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# A New Approach to (3+1) Dimensional Boiti–Leon–Manna–Pempinelli Equation

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## Abstract

In this article, some new travelling wave solutions of the (3+1) dimensional Boiti–Leon–Manna–Pempinelli (BLMP) equation are obtained using the modified exponential function method. When the solution functions obtained are examined, it is seen that functions with periodic functions are obtained. Two and three dimensional graphs of the travelling wave solutions of the BLMP equation are drawn by selecting the appropriate parameters

**Keywords:** Boiti–Leon–Manna–Pempinelli (BLMP) equation; modified exponential function method (MEFM)
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## 1 Introduction

Nonlinear partial differential equations (NPDE) have an important role to describe natural phenomenon from biology to engineering. Especially in engineering, acoustic waves, water waves, electromagnetic waves have been model via NPDE. Physics and engineering applications have concentrated to the behavior of waves, for this reason solutions of such equations have attracted the attention of many scientists for many years. Hence, there are various analytical methods in the literature used by researchers to obtain solutions for such equations. Some of these are the multipliers method [1],the simplest equation method [2], the (G/G)-expansion method [3–6], the Sine- Gordon expansion method [7–11],the extended trial equation method [12, 13], the new function method [14, 15].
In this study, we used the Modified Exponential Function Method (MEFM) to the (3+1) dimensional Boiti-Leon-Manna-Pempinelli equation (BLMP) which is used to describe incompressible liquid in fluid mechanics. The equation is given as,

$$\upsilon_{yt}+\upsilon_{zt}+\upsilon_{xxxy}+\upsilon_{xxxz}-3\upsilon_{x}\left(\upsilon_{xy}+\upsilon_{xz}\right)-3\upsilon_{xx}\left(\upsilon_{y}+\upsilon_{z}\right)=0. \tag{1}$$

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Eq.(1) is derived from (2+1)-dimensional Boiti–Leon–Manna– Pempinelli equation by Darvishi et al. [16]. They have submitted multisoliton solutions of both equations. There are several studies on the Boiti–Leon–Manna–Pempinelli equation in the literature. Authors have investigated the Lax pair of Eq.(1) by singular manifolds method [17]. Employing Hirota's bilinear method different types of lump solitons of Eq.(1) have been submitted in [18]. They have disscused Nth-order soliton solutions, rational solutions, periodic wave solutions using Pffafian tecnique, the ansatz method and the Hirota-Riemann method respectively [19]. Further researchs can be seen [?, 2, 20–33].
This paper is rested as manner; we give steps of the modified exponential function method in section 2, then an application of the mention method is given in section 3. In last section 4, we give some conclusions on the obtained wave solutions.

## 2 The Manner of the Method

In this section, we give the manner of the MEFM [34–36].
Let's consider the following general form of nonlinear partial differential equation;

$$P\left(u, u_x, u_y, u_z, u_{xy}, u_{yt}, u_{xxz}, u_{xxxz}, \cdots\right) = 0, \tag{2}$$

where $u = u(x,y,z,t)$is unknown solution function.
**Step 1.** Regarding travelling wave transformation as follows;

$$u(x,t) = U(\zeta), \zeta = x+y+z-ct, \tag{3}$$

where $c$ is a non-zero real value, required derivative terms are substituted into Eq. (2). By this way, the following nonlinear ordinary differential equation is obtained,

$$N\left(U, U', U'', \cdots\right) = 0. \tag{4}$$

**Step 2:** We think the solution function $U$in Eq. (4) as follows;

$$U(\zeta) = \frac{\sum_{i=0}^{N} A_i \left[\exp\left(-\Omega(\zeta)\right)\right]^i}{\sum_{j=0}^{M} B_j \left[\exp\left(-\Omega(\zeta)\right)\right]^j} = \frac{A_0 + A_1 \exp(-\Omega) + \cdots + A_N \exp\left(N(-\Omega)\right)}{B_0 + B_1 \exp(-\Omega) + \cdots + B_M \exp\left(M(-\Omega)\right)}, \tag{5}$$

where$A_i, B_j, (0 \le i \le N, 0 \le j \le M)$are constants. Using the balancing principle, a relationship is set between the upper limits of Eq. (5), $M$ and $N$ values. (Balancing principle; it is obtained by equalizing the term containing the highest order derivative and the highest degree nonlinear term). $A_N \neq 0, B_M \neq 0$, and $\Omega = \Omega(\zeta)$provides the following differential equation;

$$\Omega'(\zeta) = \exp\left(-\Omega(\zeta)\right) + \mu \exp\left(\Omega(\zeta)\right) + \lambda. \tag{6}$$

When the Eq. (6) is solved, the following solution families are obtained [37].
**Family 1:** When $\mu \neq 0, \lambda^2 - 4\mu > 0$,

$$\Omega(\zeta) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu}}{2\mu} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\zeta + E)\right) - \frac{\lambda}{2\mu}\right). \tag{7}$$

**Family 2:** When $\mu \neq 0, \lambda^2 - 4\mu < 0$,

$$\Omega(\zeta) = \ln\left(\frac{\sqrt{-\lambda^2 + 4\mu}}{2\mu} \tan\left(\frac{\sqrt{-\lambda^2 + 4\mu}}{2}(\zeta + E)\right) - \frac{\lambda}{2\mu}\right). \tag{8}$$

**Family 3:** When $\mu = 0, \lambda \neq 0,$ and $\lambda^2 - 4\mu > 0,$

$$\Omega(\zeta) = -\ln\left(\frac{\lambda}{\exp(\lambda(\zeta+E))-1}\right). \tag{9}$$

**Family 4:** When $\mu \neq 0, \lambda \neq 0,$ and $\lambda^2 - 4\mu = 0,$

$$\Omega(\zeta) = \ln\left(-\frac{2\lambda(\zeta+E)+4}{\lambda^2(\zeta+E)}\right). \tag{10}$$

**Family 5:** When $\mu = 0, \lambda = 0,$ and $\lambda^2 - 4\mu = 0,$

$$\Omega(\zeta) = \ln(\zeta+E). \tag{11}$$

where $A_0, A_1, \ldots, A_N, B_0, B_1, \ldots, B_M, E, \lambda, \mu$ are constants.
**Step 3:** The Eq.(6) and the solution families are written into the Eq. (5) to obtain the algebraic equation system consisting of exp $(\Omega(\zeta up))$.
The equation systems obtained are equalized to zero and $A_0, A_1, \ldots, A_N, B_0, B_1, \ldots, B_M, E, \lambda, \mu$ are obtained. These coefficients are written in Eq. (5) instead of Eq. (2) to provide the travelling wave solutions.

## 3 Applications

By using the wave transformation in equation (3) to equation (1), the following nonlinear differential equation is obtained;

$$-c\,\upsilon'' + \upsilon^{iv} - 6\,\upsilon'\,\upsilon'' = 0. \tag{12}$$

If the equation (12) is integrated,

$$-c\,\upsilon' + \upsilon''' - 3\,(\upsilon')^2 = 0. \tag{13}$$

It is described in a simpler way as follows by applying the transformation $\upsilon' = \omega$ to the nonlinear ordinary differential equation (13).

$$-c\,\omega + \omega'' - 3\,(\omega)^2 = 0. \tag{14}$$

When the balancing principle is applied to the equation (14), the following relation is found between $M$ and $N$,

$$N = M + 2.$$

$N = 3$ for $M = 1$ and values to Eq.(5) , $\omega, \omega', \omega''$ can be written as follows,

$$\begin{aligned}
\omega(\zeta) &= \tfrac{\psi}{\varphi} = \tfrac{A_0 + A_1 e^{-\Omega(\zeta)} + A_2 e^{-2\Omega(\zeta)} + A_3 e^{-3\Omega(\zeta)}}{B_0 + B_1 e^{-\Omega(\zeta)}},\\
\omega'(\zeta) &= \tfrac{\psi'\varphi - \psi\varphi'}{\varphi^2},\\
\omega''(\zeta) &= \tfrac{\psi''\varphi^3 - \varphi^2\psi'\varphi' - (\psi\varphi'' + \psi'\varphi')\varphi^2 + 2(\psi')^2\psi\varphi}{\varphi^4}.
\end{aligned} \tag{15}$$

Substituting Eq.(15) into Eq. (14), $\omega$ related solution functions are obtained. By integrating these solutions, the travelling wave solutions providing the equation (1) were obtained as follows.
**CASE 1:**

$$A_0 = 2\mu B_0, A_1 = 2(\lambda B_0 + \mu B_1), A_2 = 2(B_0 + \lambda B_1), A_3 = 2B_1, c = \lambda^2 - 4\mu. \tag{16}$$

Using the coefficients given above, the following solution families are obtained.
**Family 1:**

$$\upsilon_{1,1}(x,y,z,t) = -\frac{\lambda^3 - 4\lambda\mu + 2\sqrt{\lambda^2 - 4\mu}\,\mu\,Sinh[\tau]}{\lambda^2 - 2\mu + 2\mu\,Cosh[\tau]}, \tag{17}$$

where $\tau(x,y,z,t) = (EE+\zeta)\sqrt{\lambda^2-4\mu}$.

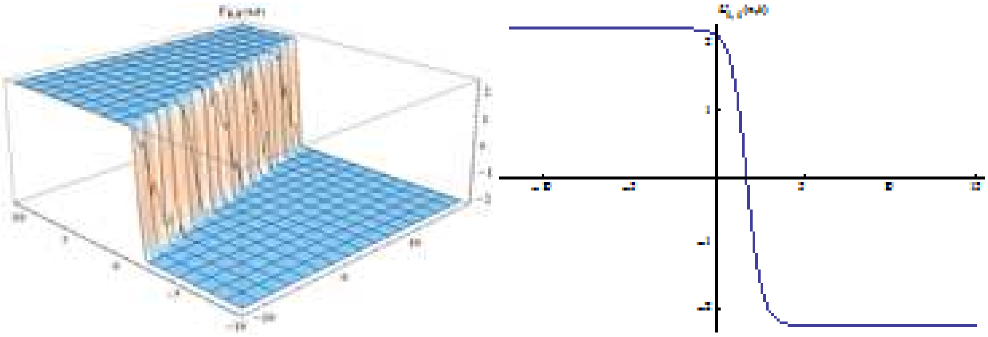

**Figure 1.** The 3D and $t = 1$ for 2D of Eq.(17)

**Family 2:**

$$\upsilon_{1,2}(x,y,z,t) = \frac{-\lambda^3+4\lambda\mu+2\mu\sqrt{-\lambda^2+4\mu}\,Sin[\xi]}{\lambda^2-2\mu+2\mu\,Cos[\xi]}, \tag{18}$$

where $\xi(x,y,z,t) = (EE+\zeta)\sqrt{-\lambda^2+4\mu}$.

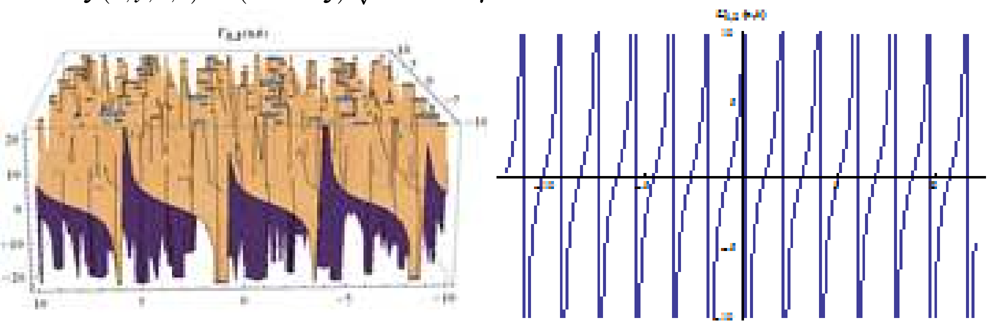

**Figure 2.** The 3D and $t = 1$ for 2D of Eq.(18)

**Family 3:**

$$\upsilon_{1,3}(x,y,z,t) = -\frac{2\lambda}{-1+e^{\lambda(EE+\zeta)}}, \tag{19}$$

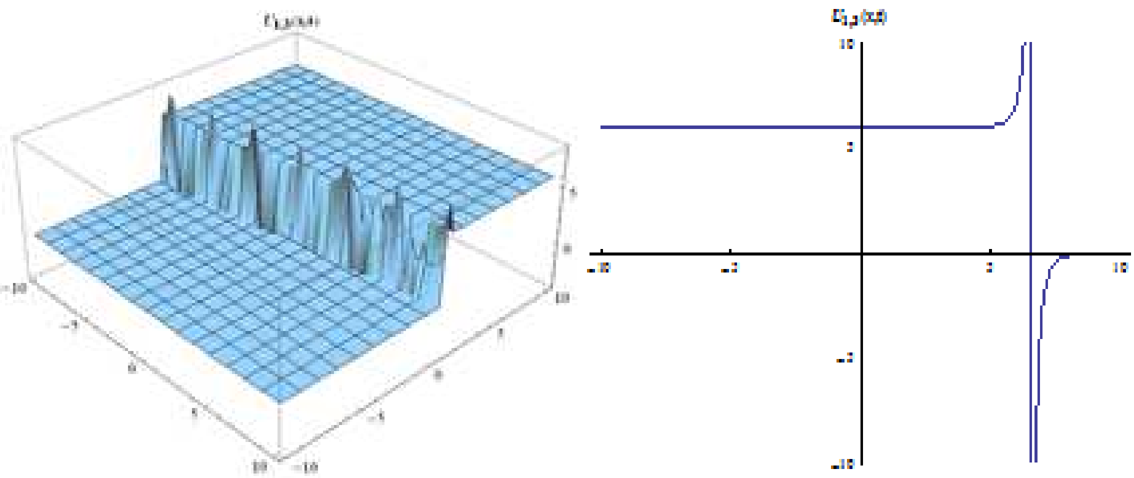

**Figure 3.** The 3D and $t = 1$for 2D of Eq.(19)

**CASE 2**:

$$\begin{aligned} &A_0 = -\tfrac{1}{3}(c-6\mu)B_0, A_1 = -2\sqrt{-c+4\mu}B_0 - \tfrac{1}{3}(c-6\mu)B_1, \\ &A_2 = 2\left(B_0 - \sqrt{-c+4\mu}B_1\right), A_3 = 2B_1, \lambda = -\sqrt{-c+4\mu}. \end{aligned} \tag{20}$$

Equations (20) the following solution families are obtained from equations.

**Family 1:**

$$\upsilon_{2,1}(x,y,z,t) = \frac{3c\sqrt{-c+4\mu} - c^2(EE+\zeta) + 2c\mu(EE+\zeta)(1+Cos[\kappa]) - 6\sqrt{c}\mu Sin[\kappa]}{3(c-2\mu-2\mu Cos[\kappa])}, \tag{21}$$

where $\kappa(x,y,z,t) = (EE+\zeta)\sqrt{c}$.

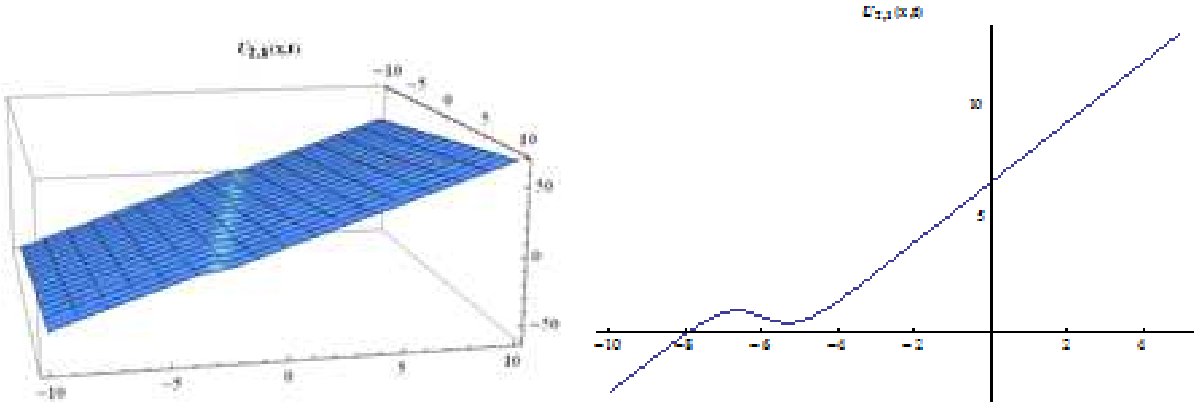

**Figure 4:** The 3D and$t=1$for 2D of Eq.(21)

**Family 2:**

$$\upsilon_{2,1}(x,y,z,t) = \frac{3c\sqrt{-c+4\mu} - c^2(EE+\zeta) + 2c\mu(EE+\zeta)(1+Cos[\kappa]) - 6\sqrt{c}\mu Sin[\kappa]}{3(c-2\mu-2\mu Cos[\kappa])}, \tag{22}$$

where $\kappa(x,y,z,t) = (EE+\zeta)\sqrt{c}$.

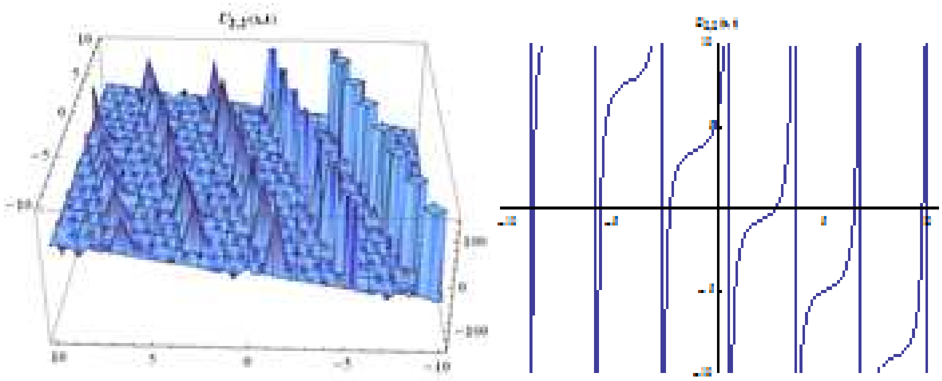

**Figure 5:** The 3D and$t=1$for 2D of Eq.(22)

**Family 3:**

$$\upsilon_{2,3}(x,y,z,t) = \left(-\frac{2\sqrt{-c}}{-1+e^{\sqrt{-c}(EE+\zeta)}} - \frac{c\zeta}{3}\right), \tag{23}$$

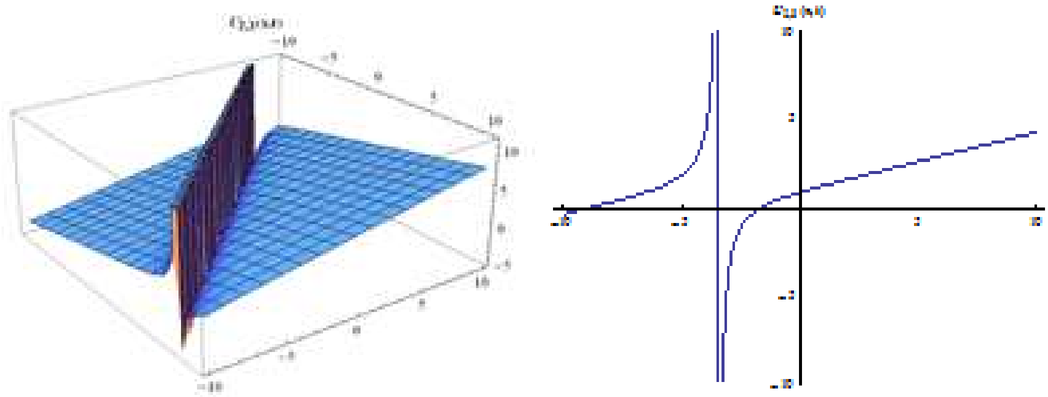

**Figure 6:** The 3D and $t = 1$ for 2D of Eq.(23)

**CASE 3**:

$$A_0 = \frac{1}{6}\left(3\lambda^2 + c\right) B_0, A_1 = 2\lambda B_0 + \frac{1}{6}\left(3\lambda^2 + c\right) B_1, A_2 = 2\left(B_0 + \lambda B_1\right),$$
$$A_3 = 2B_1, \mu = \frac{1}{4}\left(\lambda^2 + c\right). \tag{24}$$

According to the coefficients, the following solution families are get.

**Family 2:**

$$\upsilon_{3,2}(x,y,z,t) = \left( -\frac{3\lambda^2 + c\left(3 + \lambda\left(EE + \zeta\right)\right)}{3\lambda} + \frac{c + \lambda^2}{\lambda - \sqrt{c}Tan\left[\frac{1}{2}\sqrt{c}\left(EE + \zeta\right)\right]} \right), \tag{25}$$

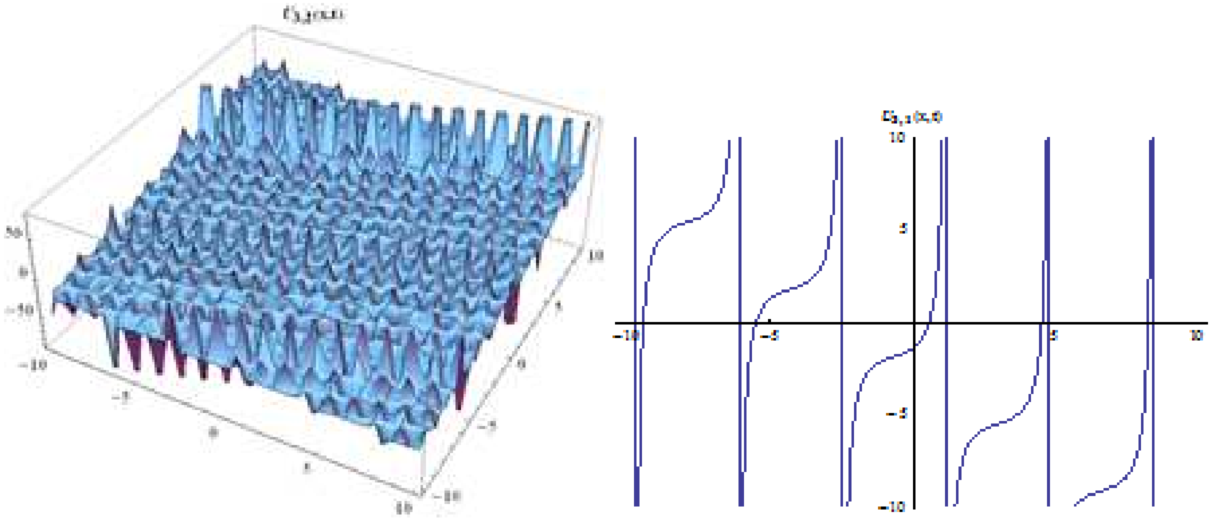

**Figure 7:** The 3D and $t = 1$ for 2D of Eq.(25)

**Remark:** Since the above cases do not meet the requirements of the conditions, no suitable solution has been found for the families.

## 4 Conclusion

In this study, new travelling wave solutions of Boiti-Leon-Manna-Pempinelli (BLMP) equation have been successfully obtained by using modified exponential function method. When we compare our results with the solutions obtained for this equation in the literature, we see that all solutions are completely different. We have drawn two and three dimensional graphs of all travelling wave solutions by selecting the suitable constants. The solutions obtained can be said to be an effective method for obtaining analytical solutions of such nonlinear

differential equations. The solutions found include trigonometric and hyperbolic functions. Such functions are also periodic functions. The advantage of such functions is that it allows us to comfortably comment on the physical behavior of the wave, regardless of the range of the graph of the resulting solution function.
The hyperbolic functions and trigonometric functions are arisen in both mathematics and physics. For example, the hyperbolic cosine functions are shape of catenary, the hyperbolic tangent functions arise in calculate to magnetic moment and rapidity of special relativity, the hyperbolic secant functions arise in the profile of a laminar jet, the hyperbolic cotangent functions arise in the Langevin function for magnetic polarization [38].

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