A New Generalization of Pochhammer Symbol and Its Applications

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Abstract

In this paper, we introduce a new generalization of the Pochhammer symbol by means of the generalization of extended gamma function (4). Using the generalization of Pochhammer symbol, we give a generalization of the extended hypergeometric functions one or several variables. Also, we obtain various integral representations, derivative formulas and certain properties of these functions.

1 Introduction

The classical Pochhammer symbol $(\lambda)_v$ is given as follows: [1, 7, 9, 10, 18, 22, 23, 26, 29, 30, 33]

$$
\begin{aligned}
(\lambda)_v &= \frac{\Gamma(\lambda + v)}{\Gamma(\lambda)} && (\lambda, v \in \mathbb{C} \setminus \mathbb{Z}_0^-) \\
&= \begin{cases} 1 & (v = 0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (v = n \in \mathbb{N}) \end{cases}.
\end{aligned}
\tag{1}
$$

and $\Gamma(\lambda)$ is the familiar Gamma function whose Euler's integral is (see, e.g., [1, 7, 9, 10, 18, 22, 23, 30, 33])

$$
\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt \quad (\Re(z) > 0).
\tag{2}
$$

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From (1) and (2), it is easy to see the following integral formula

$$(\lambda)_v = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-t} t^{\lambda+v-1} dt \quad (\Re(\lambda+v) > 0). \tag{3}$$

Here and in the following, let $\mathbb{C}$, $\mathbb{Z}_0^-$, and $\mathbb{N}$ be the sets of complex numbers, non-positive integers and positive integers, respectively and assume that $\min\{\Re(p), \Re(q), \Re(\kappa), \Re(\mu)\} > 0$. Recently, various generalization of beta functions have been introduced and investigated (see, e.g., [2–6, 9, 13–17, 19–21, 24–26, 28, 29] and the references cited therein). Very recently, Şahin et al. [31] introduced and studied following generalization of the extended gamma function as follows:

$$\Gamma_{p,q}^{(\kappa,\mu)}(z) = \int_0^\infty t^{z-1} \exp\left(-\frac{t^\kappa}{p} - \frac{q}{t^\mu}\right) dt, \tag{4}$$

$$(\Re(z) > 0, \Re(p) > 0, \Re(q) > 0, \Re(\kappa) > 0, \Re(\mu) > 0).$$

It is easily seen that the special cases of (4) returns to other forms of gamma functions. For example, $\Gamma_{1,0}^{(1,1)}(x) = \Gamma(x)$, $\Gamma_{1,q}^{(1,1)}(x) = \Gamma_q(x)$.

First, by selecting a known generalization of the gamma function (4), systematicaly, we goal to introduce new Pochhammer symbol by using the gamma function (4). Also, we give some properties for the extended Pochhammer symbol. Next, using this Pochhammer symbol, we define a new generalization of the extended hypergeometric functions one or several variables such as Gauss hypergeometric function, confluent hypergeometric function, Appell hypergeometric function, Humbert hypergeometric function. Finally, for this a new generalization of the extended hypergeometric functions, we give some properties such as integral representations, derivative formulas and recuurence relations.

2 A New Generalization of the Pochhammer Symbol

In this section, we denote a new generalization of Pochhammer symbol (5). Also, we give some useful properties.

Definition 1. Let $\lambda, \mu \in \mathbb{C}$ and $\Re(p) > 0, \Re(q) > 0, \Re(\kappa) > 0, \Re(\mu) > 0$, the generalization of the extended Pochhammer symbol $(\lambda; p, q; \kappa, \mu)_v$ is given by

$$(\lambda; p, q; \kappa, \mu)_v := \begin{cases} \dfrac{\Gamma_{p,q}^{(\kappa,\mu)}(\lambda+v)}{\Gamma(\lambda)} & , \Re(p) > 0, \Re(q) > 0, \Re(\kappa) > 0, \Re(\mu) > 0, \\ (\lambda)_v & , p = 1, q = 0, \kappa = 1, \mu = 0, \end{cases} \tag{5}$$

where $\Gamma_{p,q}^{(\kappa,\mu)}$ is the generalization of the extended gamma function (4) [31].

Theorem 1. *For the generalization of the Pochhammer symbol* (5) *following integral representation holds true:*

$$(\lambda; p, q; \kappa, \mu)_v := \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda+v-1} \exp\left(-\frac{t^\kappa}{p} - \frac{q}{t^\mu}\right) dt \tag{6}$$

$$(\Re(p) > 0, \Re(q) > 0, \Re(\kappa) > 0, \Re(\mu) > 0).$$

Proof. Using the equality (4) in the definition of the (5), we get the desired result (6).

Theorem 2. *Let* $\lambda, m, n \in \mathbb{C}$. *Then,*

$$(\lambda; p, q; \kappa, \mu)_{n+m} := (\lambda)_n (\lambda+n; p, q; \kappa, \mu)_m \tag{7}$$

Proof. From the equations (1) and (5), we obtain that

$$(\lambda; p, q; \kappa, \mu)_{m+n} := \frac{\Gamma_{p,q}^{(\kappa,\mu)}(\lambda + m + n)}{\Gamma(\lambda)} = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda + n)} \frac{\Gamma_{p,q}^{(\kappa,\mu)}(\lambda + m + n)}{\Gamma(\lambda)}$$

$$= (\lambda)_n (\lambda + n; p, q; \kappa, \mu)_m \tag{8}$$

By appealing the well-known properties of the classical Pochhammer symbol in (7) following features of generalization of the Pochhammer symbol can be easily obtained.

Corollary 3. *Let* $k, l, m, n \in \mathbb{N}_0$ *and* $N \in \mathbb{N}$. *Then,*

$$(\lambda; p, q; \kappa, \mu)_{m+n+l} := (\lambda)_m (\lambda + m)_n (\lambda + m + n; p, q; \kappa, \mu)_l$$

$$(\lambda; p, q; \kappa, \mu)_{m-n+l} := \frac{(-1)^n (\lambda)_m}{(1 - \lambda - m)_n} (\lambda + m - n; p, q; \kappa, \mu)_l$$

$$(\lambda; p, q; \kappa, \mu)_{2m+l} := 2^{2m} \left(\frac{\lambda}{2}\right)_m \left(\frac{\lambda + 1}{2}\right)_m (\lambda + 2m; p, q; \kappa, \mu)_l$$

$$(\lambda; p, q; \kappa, \mu)_{Nm+l} := N^{Nm} \left(\frac{\lambda}{N}\right)_m \left(\frac{\lambda + 1}{N}\right)_m \cdots \left(\frac{\lambda + N - 1}{N}\right)_m (\lambda + Nm; p, q; \kappa, \mu)_l$$

$$(\lambda + n; p, q; \kappa, \mu)_{n+l} := (\lambda + n)_n (\lambda + 2n; p, q; \kappa, \mu)_l = \frac{(\lambda)_{2n}}{(\lambda)_n} (\lambda + 2n; p, q; \kappa, \mu)_l$$

$$(\lambda + m; p, q; \kappa, \mu)_{n+l} := \frac{(\lambda)_n (\lambda + n)_m}{(\lambda)_m} (\lambda + m + n; p, q; \kappa, \mu)_l$$

$$(\lambda + km; p, q; \kappa, \mu)_{kn+l} := \frac{(\lambda)_{km+kn}}{(\lambda)_{km}} (\lambda + km + kn; p, q; \kappa, \mu)_l$$

$$(\lambda - n; p, q; \kappa, \mu)_{n+l} := (-1)^n (1 - \lambda)_n (\lambda; p, q; \kappa, \mu)_l$$

$$(\lambda - m; p, q; \kappa, \mu)_{n+l} := \frac{(1 - \lambda)_m (\lambda)_n}{(1 - \lambda - n)_m} (\lambda + n - m; p, q; \kappa, \mu)_l$$

$$(\lambda - km; p, q; \kappa, \mu)_{kn+l} := (-1)^{km} (\lambda)_{kn-km} (1 - \lambda)_{km} (\lambda + kn - km; p, q; \kappa, \mu)_l$$

$$(\lambda + m; p, q; \kappa, \mu)_{n-m+l} := \frac{(\lambda)_n}{(\lambda)_m} (\lambda + n; p, q; \kappa, \mu)_l$$

$$(\lambda - m; p, q; \kappa, \mu)_{n-m+l} := (-1)^m \frac{(\lambda)_n (1 - \lambda)_m}{(1 - \lambda - n)_{2m}} (\lambda + n - 2m; p, q; \kappa, \mu)_l$$

$$(-\lambda; p, q; \kappa, \mu)_{n+l} := (-1)^n (\lambda - n + 1)_n (-\lambda + n; p, q; \kappa, \mu)_l.$$

Remark 1. Taking $p = \kappa = \mu = 1$ in the *Corollary 3*, it is easily seen that the special case of extended Pochhammer symbol [29].

3 A New Generalization of the extended hypergeometric function

According to the generalization of the extended Pochhammer symbol $(\lambda; p, q; \kappa, \mu)_n$ $(n \in \mathbb{N}_0)$, a generalization of the extended hypergeometric function $_rF_s$ of r numerator parameters $a_1, \cdots, a_r$ and s denominator parameters $b_1, \cdots, b_s$ can be given as follows:

$$_rF_s \left[(a_1; p, q; \kappa, \mu), a_2, \cdots, a_r b_1, b_2, \cdots, b_s; z \right] := \sum_{n=0}^{\infty} \frac{(a_1; p, q; \kappa, \mu)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} \cdot \frac{z^n}{n!}, \tag{9}$$

on condition that the series on the right-hand side converges, it making sense that $a_j \in \mathbb{C}$ ($j = 1,...,r$) and $b_j \in \mathbb{C}$ $\setminus \mathbb{Z}_0^-$ ($j = 1,...,s; \mathbb{Z}_0^- = \{0,-1,-2,...\}$).

Particularly, the corresponding generalization of the extended confluent hypergeometric function $\Phi_{p,q}^{\kappa,\mu}$ and the Gauss hypergeometric function $F_{p,q}^{\kappa,\mu}$ are given by

$$\Phi_{p,q}^{\kappa,\mu}(a;b;z) := \sum_{n=0}^{\infty} \frac{(a;p,q;\kappa,\mu)_n}{(b)_n} \cdot \frac{z^n}{n!} \tag{10}$$

and

$$F_{p,q}^{\kappa,\mu}(a,b,c;z) := \sum_{n=0}^{\infty} \frac{(a;p,q;\kappa,\mu)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}, \tag{11}$$

respectively.

Theorem 4. *The following integral representation holds true:*

$$_rF_s\left[(a_1;p,q;\kappa,\mu),a_2,\cdots,a_r\, b_1,b_2,\cdots,b_s;z\right]$$

$$:= \frac{1}{\Gamma(a_1)} \int_0^{\infty} t^{a_1-1} \exp\left(-\frac{t^{\kappa}}{p} - \frac{q}{t^{\mu}}\right) \times_{r-1}F_s\left[a_2,\cdots,a_r\, b_1,b_2,\cdots,b_s;zt\right]dt, \tag{12}$$

$$(\Re(p) > 0, \Re(q) > 0, \Re(\kappa) > 0, \Re(\mu) > 0; \Re(b_s) > \Re(a_r) > 0,).$$

Proof. Using the integral representation given by (6) in the definition (9), we led to desired result (12).

Theorem 5. *The following integral representation holds true:*

$$_rF_s\left[(a_1;p,q;\kappa,\mu),a_2,\cdots,a_r\, b_1,b_2,\cdots,b_s;z\right] := \frac{1}{B(a_r,b_s-a_r)} \int_0^1 t^{a_r-1}(1-t)^{b_s-a_r-1}$$

$$\times\ _{r-1}F_{s-1}\left[(a_1;p,q;\kappa,\mu),a_2,\cdots,a_{r-1}\, b_1,b_2,\cdots,b_{s-1};zt\right]dt, \tag{13}$$

$$(\Re(p) > 0, \Re(q) > 0, \Re(\kappa) > 0, \Re(\mu) > 0; \Re(b_s) > \Re(a_r) > 0).$$

Proof. The classical Beta function $B(\alpha,\beta)$ defined by [1, 7, 9, 10, 18, 22, 23, 26, 29, 30, 33],

$$B(\alpha,\beta) = \begin{cases} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt & (\min\{\mathbb{R}(\alpha),\mathbb{R}(\beta)\} > 0) \\ \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\beta \in \mathbb{C}\setminus\mathbb{Z}_0^-). \end{cases} \tag{14}$$

Also, we have the following equation

$$\frac{(a_r)_n}{(b_s)_n} = \frac{1}{B(a_r,b_s-a_r)} \int_0^1 t^{a_r+n-1}(1-t)^{b_s-a_r-1}dt, \tag{15}$$

$$(\mathbb{R}(b_s) > \mathbb{R}(a_r) > 0; n \in \mathbb{N}_0)$$

Using the equalities (14), (15) in the generization of the extended hypergeometric function (9), we get the desired result (13).

Corollary 6. *Each of the integral representations hold true:*

$$\Phi_{p,q}^{\kappa,\mu}(a;b;z) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} \exp\left(-\frac{t^\kappa}{p} - \frac{q}{t^\mu}\right) {}_0F_1(-;b;zt)dt, \tag{16}$$

$$F_{p,q}^{\kappa,\mu}(a,b,c;z) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} \exp\left(-\frac{t^\kappa}{p} - \frac{q}{t^\mu}\right) {}_1F_1(b;c;zt)dt, \tag{17}$$

and

$$F_{p,q}^{\kappa,\mu}(a,b,c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} {}_1F_0((a;p,q;\kappa,\mu);-;zt)dt, \tag{18}$$

on condition that the integrals involved are convergent.

Theorem 7. *The following derivative formula holds true:*

$$\frac{d^n}{dz^n}\left\{ {}_rF_s\left[(a_1;p,q;\kappa,\mu),a_2,\cdots,a_r b_1,b_2,\cdots,b_s;z\right]\right\} := \frac{(a_1)_n(a_2)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n}$$

$$\times {}_rF_s\left[(a_1+n;p,q;\kappa,\mu),a_2+n,\cdots,a_r+n\,b_1+n,b_2+n,\cdots,b_s+n;z\right] \tag{19}$$

$$(\Re(p) > 0, \Re(q) > 0, \Re(\kappa) > 0, \Re(\mu) > 0; n \in \mathbb{N}_0).$$

Proof. Differentiating (9) with respect to z and then replacing $n \to n+1$ in the right-hand side term, we obtain

$$\frac{d}{dz}\left\{ {}_rF_s\left[(a_1;p,q;\kappa,\mu),a_2,\cdots,a_r b_1,b_2,\cdots,b_s;z\right]\right\} := \sum_{n=0}^\infty \frac{(a_1;p,q;\kappa,\mu)_{n+1}(a_2)_{n+1}\cdots(a_r)_{n+1}}{(b_1)_{n+1}(b_2)_{n+1}\cdots(b_s)_{n+1}} \frac{z^{n+1}}{(n+1)}$$

$$= \frac{a_1\cdots a_r}{b_1\cdots b_s} {}_rF_s\left[(a_1+1;p,q;\kappa,\mu),a_2+1,\cdots,a_r+1\,b_1+1,b_2+1,\cdots,b_s+1;z\right], \tag{20}$$

repeating the same procedure *n*-times gives the formula (19).

Choosing $r = s = 1$ and $r = 2, s = 1$ in (19), we have the derivative formulas for the (10) and (11), respectively.

Corollary 8. *The following derivative formulas hold true:*

$$\frac{d^n}{dz^n}\left\{\Phi_{p,q}^{\kappa,\mu}(a;b;z)\right\} = \frac{(a)_n}{(b)_n}\Phi_{p,q}^{\kappa,\mu}(a+n;b+n;z) \tag{21}$$

and

$$\frac{d^n}{dz^n}\left\{F_{p,q}^{\kappa,\mu}(a,b,c;z)\right\} = \frac{(a)_n(b)_n}{(c)_n}F_{p,q}^{\kappa,\mu}(a+n,b+n,c+n;z). \tag{22}$$

The Bessel function $J_\nu(z)$ and the modified Bessel function $I_\nu(z)$ are expressible as hypergeometric functions as follows [11, 12, 33]:

$$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)} {}_0F_1(-;\nu+1;-\frac{1}{4}z^2)$$

$$(\nu \in \mathbb{C} \setminus \mathbb{Z}^- (\mathbb{Z}^- = \{-1,-2,-3,...\})) \tag{23}$$

and

$$I_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)} {}_0F_1(-;\nu+1;\frac{1}{4}z^2)$$

$$(\nu \in \mathbb{C} \setminus \mathbb{Z}^-). \tag{24}$$

Additionally, for the incomplete gamma function $\gamma(s,x)$ defined by [10],

$$\gamma(s,x) = \int\limits_0^x t^{s-1}\exp(-t)dt \tag{25}$$

$$(\mathbb{R}(s) > 0;\ x \geq 0).$$

Also, we know that [7, 9, 10]

$$_1F_1[s;s+1;-x] = sx^{-s}\gamma(s,x). \tag{26}$$

So, we can deduce *Corollary 9* and *Corollary 10* by performing the relationships (23)-(26) in the equations (16) and (17) .

Corollary 9. *Each of the following integral representations hold true:*

$$\Phi_{p,q}^{\kappa,\mu}(a;b+1;-z) = \frac{\Gamma(b+1)}{\Gamma(a)}z^{-\frac{b}{2}}\int\limits_0^\infty t^{a-\frac{b}{2}-1}\exp(-\frac{t^\kappa}{p}-\frac{q}{t^\mu})J_b(2\sqrt{zt})dt \tag{27}$$

and

$$\Phi_{p,q}^{\kappa,\mu}(a;b+1;z) = \frac{\Gamma(b+1)}{\Gamma(a)}z^{-\frac{b}{2}}\int\limits_0^\infty t^{a-\frac{b}{2}-1}\exp(-\frac{t^\kappa}{p}-\frac{q}{t^\mu})I_b(2\sqrt{zt})dt \tag{28}$$

on condition that the integrals involves are convergent.

Corollary 10. *The following integral representation holds true:*

$$F_{p,q}^{\kappa,\mu}(a,b,b+1;-z) = \frac{bz^{-b}}{\Gamma(a)}\int\limits_0^\infty t^{a-b-1}\exp(-\frac{t^\kappa}{p}-\frac{q}{t^\mu})\gamma(b,zt)dt, \tag{29}$$

on condition that the integrals involves are convergent.

4 A New Generalization of the extended Appell hypergeometric functions

In this section, we introduce extended Appell hypergeometric series and some extended multivariable hypergeometric functions.

Let us introduce the extensions of the Appell's functions and extended Lauricella's hypergeometric function and other functions defined by

$$_{p,q}F_1^{(\kappa,\mu)}[a,b,c;d;x,y] := \sum_{m,n=0}^\infty \frac{(a;p,q;\kappa,\mu)_{m+n}(b)_m(c)_n}{(d)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \tag{30}$$

$$\max(|x|,|y|) < 1,$$

$$_{p,q}F_2^{(\kappa,\mu)}[a,b,c;d,e;x,y] := \sum_{m,n=0}^\infty \frac{(a;p,q;\kappa,\mu)_{m+n}(b)_m(c)_n}{(d)_m(e)_n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{31}$$

$$|x|+|y| < 1,$$

$$_{p,q}F_3^{(\kappa,\mu)}[a,b,c,d;e;x,y] := \sum_{m,n=0}^\infty \frac{(a;p,q;\kappa,\mu)_m(b)_n(c)_m(d)_n}{(e)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \tag{32}$$

$$\max(|x|, |y|) < 1,$$

$$_{p,q}F_4^{(\kappa,\mu)}[a,b;c,d;x,y] := \sum_{m,n=0}^{\infty} \frac{(a;p,q;\kappa,\mu)_{m+n}(b)_{m+n}}{(c)_m(d)_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{33}$$

$$\sqrt{|x|} + \sqrt{|y|} < 1,$$

$$_{p,q}F_D^{(\kappa,\mu;3)}[a,b,c,d;e;x,y,z] := \sum_{m,n,r=0}^{\infty} \frac{(a;p,q;\kappa,\mu)_{m+n+r}(b)_m(c)_n(d)_r}{(e)_{m+n+r}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^r}{r!} \tag{34}$$

$$\max(|x|, |y|, |z|) < 1,$$

$$_{p,q}\Phi_1^{(\kappa,\mu)}[a,b;d;x,y] := \sum_{m,n=0}^{\infty} \frac{(a;p,q;\kappa,\mu)_{m+n}(b)_m}{(d)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \tag{35}$$

$$\max(|x|, |y|) < 1,$$

$$_{p,q}\Psi_1^{(\kappa,\mu)}[a,b;d,e;x,y] := \sum_{m,n=0}^{\infty} \frac{(a;p,q;\kappa,\mu)_{m+n}(b)_m}{(d)_m(e)_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{36}$$

$$|x| + |y| < 1,$$

$$_{p,q}\Xi_1^{(\kappa,\mu)}[a,c,d;e;x,y] := \sum_{m,n=0}^{\infty} \frac{(a;p,q;\kappa,\mu)_m(c)_m(d)_n}{(e)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \tag{37}$$

$$\max(|x|, |y|) < 1$$

and

$$_{p,q}\Phi_D^{(\kappa,\mu;3)}[a,b,c;e;x,y,z] := \sum_{m,n,r=0}^{\infty} \frac{(a;p,q;\kappa,\mu)_{m+n+r}(b)_m(c)_n}{(e)_{m+n+r}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^r}{r!} \tag{38}$$

$$\max(|x|, |y|, |z|) < 1,$$

respectively. Note that taking $p = 1$, $q = 0$, $\kappa = 0$ and $\mu = 0$ gives the original ones [1, 7–10, 18, 22, 23, 26, 29, 30, 32, 33]. Now, we obtain the integral representations of the functions (30)-(34).

Theorem 11. *The following integral representations for* (30) *hold true:*

$$_{p,q}F_1^{(\kappa,\mu)}[a,b,c;d;x,y] = \frac{1}{\Gamma(a)} \int_0^{\infty} t^{a-1} \exp(-\frac{t^\kappa}{p} - \frac{q}{t^\mu}) \Phi_2[b,c;d;xt,yt]dt \tag{39}$$

and

$$_{p,q}F_1^{(\kappa,\mu)}[a,b,c;d;x,y] = \frac{1}{\Gamma(c)} \int_0^{\infty} t^{c-1} \exp(-t) \, _{p,q}\Phi_1^{(\kappa,\mu)}[a,b;d;x,yt]dt. \tag{40}$$

Proof. Using the generalization of the extended Pochhammer symbol $(a_1;p,q;\kappa,\mu)$ in the definition (30) by its integral representation given by (6), we led to desired result (39). Similar way, we can prove the (40).

Theorem 12. *The following integral representations for* (31) *hold true:*

$$_{p,q}F_2^{(\kappa,\mu)}[a,b,c;d,e;x,y] = \frac{1}{\Gamma(a)} \int_0^{\infty} t^{a-1} \exp(-\frac{t^\kappa}{p} - \frac{q}{t^\mu}) \, _1F_1[b;d;xt] \, _1F_1[c;e;yt]dt \tag{41}$$

and

$$_{p,q}F_2^{(\kappa,\mu)}[a,b,c;d,e;x,y] = \frac{1}{\Gamma(c)} \int_0^{\infty} t^{c-1} \exp(-t) \, _{p,q}\Psi_1^{(\kappa,\mu)}[a,b;d,e;x,yt]dt. \tag{42}$$

Proof. Using the generalization of the extended Pochhammer symbol $(a_1; p, q; \kappa, \mu)$ in the definition (31) by its integral representation given by (6), we led to desired result (41). Similar way, we can prove the (42).

Theorem 13. *The following integral representation for* (32) *holds true:*

$$_{p,q}F_3^{(\kappa,\mu)}[a,b,c,d;e;x,y] = \frac{1}{\Gamma(b)} \int_0^\infty t^{b-1} \exp(-t) \,_{p,q}\Xi_1^{(\kappa,\mu)}[a,c,d;e;x,yt]dt. \tag{43}$$

Proof. Using the generalization of the extended Pochhammer symbol $(a_1; p, q; \kappa, \mu)$ in the definition (32) by its integral representation given by (3), we led to desired result (43).

Theorem 14. *The following integral representation for* (33) *holds true:*

$$_{p,q}F_4^{(\kappa,\mu)}[a,b;c,d;x,y] = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} \exp(-\frac{t^\kappa}{p} - \frac{q}{t^\mu}) \Psi_2[b;c,d;xt,yt]dt. \tag{44}$$

Proof. Using the generalization of the extended Pochhammer symbol $(a_1; p, q; \kappa, \mu)$ in the definition (33) by its integral representation given by (6), we led to desired result (44).

Theorem 15. *The following integral representations for* (34) *hold true:*

$$_{p,q}F_D^{(\kappa,\mu;3)}[a,b,c,d;e;x,y,z] = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} \exp(-\frac{t^\kappa}{p} - \frac{q}{t^\mu}) \Phi_2^{(3)}[b,c,d;e;xt,yt,zt]dt \tag{45}$$

and

$$_{p,q}F_D^{(\kappa,\mu;3)}[a,b,c,d;e;x,y,z] = \frac{1}{\Gamma(d)} \int_0^\infty t^{d-1} \exp(-t) \,_{p,q}\Phi_D^{(\kappa,\mu;3)}[a,b,c;e;x,y,zt]dt. \tag{46}$$

Proof. Using the generalization of the extended Pochhammer symbol $(a_1; p, q; \kappa, \mu)$ in the definition (34) by its integral representation given by (6), we led to desired result (45). Similar way, we can prove the (46).

Theorem 16. *The following derivative formulas for* (30)-(34) *hold true:*

$$D_{x,y}^{m,n}\left\{ _{p,q}F_1^{(\kappa,\mu)}[a,b,c;d;x,y] \right\} := \frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}} \,_{p,q}F_1^{(\kappa,\mu)}[a+m+n,b+m,c+n;d+m+n;x,y], \tag{47}$$

$$D_{x,y}^{m,n}\left\{ _{p,q}F_2^{(\kappa,\mu)}[a,b,c;d,e;x,y] \right\} := \frac{(a)_{m+n}(b)_m(c)_n}{(d)_m(e)_n} \,_{p,q}F_2^{(\kappa,\mu)}[a+m+n,b+m,c+n;d+m,e+n;x,y], \tag{48}$$

$$D_{x,y}^{m,n}\left\{ _{p,q}F_3^{(\kappa,\mu)}[a,b,c,d;e;x,y] \right\} := \frac{(a)_m(b)_n(c)_m(d)_n}{(e)_{m+n}} \,_{p,q}F_3^{(\kappa,\mu)}[a+m,b+n,c+m,d+n;e+m+n;x,y], \tag{49}$$

$$D_{x,y}^{m,n}\left\{ _{p,q}F_4^{(\kappa,\mu)}[a,b;c,d;x,y] \right\} := \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c)_n} \,_{p,q}F_4^{(\kappa,\mu)}[a+m+n,b+m+n;c+m,d+n;x,y] \tag{50}$$

and

$$D_{x,y,z}^{m,n,r}\left\{ _{p,q}F_D^{(\kappa,\mu;3)}[a,b,c,d;e;x,y] \right\} := \frac{(a)_{m+n+r}(b)_m(c)_n(d)_r}{(e)_{m+n+r}}$$
$$\times \,_{p,q}F_D^{(\kappa,\mu;3)}[a+m+n+r,b+m,c+n,d+r;e+m+n+r;x,y,z]. \tag{51}$$

Proof. Differentiating (30)-(33) with respect to x and y, then repeating same procedure n-times and making some simple calculation, we can obtain the (47)-(50) results. Similiarly, taking differentiation (34) with respect to x, y and z, we can get the derivative formula (51)

Theorem 17. *The following derivative formulas for* (30) *hold true:*

$$D_y^n \left\{ y^{c+n-1} \,_{p,q}F_1^{(\kappa,\mu)}[a,b,c;d;x,y] \right\} := (c)_n y_{p,q}^{c-1} F_1^{(\kappa,\mu)}[a,b,c+n;d;x,y], \tag{52}$$

$$D_y^n \left\{ y^{d-1} \,_{p,q}F_1^{(\kappa,\mu)}[a,b,c;d;x,y] \right\} := (-1)^n (1-d)_n y^{d-n-1} \,_{p,q}F_1^{(\kappa,\mu)}[a,b,c;d-n;x,y] \tag{53}$$

and

$$D_y^n \left\{ y^{d-b-1} \,_{p,q}F_1^{(\kappa,\mu)}[a,b,c;d;x,y] \right\} := (-1)^n (b-d-1)_n y^{d-b-n-1}$$
$$\times \sum_{n=0}^{m} \binom{m}{n} \frac{(a)_n (c)_n y^n}{(d)_n (d-b-m)_n} \,_{p,q}F_1^{(\kappa,\mu)}[a+n,b,c+n;d+n;x,y]. \tag{54}$$

Proof. Multiplying the (30) with y^{c+n-1} and taking the derivative *n*-times with respect to *y*, we have

$$D_y^n \left\{ y^{c+n-1} \,_{p,q}F_1^{(\kappa,\mu)}[a,b,c;d;x,y] \right\} := D_y^n \left\{ y^{c+n-1} \sum_{m,n=0}^{\infty} \frac{(a;p,q;\kappa,\mu)_{m+n}(b)_m (c)_n}{(d)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \right\}$$
$$= \sum_{m=0}^{\infty} \frac{(a)_m (b)_m x^m}{(d)_m m!} D_y^n \left\{ y^{c+n-1} \sum_{n=0}^{\infty} \frac{(a+m;p,q;\kappa,\mu)_n (c)_n}{(d+m)_n n!} y^n \right\}$$
$$= \sum_{m=0}^{\infty} \frac{(a)_m (b)_m x^m}{(d)_m m!} (c)_n y^{c-1} \sum_{n=0}^{\infty} \frac{(a+m;p,q;\kappa,\mu)_n (c+n)_n}{(d+m)_n n!} y^n$$
$$= (c)_n y^{c-1} \sum_{m,n=0}^{\infty} \frac{(a;p,q;\kappa,\mu)_{m+n}(b)_m (c+n)_n}{(d)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}. \tag{55}$$

Thus, we obtain the (52) result. Similar way, we can prove the equations (53) and (54).

Theorem 18. *The following derivative formulas for* (31) *hold true:*

$$D_y^n \left\{ y^{c+n-1} \,_{p,q}F_2^{(\kappa,\mu)}[a,b,c;d,e;x,y] \right\} := (c)_n y_{p,q}^{c-1} F_2^{(\kappa,\mu)}[a,b,c+n;d,e;x,y] \tag{56}$$

and

$$D_y^n \left\{ y^{e-1} \,_{p,q}F_2^{(\kappa,\mu)}[a,b,c;d,e;x,y] \right\} := (-1)^n (1-e)_n y^{e-n-1} \,_{p,q}F_2^{(\kappa,\mu)}[a,b,c;d,e-n;x,y]. \tag{57}$$

Proof. The proof of theorem would be parallel to those of the *Theorem 17*.

Theorem 19. *The following derivative formulas for* (32) *hold true:*

$$D_y^n \left\{ y^{d+n-1} \,_{p,q}F_3^{(\kappa,\mu)}[a,b,c,d;e;x,y] \right\} := (d)_n y_{p,q}^{d-1} F_3^{(\kappa,\mu)}[a,b,c,d+n;e;x,y], \tag{58}$$

$$D_y^n \left\{ (1-y)^{b+n-1} \,_{p,q}F_3^{(\kappa,\mu)}[a,b,e-c,c;e;x,y] \right\} := (-1)^n \frac{(b)_n (e-c)_n}{(e)_n} (1-y)^{b-1}$$
$$\times \,_{p,q}F_3^{(\kappa,\mu)}[a,b+n,e-c+n,d;e+n;x,y] \tag{59}$$

and

$$D_y^n \left\{ y^{e-c-1} \,_{p,q}F_3^{(\kappa,\mu)}[a,b,c,d;e;x,y] \right\} := (-1)^n (c-e-1)_n y^{e-c-n-1}$$
$$\times \sum_{n=0}^{m} \binom{m}{n} \frac{(b)_n (d)_n y^n}{(e)_n (e-c-m)_n} \,_{p,q}F_3^{(\kappa,\mu)}[a,b+n,c,d+n;e+n;x,y]. \tag{60}$$

Proof. The proof of theorem would be parallel to those of the *Theorem 17*.

Theorem 20. *The following derivative formulas for* (33) *hold true:*

$$D_x^n \left\{ x^{c-1} {}_{p,q}F_4^{(\kappa,\mu)}[a,b;c,d;x,y] \right\} := (-1)^n (1-c)_n x_{p,q}^{c-n-1} F_4^{(\kappa,\mu)}[a,b;c-n,d;x,y] \tag{61}$$

and

$$D_y^n \left\{ y^{d-1} {}_{p,q}F_4^{(\kappa,\mu)}[a,b;c,d;x,y] \right\} := (-1)^n (1-d)_n y_{p,q}^{d-n-1} F_4^{(\kappa,\mu)}[a,b;c,d-n;x,y]. \tag{62}$$

Proof. The proof of theorem would be parallel to those of the *Theorem 17*.

5 Recursion Formulas for Extended Appell Hypergeometric Functions

In this section, we present some recursion formulas for Appell hypergeometric functions. Let's we start following theorem.

Theorem 21. *The following recursion formulas for* (30) *hold true:*

$$_{p,q}F_1^{(\kappa,\mu)}[a,b+n,c;d;x,y] =_{p,q} F_1^{(\kappa,\mu)}[a,b,c;d;x,y] + \frac{ax}{d} \sum_{k=1}^{n} {}_{p,q}F_1^{(\kappa,\mu)}[a+1,b+k,c;d+1;x,y], \tag{63}$$

$$_{p,q}F_1^{(\kappa,\mu)}[a,b-n,c;d;x,y] =_{p,q} F_1^{(\kappa,\mu)}[a,b,c;d;x,y] - \frac{ax}{d} \sum_{k=0}^{n-1} {}_{p,q}F_1^{(\kappa,\mu)}[a+1,b-k,c;d+1;x,y], \tag{64}$$

and

$$\begin{aligned} {}_{p,q}F_1^{(\kappa,\mu)}[a,b,c;d-n;x,y] =&_{p,q} F_1^{(\kappa,\mu)}[a,b,c;d;x,y] \\ &+ abx \sum_{k=1}^{n} \frac{{}_{p,q}F_1^{(\kappa,\mu)}[a+1,b+1,c;d+2-k;x,y]}{(d-k)(d-k-1)} \\ &+ acy \sum_{k=1}^{n} \frac{{}_{p,q}F_1^{(\kappa,\mu)}[a+1,b,c+1;d+2-k;x,y]}{(d-k)(d-k-1)}. \end{aligned} \tag{65}$$

Proof. Applying the transformation formula $(b+1)_m = (b)_m \times (1 + \frac{m}{b})$ in the definition of the extension of the Appell hypergeometric function ${}_{p,q}F_1^{(\kappa,\mu)}(.)$ in (30) and we have following contiguous formula:

$$p,qF_1^{(\kappa,\mu)}[a,b+1,c;d;x,y] =_{p,q} F_1^{(\kappa,\mu)}[a,b,c;d;x,y] + \frac{ax}{d} {}_{p,q}F_1^{(\kappa,\mu)}[a+1,b+1,c;d+1;x,y]. \tag{66}$$

Calculating the function ${}_{p,q}F_1^{(\kappa,\mu)}(.)$ with the parameter $b+n$ by equation (66) for *n* times, we obtain the required result (63). Setting the $b = b-n$ in the equation (66) and making same calculation as above equation, we can be yield the desired result (64). The proof of (65) is omitted to readers because it is similar to the proof of (63).

Theorem 22. *The following recursion formulas for* (31) *hold true:*

$$_{p,q}F_2^{(\kappa,\mu)}[a,b+n,c;d,e;x,y] =_{p,q} F_2^{(\kappa,\mu)}[a,b,c;d,e;x,y] + \frac{ax}{d} \sum_{k=1}^{n} {}_{p,q}F_2^{(\kappa,\mu)}[a+1,b+k,c;d+1,e;x,y], \tag{67}$$

$$_{p,q}F_2^{(\kappa,\mu)}[a,b-n,c;d,e;x,y] =_{p,q} F_2^{(\kappa,\mu)}[a,b,c;d,e;x,y] - \frac{ax}{d} \sum_{k=0}^{n-1} {}_{p,q}F_2^{(\kappa,\mu)}[a+1,b-k,c;d+1,e;x,y], \tag{68}$$

and

$$_{p,q}F_2^{(\kappa,\mu)}[a,b,c;d-n,e;x,y] =_{p,q} F_2^{(\kappa,\mu)}[a,b,c;d,e;x,y] + abx \sum_{k=1}^{n} \frac{{}_{p,q}F_2^{(\kappa,\mu)}[a+1,b+1,c;d+2-k,e;x,y]}{(d-k)(d-k-1)}. \tag{69}$$

Proof. The proof of the *Theorem 22* is similar to the proof of *Theorem 21*.

Theorem 23. *The following recursion formula for* (32) *holds true:*

$$
\begin{aligned}
{}_{p,q}F_3^{(\kappa,\mu)}[a,b,c,d;e-n;x,y] = {}&_{p,q}F_3^{(\kappa,\mu)}[a,b,c,d;e;x,y] \\
&+ acx \sum_{k=1}^{n} \frac{{}_{p,q}F_3^{(\kappa,\mu)}[a+1,b,c+1,d;e+2-k;x,y]}{(e-k)(e-k-1)} \\
&+ bdy \sum_{k=1}^{n} \frac{{}_{p,q}F_3^{(\kappa,\mu)}[a,b+1,c,d+1;e+2-k;x,y]}{(e-k)(e-k-1)}.
\end{aligned}
\tag{70}
$$

Proof. The proof of the *Theorem 23* is parallel to the proof of *Theorem 21*.

Theorem 24. *The following recursion formula for* (33) *holds true:*

$$
{}_{p,q}F_4^{(\kappa,\mu)}[a,b;c-n,d;x,y] = {}_{p,q}F_4^{(\kappa,\mu)}[a,b;c,d;x,y] + abx \sum_{k=0}^{n-1} \frac{{}_{p,q}F_4^{(\kappa,\mu)}[a+1,b+1;c+1-k,d;x,y]}{(c-k)(c-k-1)}.
\tag{71}
$$

Proof. The proof of the *Theorem 24* is same as the proof of *Theorem 21*.

Remark 2. Taking $p=1$ and $q=\kappa=\mu=0$ in the relation *Theorem 21-Theorem 24*, it is easily seen that the special case of recursion formulas of Appell hypergeometric functions [32].

6 Conclusions

We may also give point to that results obtained in this work are of general character and can appropriate to give a new generalization of the Pochhammer symbol by means of the generalization of extended gamma function (4) [31]. Using the generalization of Pochhammer symbol, we give a generalization of the extended hypergeometric functions one or several variables. Also, we obtain various integral representations, derivative formulas and certain properties of these functions.

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