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Special Curves According to Bishop Frame in Minkowski 3-Space

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Abstract

Pseudo null curves were studied by some geometers in both Euclidean and Minkowski spaces, but some special characters of the curve are not considered. In this paper, we study weak $AW(k)$ -type and $AW(k)$ -type pseudo null curve in Minkowski 3-space E_1^3 . We define helix and slant helix according to Bishop frame in E_1^3 . Furthermore, the necessary and sufficient conditions for the slant helix and helix in Minkowski 3-space are obtained.

Keywords: Bishop frame, Pseudo null curve, slant helices

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1 Introduction

The ability to "ride" along a three-dimensional space curve and illustrate the properties of the curve, such as curvature and torsion, would be a great asset to Mathematicians. The classic Serret-Frenet frame provides such ability. The tangent normal, and binormal vector fields are called the Frenet-Serret frame or TNB frame. But, the curve might not be continuous at some points, which is undefined when the second derivative of the curve vanishes [9]. In 1975, Richard Lawrence Bishop first introduced the parallel frame as a new frame which is well defined even if the curve has vanishing second derivative, then the parallel frame came to be called the Bishop frame [8, 9, 10]. Bishop frame contains the tangential vector field T and two normal vector fields N_1 and N_2 . The Bishop frame may have applications in the area of Biology and Computer Graphics. For example, it may be possible to compute information about the shape of sequences of DNA using a curve defined by the Bishop frame. It also provides a new way to control virtual cameras in computer animation [12]. Some applications of the Bishop frames in Minkowski spaces can be found in [3, 4].

In differential geometry, a general helix or a curve of constant slope in Euclidean 3-space E^3 is defined in such a way that the tangent makes a constant angle with a fixed direction. A classical result stated by M. A. Lancret in 1802 and first proved by B. de Saint Venant in 1845, [5, 7, 16, 18]. For helical structures in nature, helices arise in nano-springs, carbon nano-tubes, DNA double and collagen triple helix, lipid bilayers, bacterial

flagella in salmonella and escherichia coli, aerial hyphae in actinomycetes, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, helical staircases and seashells [1, 2, 11]. Helical structures are used in fractal geometry, for instance, hyper-helices.

In [7], a slant helix in Euclidean 3-space was defined by the property that the principal normal makes a constant angle with a fixed direction. Moreover, Izumiya and Takeuchi showed that α is a slant helix in E^3 if and only if the geodesic curvature of the principal normal of a space curve α is a constant function [15, 17, 19].

2 Preliminaries

Definition 1. The Minkowski 3-space E_1^3 is the real vector space E^3 which is endowed with the standard indefinite flat metric $\langle \cdot, \cdot \rangle$ defined by

$$\langle u, v \rangle = -u_1v_1 + u_2v_2 + u_3v_3, \quad (2.1)$$

for any two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in E_1^3 . Since $\langle \cdot, \cdot \rangle$ is an indefinite metric, an arbitrary vector $u \in E_1^3 \setminus \{0\}$ can have one of three causal characters:

- i) it can be space-like, if $\langle u, u \rangle_1 > 0$,
- ii) time-like, if $\langle u, u \rangle_1 < 0$ or
- iii) light-like or isotropic or null vector, if $\langle u, u \rangle_1 = 0$ but $u \neq 0$.

In particular, the norm (length) of a non-lightlike vector $u \in E_1^3$ is given by

$$\|u\| = \sqrt{|\langle u, u \rangle|}.$$

Given a regular curve $\alpha : I \rightarrow E_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(t)$ satisfy $\langle \alpha'(t), \alpha'(t) \rangle_1 > 0$, $\langle \alpha'(t), \alpha'(t) \rangle_1 < 0$ or $\langle \alpha'(t), \alpha'(t) \rangle_1 = 0$ respectively, at any $t \in I$, where $\alpha'(t) = \frac{d\alpha}{dt}$.

Definition 2. A spacelike curve $\alpha : I \rightarrow E_1^3$ is called a pseudo null curve, if its principal normal vector field N and binormal vector field B are null vector fields satisfying the condition $\langle N, B \rangle = 1$. The Frenet formulae of a non-geodesic pseudo null curve α has the form

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ 0 & -\tau & 0 \\ -k & 0 & -\tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad (2.2)$$

where the curvature $k(s) = 1$ and the torsion $\tau(s)$ is an arbitrary function in arc-length parameter s of α . The Frenet's frame vectors satisfy the equations

$$\begin{aligned} \langle N, B \rangle &= 1, \quad \langle T, N \rangle = \langle T, B \rangle = 0, \\ \langle T, T \rangle &= 1, \quad \langle N, N \rangle = \langle B, B \rangle = 0 \end{aligned} \quad (2.3)$$

and

$$T \times N = N, \quad N \times B = T, \quad B \times T = B. \quad (2.4)$$

The frame $\{T, N, B\}$ is positively oriented, if $\det(T, N, B) = [T, N, B] = 1$ [10].

2.1 The Bishop Frame

The Bishop frame or relatively parallel adapted frame $\{T, N_1, N_2\}$ of a regular curve in Euclidean 3-space contains a tangential vector field T and two normal vector fields N_1 and N_2 , which can be obtained by rotating the Frenet vectors N and B in the normal plane $T^\perp$ of the curve, in such a way that they become relatively parallel. This means that their derivatives N_1' and N_2' with respect to the arc-length parameter s of the curve are **collinear** with the tangential vector field T [10].

Remark 1. We can also define N_1 and N_2 to be relatively parallel if the normal component $T_1^\perp = \text{span}\{N_1, N_2\}$ of their derivatives N_1' and N_2' is zero, which implies that the mentioned derivatives are collinear with T_1 .

2.2 The Bishop frame of a pseudo null curve in E_1^3

The Bishop frame $\{T_1, N_1, N_2\}$ of a pseudo null curve in E_1^3 is positively oriented pseudo orthonormal frame consisting of the tangential vector field T_1 and two relatively parallel lightlike normal vector fields N_1 and N_2 . Bishop vector N_1 (of the first Bishop frame) can be obtained by applying the hyperbolic rotation to the principal normal vector N , while the normal Bishop vector N_2 (of the first Bishop frame) can be obtained by applying the composition of three rotations about two lightlike and one spacelike axis to the binormal vector B [10].

Theorem 1. Let α be a pseudo null curve in E_1^3 parameterized by arc-length parameter s with the curvature $k(s) = 1$ and the torsion $\tau(s)$. Then the Bishop frame $\{T_1, N_1, N_2\}$ and the Frenet frame $\{T, N, B\}$ of α are related by

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{k_2} & 0 \\ 0 & 0 & k_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad (2.5)$$

and the Frenet equations of α according to the Bishop frame read

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_2 & k_1 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}, \quad (2.6)$$

where $k_1(s) = 0$ and $k_2(s) = c_0 e^{\int \tau(s) ds}$, $c_0 \in \mathbb{R}_0^+$.

Which satisfies the conditions

$$\begin{aligned} \langle T_1, T_1 \rangle &= 1, \quad \langle N_1, N_1 \rangle = \langle N_2, N_2 \rangle = 0, \\ \langle N_1, N_2 \rangle &= 1, \quad \langle T_1, N_2 \rangle = \langle T_1, N_1 \rangle = 0 \quad [10]. \end{aligned}$$

3 Curves of AW(k)- type

Proposition 1. Let α be a Frenet curve of osculating order 3 in E_1^3 , by using the Bishop frame of pseudo null curve (2.6), then we have

$$\begin{aligned} \alpha'(s) &= T_1(s), \\ \alpha''(s) &= k_2 N_1 + k_1 N_2, \\ \alpha'''(s) &= -2k_1 k_2 T_1 + k_2' N_1 + k_1' N_2, \\ \alpha''''(s) &= (-3k_1' k_2 - 3k_1 k_2') T_1 + (k_2'' - 2k_1 k_2'') N_1 + (k_1'' - 2k_1' k_2') N_2. \end{aligned}$$

Notation 1. Let us write

$$M_1(s) = k_2 N_1 + k_1 N_2, \quad (3.1)$$

$$M_2(s) = k_2' N_1 + k_1' N_2, \quad (3.2)$$

$$M_3(s) = (k_2'' - 2k_1 k_2'') N_1 + (k_1'' - 2k_1' k_2') N_2. \quad (3.3)$$

Corollary 1. $\alpha'(s)$, $\alpha''(s)$, $\alpha'''(s)$ and $\alpha''''(s)$ are linearly dependent if and only if $M_1(s)$, $M_2(s)$ and $M_3(s)$ are linearly dependent.

Definition 3. Frenet curves of osculating order 3 are :

i) of type weak AW(2) if they satisfy

$$M_3(s) = \langle M_3(s), M_2^*(s) \rangle M_2^*(s), \quad (3.4)$$

ii) of type weak AW(3) if they satisfy

$$M_3(s) = \langle M_3(s), M_1^*(s) \rangle M_1^*(s), \quad (3.5)$$

where

$$M_1^*(s) = \frac{M_1(s)}{\|M_1(s)\|}, \quad (3.6)$$

$$M_2^*(s) = \frac{M_2(s) - \langle M_2(s), M_1^*(s) \rangle M_1^*(s)}{\|M_2(s) - \langle M_2(s), M_1^*(s) \rangle M_1^*(s)\|}, \quad (3.7)$$

iii) of type AW(1) if they satisfy

$$M_3(s) = 0, \quad (3.8)$$

iv) of type AW(2) if they satisfy

$$\|M_2(s)\|^2 M_3(s) = \langle M_3(s), M_2(s) \rangle M_2(s), \quad (3.9)$$

v) of type AW(3) if they satisfy

$$\|M_1(s)\|^2 M_3(s) = \langle M_3(s), M_1(s) \rangle M_1(s). \quad (3.10)$$

Proposition 2. Suppose that α is a Frenet curve of osculating order 3 in E_1^3 , then α is AW(1)-type if and only if

$$k_1'' = 2k_1^2 k_2, \quad (3.11)$$

$$k_2'' = 2k_1 k_2^2. \quad (3.12)$$

Proof. Since α is a curve of AW(1)-type, then α must satisfy (3.8)

$$\begin{aligned} M_3(s) &= (k_2'' - 2k_1 k_2^2) N_1 + (k_1'' - 2k_1^2 k_2) N_2, \\ 0 &= (k_2'' - 2k_1 k_2^2) N_1 + (k_1'' - 2k_1^2 k_2) N_2. \end{aligned}$$

Since N_1 and N_2 are linearly independent, then we have

$$\begin{aligned} k_2'' - 2k_1 k_2^2 &= 0 \\ k_2'' &= 2k_1 k_2^2, \end{aligned}$$

and

$$\begin{aligned} k_1'' - 2k_1^2 k_2 &= 0 \\ k_1'' &= 2k_1^2 k_2, \end{aligned}$$

which completes the proof of the proposition. □

Proposition 3. Let α be a Frenet curve of osculating order 3 in E_1^3 , then α is AW(2)-type if and only if

$$k_1'' k_2' - 2k_1^2 k_2 k_2' = k_1' k_2'' - 2k_1 k_1' k_2^2. \quad (3.13)$$

Proof. Suppose that α is a Frenet curve of osculating order 3. From (2.14) and (2.15) we can write

$$\begin{aligned}M_2(s) &= \beta(s)N_1 + \gamma(s)N_2, \\M_3(s) &= \delta(s)N_1 + \eta(s)N_2,\end{aligned}$$

where $\beta(s)$, $\gamma(s)$, $\delta(s)$ and $\eta(s)$ are differential functions. Since $M_2(s)$ and $M_3(s)$ are linearly dependent, then the determinant of the coefficients of N_1 and N_2 is equal to zero and hence one can write

$$\begin{vmatrix} \beta(s) & \gamma(s) \\ \delta(s) & \eta(s) \end{vmatrix} = 0, \quad (3.14)$$

where

$$\begin{aligned}\beta(s) &= k_2', \quad \gamma(s) = k_1', \\ \delta(s) &= k_2'' - 2k_1k_2^2, \\ \eta(s) &= k_1'' - 2k_1^2k_2.\end{aligned} \quad (3.15)$$

By substituting (3.15) in (3.14), we can obtain (3.13), which completes the proof of the proposition. $\square$

Proposition 4. Let α be a Frenet curve of osculating order 3 in E_1^3 , then α is AW(3)-type if and only if

$$k_1''k_2 = k_1k_2''. \quad (3.16)$$

Proof. Suppose that α is a Frenet curve of osculating order 3, then

$$\begin{aligned}M_1(s) &= \beta(s)N_1 + \gamma(s)N_2, \\M_3(s) &= \delta(s)N_1 + \eta(s)N_2,\end{aligned}$$

where $\beta(s)$, $\gamma(s)$, $\delta(s)$ and $\eta(s)$ are differential functions. Since $M_2(s)$ and $M_3(s)$ are linearly dependent, then the determinant of the coefficients of N_1 and N_2 is equal to zero and hence we can write

$$\beta(s)\eta(s) - \gamma(s)\delta(s) = 0 \quad (3.17)$$

where

$$\begin{aligned}\beta(s) &= k_2, \quad \gamma(s) = k_1, \\ \delta(s) &= k_2'' - 2k_1k_2^2, \\ \eta(s) &= k_1'' - 2k_1^2k_2.\end{aligned}$$

Considering these equations in (3.17), we get (3.16), which completes the proof of the proposition. $\square$

Proposition 5. Let α be a Frenet curve of osculating order 3 in E_1^3 , then α is of weak AW(2)-type if and only if

$$(k_2'' - 2k_1k_2^2) = \frac{p}{p^2 + q^2} \left[(k_2'' - 2k_1k_2^2)q + (k_1'' - 2k_1^2k_2)p \right], \quad (3.18)$$

$$(k_1'' - 2k_1^2k_2) = \frac{q}{p^2 + q^2} \left[(k_2'' - 2k_1k_2^2)q + (k_1'' - 2k_1^2k_2)p \right], \quad (3.19)$$

where

$$\begin{aligned}p &= (k_2^2 + k_1^2)k_2' - (k_1k_2)'k_2 \text{ and} \\ q &= (k_2^2 + k_1^2)k_1' - (k_1k_2)'k_1.\end{aligned}$$

Proposition 6. Let α be a Frenet curve of osculating order 3 in E_1^3 , then α is of weak AW(3)-type if and only if

$$k_2'' - 2k_1k_2^2 = \frac{k_2}{k_2^2 + k_1^2} \left[k_1k_2'' + k_1''k_2 - 4k_1^2k_2^2 \right], \quad (3.20)$$

$$k_1'' - 2k_1^2k_2 = \frac{k_1}{k_2^2 + k_1^2} \left[k_1k_2'' + k_1''k_2 - 4k_1^2k_2^2 \right]. \quad (3.21)$$

4 The Slant helices according to Bishop frame of the pseudo null curve in Minkowski 3-space.

Definition 4. Helix is a curve whose tangent lines make a constant angle with a fixed direction. Helices are characterized by the fact that the ratio $\frac{k_1}{k_2}$ is constant along the curve.

Definition 5. A unit speed curve α is called a slant helix if there exists a non-zero constant vector field $U \in E_1^3$ such that the function $\langle N(s), U \rangle$ is constant.

It is important to point out, in contrast to what happens in E^3 , we cannot define the angle between two vectors (except that both vectors are of time-like). For this reason, we avoid to say about the angle between the normal vector field $N(s)$ and U [14].

Theorem 2. Let α be a pseudo null curve in E_1^3 , then α is a general helix if and only if $\frac{k_1}{k_2}$ is constant.

Proof. Let α be a general helix. The slope axis of the curve α is shown as $sp\{U\}$. note that

$$\langle T, U \rangle = c \quad (c \text{ is constant}). \quad (4.1)$$

If we differentiate both sides of the equation (4.1), then we have

$$\langle T', U \rangle = 0. \quad (4.2)$$

By using (2.10) and (4.44)

$$\begin{aligned} \langle k_2 N_1 + k_1 N_2, U \rangle &= 0, \\ k_2 \cos \theta + k_1 \sin \theta &= 0, \\ \frac{k_1}{k_2} &= -\cot \theta \quad (\text{constant}), \end{aligned} \quad (4.3)$$

as desired. $\square$

Theorem 3. Let α be a pseudo null curve in E_1^3 , then α is a slant helix if and only if $\frac{k_1}{k_2}$ is constant.

Proof. Let α be a slant helix in E_1^3 and $\langle N(s), U \rangle$ is constant. Then α is a slant helix, from the definition we have

$$\langle N(s), U \rangle = c \quad (c \text{ is a constant}), \quad (4.4)$$

where U is a constant vector in E_1^3 . By differentiating (4.4) and using (2.6)

$$\begin{aligned} \langle N'(s), U \rangle &= 0, \\ -k_1 \langle T_1, U \rangle &= 0, \quad k_1 \neq 0. \end{aligned}$$

Hence

$$\langle T_1, U \rangle = 0, \quad (4.5)$$

$u \in sp\{N_1, N_2\}$, therefore $u = \cos \theta N_1 + \sin \theta N_2$. U is a linear combination of N_1 and N_2 . By differentiating (4.5) and using (2.6)

$$\begin{aligned} \langle T', U \rangle &= 0, \\ k_2 \cos \theta + k_1 \sin \theta &= 0, \\ \frac{k_1}{k_2} &= -\cot \theta, \end{aligned}$$

as desired. $\square$

Theorem 4. Let α be a pseudo null curve in E_1^3 , then α is a slant helix if and only if

$$\det(N_1', N_1'', N_1''') = 0. \quad (4.6)$$

Proof. ($\implies$) Suppose that $\frac{k_1}{k_2}$ be constant. We have equalities as

$$\begin{aligned} N_1' &= -k_1 T, \\ N_1'' &= -k_1' T - k_1 k_2 N_1 - k_1^2 N_2, \\ N_1''' &= (2k_1^2 k_2 - k_1'') T + (-2k_1' k_2 - k_1 k_2') N_1 - 3k_1 k_1' N_2. \end{aligned}$$

So we get

$$\begin{aligned} \det(N_1', N_1'', N_1''') &= \begin{vmatrix} -k_1 & 0 & 0 \\ -k_1' & -k_1 k_2 & -k_1^2 \\ (2k_1^2 k_2 - k_1'') & -(2k_1' k_2 + k_1 k_2') & -3k_1 k_1' \end{vmatrix}, \\ \det(N_1', N_1'', N_1''') &= -k_1^3 k_2^2 \left(\frac{k_1}{k_2}\right)'. \end{aligned}$$

Since α is a slant helix and $\frac{k_1}{k_2}$ is constant. Hence, we have

$$\det(N_1', N_1'', N_1''') = 0, \text{ but } k_2 \neq 0.$$

($\impliedby$) Suppose that $\det(N_1', N_1'', N_1''') = 0$, then it is clear that the $\frac{k_1}{k_2}$ is constant, since $\left(\frac{k_1}{k_2}\right)'$ is zero. Hence the theorem is proved. $\square$

Theorem 5. Let α be a pseudo null curve in E_1^3 , then α is a slant helix if and only if

$$\det(N_2', N_2'', N_2''') = 0. \quad (4.7)$$

Proof. ($\implies$) Suppose that $\frac{k_1}{k_2}$ be constant. From (2.6) we have

$$N_2' = -k_2 T,$$

therefore

$$\begin{aligned} N_2'' &= -k_2' T - k_2^2 N_1 - k_1 k_2 N_2, \\ N_2''' &= (2k_1 k_2^2 - k_2'') T + (-k_2^2 - 2k_2 k_2') N_1 + (-k_1 k_2 - k_1' k_2 - k_1 k_2') N_2. \end{aligned}$$

So we get

$$\begin{aligned} \det(N_2', N_2'', N_2''') &= \begin{vmatrix} -k_2 & 0 & 0 \\ -k_2' & -k_2^2 & -k_1 k_2 \\ (k_1 k_2^2 - k_2'') & (-k_2^2 - 2k_2 k_2') & (-k_1 k_2 - k_1' k_2 - k_1 k_2') \end{vmatrix}, \\ \det(N_2', N_2'', N_2''') &= -k_2^5 \left(\frac{k_1}{k_2}\right)'. \end{aligned}$$

Since α is a slant helix and $\frac{k_1}{k_2}$ is constant. Hence, we have

$$\det(N'_2, N''_2, N'''_2) = 0, \text{ but } k_2 \neq 0.$$

($\Leftarrow$) Suppose that $\det(N'_2, N''_2, N'''_2) = 0$, then it is clear that the $\frac{k_1}{k_2}$ is constant, since $\left(\frac{k_1}{k_2}\right)'$ is zero. Hence the theorem is proved. $\square$

From (2.6)

$$\begin{aligned}\alpha(s) &= T, \\ D_T T &= k_2 N_1 + k_1 N_2, \\ D_T N_1 &= -k_1 T_1, \\ D_T N_2 &= -k_2 T_1,\end{aligned}\tag{4.8}$$

Theorem 6. Let $\alpha : I \rightarrow E_1^3$ be a unit speed pseudo null curve on M_1 then α is a general slant helix if and only if

$$D_T(D_T D_T N_1) + 3k'_1 D_T T = \left(k''_1 - 2k_1^2 k_2\right) \frac{1}{k_1} D_T N_1.\tag{4.9}$$

Proof. ($\Rightarrow$) Suppose that α is a general slant helix. Then, from (4.8), we have

$$\begin{aligned}D_T N_1 &= -k_1 T_1, \\ D_T(D_T N_1) &= -k'_1 T_1 - k_1 k_2 N_1 - k_1^2 N_2,\end{aligned}\tag{4.10}$$

$$\begin{aligned}D_T(D_T D_T N_1) &= \left(k_1^2 k_2 - k_1''\right) T_1 - k'_1 D_T T - \left(k'_1 k_2 + k_1 k'_2\right) N_1 \\ &\quad - k_1 k_2 D_T N_1 - 2k_1 k'_1 N_2.\end{aligned}\tag{4.11}$$

Since α is a general helix

$$\frac{k_1}{k_2} = c \quad c \text{ is constant},\tag{4.12}$$

by differentiating (4.12), we get

$$(k_1 k_2)' = 2k'_1 k_2,\tag{4.13}$$

but

$$\begin{aligned}D_T N_1 &= N'_1 = -k_1 T, \\ T &= -\frac{1}{k_1} D_T N_1.\end{aligned}\tag{4.14}$$

By substituting (4.13) and (4.14) in (4.11) we get

$$\begin{aligned}D_T(D_T D_T N_1) &= \left(k''_1 - 2k_1^2 k_2\right) \left(\frac{1}{k_1} D_T N_1\right) - 3k'_1 D_T T, \\ D_T(D_T D_T N_1) + 3k'_1 D_T T &= \left(k''_1 - 2k_1^2 k_2\right) \left(\frac{1}{k_1} D_T N_1\right).\end{aligned}\tag{4.15}$$

($\Leftarrow$) We will now show that pseudo null curve α is a slant helix. By differentiating (4.14) covariantly

$$T = -\frac{1}{k_1} D_T N_1$$

$$D_T T = \frac{k_1'}{k_1^2} D_T N_1 - \frac{1}{k_1} D_T D_T N_1, \quad (4.16)$$

$$D_T D_T T = \left(\frac{k_1'}{k_1^2} \right)' D_T N_1 + \frac{2k_1'}{k_1^2} D_T D_T N_1 - \frac{1}{k_1} D_T D_T D_T N_1. \quad (4.17)$$

By substituting (4.15) in (4.17) we get

$$\begin{aligned} D_T D_T T &= \left[\left(\frac{k_1'}{k_1^2} \right)' - \frac{1}{k_1^2} (k_1'' - 2k_1^2 k_2) \right] D_T N_1 \\ &\quad + \frac{2k_1'}{k_1^2} D_T D_T N_1 + \frac{3k_1'}{k_1} D_T T, \end{aligned} \quad (4.18)$$

by substituting (4.8) and (4.10) in (4.18) we get

$$\begin{aligned} D_T D_T T &= \left[\left(\frac{k_1'}{k_1^2} \right)' - \frac{1}{k_1^2} (k_1'' - 2k_1^2 k_2) \right] D_T N_1 \\ &\quad - 2 \left(\frac{k_1'}{k_1} \right)^2 T + \frac{k_1' k_2}{k_1} N_1 + k_1' N_2. \end{aligned} \quad (4.19)$$

From (4.8)

$$\begin{aligned} D_T T &= k_2 N_1 + k_1 N_2, \\ D_T D_T T &= k_2' N_1 + k_2 D_T N_1 + k_1' N_2 - k_1 k_2 T_1. \end{aligned} \quad (4.20)$$

By comparing (4.19) and (4.20)

$$\begin{aligned} \frac{k_1' k_2}{k_1} &= k_2', \\ \frac{k_1'}{k_1} &= \frac{k_2'}{k_2}, \end{aligned} \quad (4.21)$$

by integrating (4.21) we get

$$\frac{k_1}{k_2} = e^c \quad (\text{constant}).$$

Hence α is a general slant helix. □

Theorem 7. Let $\alpha : I \longrightarrow E_1^3$ be a unit speed pseudo null curve on M_1 , then α is a general slant helix if and only if

$$D_T (D_T D_T N_2) + 3k_2' D_T T = \left(\frac{k_2''}{k_2} - 2k_1 k_2 \right) D_T N_2. \quad (4.22)$$

Proof. ($\implies$) Suppose that α is a general slant helix. Then, from (4.8), we have

$$\begin{aligned} D_T N_2 &= -k_2 T_1, \\ D_T D_T N_2 &= -k_2' T_1 - k_2^2 N_1 - k_1 k_2 N_2, \end{aligned} \quad (4.23)$$

$$\begin{aligned} D_T (D_T D_T N_2) &= (k_2^2 k_1 - k_2'') T_1 - k_2' D_T T - 2k_2 k_2' N_1 \\ &\quad - (k_1' k_2 + k_1 k_2') N_2 - k_1 k_2 D_T N_2. \end{aligned} \quad (4.24)$$

Since α is a general helix

$$\frac{k_1}{k_2} = c \quad c \text{ is constant,}$$

by differentiating the above equation we get

$$(k_1 k_2)' = 2k_1 k_2', \quad (4.25)$$

but

$$T = -\frac{1}{k_2} D_T N_2. \quad (4.26)$$

By substituting (4.25) and (4.26) in (4.24) we get

$$\begin{aligned} D_T (D_T D_T N_2) &= \left(\frac{k_2''}{k_2} - 2k_1 k_2' \right) D_T N_2 - 3k_2' D_T T, \\ D_T (D_T D_T N_2) + 3k_2' D_T T &= \left(\frac{k_2''}{k_2} - 2k_1 k_2' \right) D_T N_2. \end{aligned} \quad (4.27)$$

($\Leftarrow$) We will show that pseudo null curve α is a slant helix. So by differentiating (4.26) covariantly we get

$$D_T T = \frac{k_2'}{k_2^2} D_T N_2 - \frac{1}{k_2} D_T D_T N_2, \quad (4.28)$$

$$\begin{aligned} D_T D_T T &= \left(\frac{k_2'}{k_2^2} \right)' D_T N_2 + \frac{2k_2'}{k_2^2} D_T D_T N_2 \\ &\quad - \frac{1}{k_2} D_T D_T D_T N_2. \end{aligned} \quad (4.29)$$

By substituting (4.27) in (4.29) we get

$$\begin{aligned} D_T D_T T &= \left[\left(\frac{k_2'}{k_2^2} \right)' - \left(\frac{k_2''}{k_2} - 2k_1 k_2' \right) \frac{1}{k_2} \right] D_T N_2 \\ &\quad + \frac{2k_2'}{k_2^2} D_T D_T N_2 + \frac{3k_2'}{k_2} D_T T, \end{aligned} \quad (4.30)$$

by substituting (4.8) and (4.23) in (4.30) we get

$$\begin{aligned} D_T D_T T &= \left[\left(\frac{k_2'}{k_2^2} \right)' - \left(\frac{k_2''}{k_2} - 2k_1 k_2' \right) \frac{1}{k_2} \right] D_T N_2 \\ &\quad - 2 \left(\frac{k_2'}{k_2} \right)^2 T_1 + k_2' N_1 + \frac{k_1 k_2'}{k_2} N_2. \end{aligned} \quad (4.31)$$

From (4.8)

$$\begin{aligned} D_T T &= k_2 N_1 + k_1 N_2, \\ D_T D_T T &= k_2' N_1 + k_2 D_T N_1 + k_1' N_2 - k_1 k_2 T_1. \end{aligned} \quad (4.32)$$

By comparing (4.31) and (4.32)

$$\begin{aligned}\frac{k_1 k_2'}{k_2} &= k_1' \\ \frac{k_1'}{k_1} &= \frac{k_2'}{k_2},\end{aligned}\tag{4.33}$$

by integrating (4.33) we get

$$\frac{k_1}{k_2} = e^c \quad \text{constant.}$$

Hence α is a general slant helix, the theorem is now proved. $\square$

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