu_{n-1}(t)\| d\tau, \\ \|\varepsilon_n(t)\| &\leq \frac{L_4}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|\varepsilon_{n-1}(t)\| d\tau,\end{aligned} \tag{23}$$

after the above results, let us give a new theorem for solutions of model.
Theorem 2 : The nuclear family model with the Caputo fractional derivative (9) has a unique solution under the conditions that we can find $t_{\max}$ satisfying

$$\frac{t_{\max}^{\alpha}}{\Gamma(\alpha)} L_i < 1, \text{ for } i = 1, 2, 3, 4. \tag{24}$$

Proof : We know that the functions $B(t), F(t), M(t)$ and $B_1(t)$ are bounded. Also we have shown that their kernels satisfy the Lipschitz condition. So from the equality (22)-(23), we obtain the succeeding relations as follows:

$$\begin{aligned}\|\phi_n(t)\| &\leq \|B_0\| \left[\frac{t_{\max}^{\alpha}}{\Gamma(\alpha)} L_1\right]^n, \\ \|\psi_n(t)\| &\leq \|F_0\| \left[\frac{t_{\max}^{\alpha}}{\Gamma(\alpha)} L_2\right]^n, \\ \|\mu_n(t)\| &\leq \|M_0\| \left[\frac{t_{\max}^{\alpha}}{\Gamma(\alpha)} L_3\right]^n, \\ \|\varepsilon_n(t)\| &\leq \|B_{10}\| \left[\frac{t_{\max}^{\alpha}}{\Gamma(\alpha)} L_4\right]^n.\end{aligned} \tag{25}$$

Thus equality (19) exists and is a smooth function. To show that the above functions are the solutions of the model, let we assume

$$\begin{aligned} B(t)-B_0 &= B_n(t)-b_n(t), \\ F(t)-F_0 &= F_n(t)-c_n(t), \\ M(t)-M_0 &= M_n(t)-d_n(t), \\ B_1(t)-B_{10} &= B_{1n}(t)-e_n(t). \end{aligned} \tag{26}$$

Our aim here is to show that the term at infinity goes $\|b_\infty(t)\| \longrightarrow 0$. Therefore we have

$$\begin{aligned} \|b_n(t)\| &\leq \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(F_1(\tau,B(\tau)) - F_1(\tau,B_{n-1}(\tau))\right) d\tau \right\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left\|F_1(\tau,B(\tau)) - F_1(\tau,B_{n-1}(\tau))\right\| d\tau \\ &\leq \frac{t^\alpha L_1}{\Gamma(\alpha)} \left\|B - B_{n-1}\right\|. \end{aligned} \tag{27}$$

Repeating this process recursively, we obtain

$$\|b_n(t)\| \leq \|B_0\| \left[\frac{t^\alpha}{\Gamma(\alpha)}\right]^{n+1} L_1^n M. \tag{28}$$

Then at $t_{\max}$ we have

$$\|b_n(t)\| \leq \|B_0\| \left[\frac{t_{\max}^\alpha}{\Gamma(\alpha)}\right]^{n+1} L_1^n M. \tag{29}$$

With applying the limit on both sides as n tends to infinity, we obtain $\|b_\infty(t)\| \longrightarrow 0$. This completes the proof.

3.2 Uniqueness of the special Solution

Another important application is to prove the uniqueness of the system of solutions. So we assume by contraction that there exists another system of solutions of (9), $B_2(t)$, $F_2(t)$, $M_2(t)$ and $B_{12}(t)$. Then

$$\|B(t) - B_2(t)\| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(F_1(\tau,B(\tau)) - F_1(\tau,B_2(\tau))\right) d\tau. \tag{30}$$

Applying the norm to eq. (30), we get

$$\|B(t) - B_2(t)\| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left\|F_1(\tau,B(\tau)) - F_1(\tau,B_2(\tau))\right\| d\tau. \tag{31}$$

By using the Lipschitz condition properties of the kernel, we have

$$\|B(t) - B_2(t)\| \leq \frac{t^\alpha L_1}{\Gamma(\alpha)} \|B(t) - B_2(t)\|. \tag{32}$$

This gives

$$\|B(t) - B_2(t)\| \left(1 - \frac{t^\alpha L_1}{\Gamma(\alpha)}\right) \leq 0, \tag{33}$$

$$\|B(t) - B_2(t)\| = 0 \longrightarrow B(t) = B_2(t). \tag{34}$$

So the equation has a unique solution. It is clear that we can show the same results for other solutions of $F(t)$, $M(t)$ and $B_1(t)$.

4 Two-step Adams-Bashforth scheme with fractional order Caputo derivative

In this section we consider the two-step Adams-Bashforth scheme with Caputo derivative which is given by Atangana and Owolabi in [9]. Let us give fractional differential equation with fractional order Caputo derivative as below:

$$\begin{aligned} {}_0^C D_t^{\alpha} x(t) &= F(t,x(t)), \\ x(0) &= x_{0.} \end{aligned} \tag{35}$$

The above fractional order Caputo equation is equal to integral equation as below:

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t F(\tau,x(\tau))(t-\tau)^{\alpha-1} d\tau. \tag{36}$$

With using the fundamental theorem of calculus and taking $t = t_{n+1}$, we have

$$x(t_{n+1}) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} F(\tau,x(\tau))(t_{n+1}-\tau)^{\alpha-1} d\tau. \tag{37}$$

When $t = t_n$, we have

$$x(t_n) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_n} F(\tau,x(\tau))(t_n-\tau)^{\alpha-1} d\tau. \tag{38}$$

Then we can write follows that

$$\begin{aligned} x(t_{n+1}) - x(t_n) = {} & \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} F(\tau,x(\tau))(t_{n+1}-\tau)^{\alpha-1} d\tau \\ & - \frac{1}{\Gamma(\alpha)} \int_0^{t_n} F(\tau,x(\tau))(t_n-\tau)^{\alpha-1} d\tau. \end{aligned} \tag{39}$$

To get the value of integrals

$$\int_0^{t_{n+1}} F(\tau,x(\tau))(t_{n+1}-\tau)^{\alpha-1} d\tau \tag{40}$$

and

$$\int_0^{t_n} F(\tau,x(\tau))(t_n-\tau)^{\alpha-1} d\tau, \tag{41}$$

we can use the polynomial interpolation $p(\tau)$ as an approximation of $F(\tau,x(\tau))$. Then the interpolation is taking as with Lagrange polynomial

$$p(\tau) = F(\tau,x(\tau)) = \frac{\tau-\tau_{n-1}}{\tau_n-\tau_{n-1}} F(\tau_n,x(\tau_n)) + \frac{\tau-\tau_n}{\tau_{n-1}-\tau_n} F(\tau_{n-1},x(\tau_{n-1})). \tag{42}$$

If we integrate and simplify the right side of equality, then we get

$$\begin{aligned}
&\int_0^{t_{n+1}} \left(\begin{array}{c} \frac{\tau-\tau_{n-1}}{\tau_n-\tau_{n-1}} F(\tau_n, x(\tau_n)) \\ +\frac{\tau-\tau_n}{\tau_{n-1}-\tau_n} F(\tau_{n-1}, x(\tau_{n-1})) \end{array} \right) (t_{n+1}-\tau)^{\alpha-1} d\tau \\
&= \frac{F(t_n, x(t_n))}{h\Gamma(\alpha)} \left(\frac{2h}{\alpha} t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} \right) \\
&\quad - \frac{F(t_{n-1}, x(t_{n-1}))}{h\Gamma(\alpha)} \left(\frac{h}{\alpha} t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} \right),
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
&\int_0^{t_n} \left(\begin{array}{c} \frac{\tau-\tau_{n-1}}{\tau_n-\tau_{n-1}} F(\tau_n, x(\tau_n)) \\ +\frac{\tau-\tau_n}{\tau_{n-1}-\tau_n} F(\tau_{n-1}, x(\tau_{n-1})) \end{array} \right) (t_n-\tau)^{\alpha-1} d\tau \\
&= \frac{F(t_n, x(t_n))}{h\Gamma(\alpha)} \left(\frac{h}{\alpha} t_n^{\alpha} - \frac{t_n^{\alpha+1}}{\alpha+1} \right) \\
&\quad + \frac{F(t_{n-1}, x(t_{n-1}))}{h\Gamma(\alpha+1)} t_n^{\alpha+1}.
\end{aligned} \tag{44}$$

Here t_{n-1}, t_n and t_{n+1} are equally spaced then we take

$$\begin{aligned}
t_n - t_{n-1} &= h, \\
t_{n+1} - t_n &= h.
\end{aligned} \tag{45}$$

Therefore finally we get

$$\begin{aligned}
x(t_{n+1}) = x(t_n) &+ \frac{F(t_n, x(t_n))}{h\Gamma(\alpha)} \left(\frac{2h}{\alpha} t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{h}{\alpha} t_n^{\alpha} - \frac{t_n^{\alpha+1}}{\alpha} \right) \\
&+ \frac{F(t_{n-1}, x(t_{n-1}))}{h\Gamma(\alpha)} \left(\frac{h}{\alpha} t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{t_n^{\alpha}}{\alpha+1} \right) + R_n^{\alpha}(t).
\end{aligned} \tag{46}$$

Here $R_n^{\alpha}(t)$ is error term for two step Adams-Bashforth scheme and calculated as below:

$$\begin{aligned}
R_n^{\alpha}(t) &= \frac{F^{(n+1)}(t, x(t))}{(n+1)!} \prod_{i=0}^{n} (t - t_i) \\
&< \frac{h^{3+\alpha} t_{\max}}{12\Gamma(\alpha+1)} ((n+1)^{\alpha} + n^2).
\end{aligned} \tag{47}$$

Readers can be found detailed analysis of method in paper [9].

4.1 Application of the two-step fractional Adams-Bashforth method on fractional order nuclear family model via Caputo derivative

Let us consider the fractional order nuclear family model as below:

$$\begin{aligned}
{}_a^C D_t^{\alpha} B(t) &= F_1(t, B(t)), \\
{}_a^C D_t^{\alpha} F(t) &= F_2(t, F(t)), \\
{}_a^C D_t^{\alpha} M(t) &= F_3(t, M(t)), \\
{}_a^C D_t^{\alpha} B_1(t) &= F_4(t, B_1(t)),
\end{aligned} \tag{48}$$

with initial conditions $B(t_0) = B_0$, $F(t_0) = F_0$, $M(t_0) = M_0$ and $B_1(t_0) = B_{10}$.

Here

$$\begin{aligned} F_1(t,B(t)) &= aB(t) + b(F(t) - M(t))(c - (F(t) - M(t)) + \gamma_1, \\ F_2(t,F(t)) &= eF(t) + gB(t)(h - B(t)) + jM(t) + \gamma_2, \\ F_3(t,M(t)) &= kM(t) + mB_1(t)(n - B_1(t)) + pF(t) + \gamma_3, \\ F_4(t,B_1(t)) &= aB_1(t) + b(M(t) - F(t))(d - (M(t) - F(t)) + \gamma_4. \end{aligned} \tag{49}$$

By using the numerical shcheme of above (46)-(47) then we have

$$\begin{aligned} B(t_{n+1}) &= B(t_n) + \frac{F_1(t_n,B(t_n))}{h\Gamma(\alpha)}\left(\frac{2h}{\alpha}t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{h}{\alpha}t_n^{\alpha} - \frac{t_n^{\alpha+1}}{\alpha}\right) \\ &\quad + \frac{F_1(t_{n-1},B(t_{n-1}))}{h\Gamma(\alpha)}\left(\frac{h}{\alpha}t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{t_n^{\alpha}}{\alpha+1}\right) + R_n^{\alpha}(t), \end{aligned} \tag{50}$$

$$\begin{aligned} F(t_{n+1}) &= F(t_n) + \frac{F_2(t_n,F(t_n))}{h\Gamma(\alpha)}\left(\frac{2h}{\alpha}t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{h}{\alpha}t_n^{\alpha} - \frac{t_n^{\alpha+1}}{\alpha}\right) \\ &\quad + \frac{F_2(t_{n-1},F(t_{n-1}))}{h\Gamma(\alpha)}\left(\frac{h}{\alpha}t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{t_n^{\alpha}}{\alpha+1}\right) + R_n^{\alpha}(t), \end{aligned}$$

$$\begin{aligned} M(t_{n+1}) &= M(t_n) + \frac{F_3(t_n,M(t_n))}{h\Gamma(\alpha)}\left(\frac{2h}{\alpha}t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{h}{\alpha}t_n^{\alpha} - \frac{t_n^{\alpha+1}}{\alpha}\right) \\ &\quad + \frac{F_3(t_{n-1},M(t_{n-1}))}{h\Gamma(\alpha)}\left(\frac{h}{\alpha}t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{t_n^{\alpha}}{\alpha+1}\right) + R_n^{\alpha}(t), \end{aligned}$$

$$\begin{aligned} B_1(t_{n+1}) &= B_1(t_n) + \frac{F_4(t_n,B_1(t_n))}{h\Gamma(\alpha)}\left(\frac{2h}{\alpha}t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{h}{\alpha}t_n^{\alpha} - \frac{t_n^{\alpha+1}}{\alpha}\right) \\ &\quad + \frac{F_4(t_{n-1},B_1(t_{n-1}))}{h\Gamma(\alpha)}\left(\frac{h}{\alpha}t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{t_n^{\alpha}}{\alpha+1}\right) + R_n^{\alpha}(t). \end{aligned}$$

5 Conclusion

In this paper fractional order nuclear family model is considered. Here, we generalize the previous model by considering the order as fractional order. As we saw that, the fractional order model is much more efficient in modeling than its integer order version. The detailed analysis such as existence and uniqueness results of the solution and efficient numerical scheme for model are presented.

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