Applied Mathematics and Nonlinear Sciences 5(1) (2020) 195–210

Applied Mathematics and Nonlinear Sciences

https://www.sciendo.com

Global Attractors for the Higher-Order Evolution Equation

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Submission Info

Communicated by Hacı Mehmet Baskonus
Received April 30th 2019
Accepted June 18th 2019
Available online March 31st 2020

Abstract

In this paper, we obtain the existence of a global attractor for the higher-order evolution type equation. Moreover, we discuss the asymptotic behavior of global solution.

Keywords: Global attractor, existence, asymptotic behavior.
AMS 2010 codes: 35L35, 35B40, 35B41.

1 Introduction

We consider the following nonlinear evolution equation

$$\begin{cases} u_{tt}+(-\Delta)^m u+(-\Delta)^m u_t+(-\Delta)^m u_{tt}+g(x,u)=f(x), & (x,t)\in\Omega\times(0,\infty), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in\Omega, \\ \frac{\partial^i u(x,t)}{\partial v^i}=0,\ i=1,2,\dots,m-1, & (x,t)\in\partial\Omega\times[0,\infty), \end{cases} \tag{1.1}$$

where in a bounded domain $\Omega\subset R^n$ with smooth boundary $\partial\Omega$, the assumption on f, g, u_0 and u_1 will be made below.

When $m=1$, the equation (1.1) is following form

$$u_{tt}-\Delta u-\Delta u_t-\Delta u_{tt}+g(x,u)=f(x). \tag{1.2}$$

Chen and Wang [19] proved the existence of global attractor for the problem (1.2). Lately, Xie and Zhong in [8] studied the existence of global attractor of solution for the problem (1.1) with $f=0$. Also, there are some authors studied the existence and nonexistence, asymptotic behavior of global solution for (1.2) (see [2–7] for more details). Nakao and Yang in [9] showed the global attractor of the Kirchhoff type wave equation.

In this paper, we improve our result by adopting and modifying the method of [19], we studied more general form of the equation.

ISSN 2444-8656 doi:10.2478/AMNS.2020.1.00018

This paper is organized as follows: In section 2, we give some assumptions and state the main results. In section 3, we prove the global existence of solution using the Faedo-Galerkin method. Also, we write some important estimates for the solution. In section 4, the existence of the global attractor is proved. In Section 5, the proof of decay property for solution is showed.

2 Preliminaries and main results

We write the Sobolev space $H^k(\Omega) = W^{k,2}(\Omega)$, $H_0^k(\Omega) = W_0^{k,2}(\Omega)$. Furthermore, we show by $(.,.)$ the inner product of $L^2(\Omega)$, by $\|.\|_p$ the norm of $L^p(\Omega)$, $p \geq 1$ and by $\|.\|_E$ the norm of any other Banach space E. As usual, we give $u(t)$ instead of $u(x,t)$, and $u'(t)$ for $u_t(t)$ and so on.

We write the following assumptions on f and g.

(A_1) Assume $f(x) \in L^2(\Omega)$ and show $F = \|f\|_2$;

(A_2) Suppose $g(x,u) \in C^1(\Omega \times R^1)$ and $\exists k_1, k_2 > 0$, $h_1(x) \in L^2(\Omega)$, $h_2(x) \in L^2(\Omega) \cap L^{n/2}(\Omega)$ such that

$$g(x,u)u + h_1(x)|u| \geq k_1(G(x,u) + h_1(x)|u|) \geq 0, \ (x,u) \in \Omega \times R^1 \tag{2.1}$$

and the growth condition in u

$$|g(x,u)| \leq k_2(|u|^\alpha + h_2(x)), \ |g_u(x,u)| \leq k_2\left(|u|^{\alpha-1} + h_2(x)\right), \ (x,u) \in \Omega \times R^1 \tag{2.2}$$

with $\alpha \geq 1$, $(n = 1,2)$, and $1 \leq \alpha \leq \frac{n+2}{n-2}$, $(n \geq 3)$, $G(x,u) = \int_0^u g(x,s)\,ds$.

Later, we assume $H_1 = \|h_1\|_2$, $H_2 = \max\left\{\|h_2\|_2, \ \|h_2\|_{n/2}\right\}$.

Clearly, the function $g(x,u) = a(x)|u|^{\alpha-1}u - b(x)|u|^{\beta-1}u \, (1 \leq \beta < \alpha)$ supplies (2.1) and (2.2) for some $a(x), b(x)$.

Next, we show the definition and lemmas relating to the global attractor, (see [9, 11, 12]).

Definition 1. *Suppose that E is Banach space and $\{S(t)\}_{t\geq 0}$ a semigroup on E. A set $A \subset E$ is said a $(E,E)-$global attractor if and only iff*

(1) A is never changing (invariant), namely, $S(t)A = A$ for whole $t \geq 0$;

(2) A is compact in E;

(3) A is a bounded set in E and absorbs all bounded subset B in E relating with E topology, that is, for whichever bounded subset $B \subset E$,

$$dist_E(S(t)B, A^*) = \sup_{y\in B}\inf_{x\in A^*} \|S(t)y - x\|_E \to 0 \text{ as } t \to \infty. \tag{2.3}$$

Lemma 2. *Assume E is Banach space and $\{S(t)\}_{t\geq 0}$ is a semigroup of continuous operators on E. Then, there exists $(E,E)-$global attractor A if the following conditions are supplied:*

(1) There exists a bounded absorbing set B_0 in E, that is, for whichever bounded subset $B \subset E$, there is a $T = T(B)$ such that $S(t)B \subset B_0$ for any $t \geq T$.

(2) $\{S(t)\}_{t\geq 0}$ as asymptotically compact in E, that is, for any bounded sequence $\{y_n\}$ in E and $t_n \to \infty$ as $n \to \infty$, $\{S(t_n)y_n\}_{n=1}^\infty$ has a convergent subsequence relating to E topology.

We show the basic results now.

Theorem 3. *Suppose (A_1)-(A_2) satisfy and $(u_0, u_1) \in X$. Then, the problem (1.1) admits a unique weak solution $u(t)$ in the class*

$$C^1([0,\infty); H_0^m) \cap C([0,\infty); H^{2m} \cap H_0^m) \cap W^{2,\infty}([0,\infty); H_0^m) \cap W^{1,\infty}\left(([0,\infty); H^{2m}\right) \tag{2.4}$$

holds.

$$\left\|P^{\frac{1}{2}}u(t)\right\|_2^2+\left\|P^{\frac{1}{2}}u_t(t)\right\|_2^2\leq C_1e^{-\lambda_1 t}+C_2,\ t\geq 0 \tag{2.5}$$

$$\|u_{tt}(t)\|_2^2+\left\|P^{\frac{1}{2}}u_t(t)\right\|_2^2+\left\|P^{\frac{1}{2}}u_{tt}(t)\right\|_2^2\leq C_3e^{-\lambda_2 t}+C_4,\ t\geq 0 \tag{2.6}$$

and

$$\left\|P^{\frac{1}{2}}u_t(t)\right\|_2^2+\|Pu(t)\|_2^2+\|Pu_t(t)\|_2^2\leq C_5e^{-\lambda_3 t}+C_4,\ t\geq 0 \tag{2.7}$$

with some $\lambda_1,\ \lambda_2,\lambda_3>0$. *In this theorem* $C_1=C_1\left(\left\|P^{\frac{1}{2}}u_0\right\|_2,\left\|P^{\frac{1}{2}}u_1\right\|_2\right),C_2=C_2(F,H_1),\ C_3=C_3\left(\left\|P^{\frac{1}{2}}u_0\right\|_2,\left\|P^{\frac{1}{2}}u_1\right\|_2,F,H_1,H_2\right),C_4=C_4(F,H_1,H_2),C_5=C_5(\|Pu_0\|_2,\|Pu_1\|_2,F,H_1,H_2)$.

Show the solution in Theorem 1 by $S(t)(u_0,u_1)=(u(t),u_t(t))$. We are now in a position to prove some continuity of $S(t)$ relating to the initial data (u_0,u_1), which will be needed for the proof of the existence of global attractor.

Theorem 4. *Suppose whole conditions in Theorem 3. Assume $S(t)(u_{0k},u_{1k})$ and $S(t)(u_0,u_1)$ are the solutions of the problem (1.1) with the initial data (u_{0k},u_{1k}) and (u_0,u_1). If $(u_{0k},u_{1k})\to(u_0,u_1)$ in X as $k\to\infty$, then $S(t)(u_{0k},u_{1k})\to S(t)(u_0,u_1)$ in X as $k\to\infty$.*

Theorem 4 denotes that the semigroup $S(t):X\to X$ is continuous on X.

Theorem 5. *Assume every assumptions in Theorem 3 be provided. Then, the semigroup $\{S(t)\}_{t\geq 0}$ related with the solution of the problem (1.1) accepts a $(X,X)-$global attractor A.*

For the decay property of solution $u(t)$ for the problem (1.1), we get

Theorem 6. *Suppose u is a weak solution in Theorem 3 with $f=0$ and $g(x,u)=g(u)$. Besides, suppose $0\leq 2G(u)\leq ug(u)$. Then, for whichever $q>0$, there is $C_1=C_1\left(\left\|P^{\frac{1}{2}}u_0\right\|_2,\left\|P^{\frac{1}{2}}u_1\right\|_2\right)$ such that*

$$E(t)=\frac{1}{2}\left(\|u(t)\|_2^2+\left\|P^{\frac{1}{2}}u(t)\right\|_2^2+\left\|P^{\frac{1}{2}}u_t(t)\right\|_2^2\right)+\int_\Omega G(u(t))\,dx\leq C_1(1+t)^{-1/q}. \tag{2.8}$$

3 The Proof of Theorem 3

In this section, we suppose that all assumptions in Theorem 3 are supplied. Firstly, we establish the global existence of a solution to problem (1.1) with Fadeo-Galerkin method as in [16, 17].

Assume $\omega_j(x)$ $(j=1,2,...)$ is the complete set of properly normalized eigenfunctions for the operator $(-\Delta)^m$ in $H_0^m(\Omega)$. Then, the family $\{\omega_1,\omega_2...,\omega_k,...\}$ holds an orthogonal basis for both $H_0^m(\Omega)$ and $L^2(\Omega)$, see [16, 17]. For each positive integer k, show $V_k=span\{\omega_1,\omega_2...,\omega_k,...\}$. We search for an approximation solution $u_k(t)$ to the problem (1.1) in the form

$$u_k(t)=\sum_{j=1}^{k}d_{jk}(t)\,\omega_j$$

where $d_{jk}(t)$ are the solution of the nonlinear ordinary differential equation (ODE) system in the variant t:

$$\left(u_k'',\omega_j\right)-\left(Pu_k,\omega_j\right)-\left(Pu_k',\omega_j\right)-\left(Pu_k'',\omega_j\right)+\left(g,\omega_j\right)=\left(f,\omega_j\right),\quad j=1,2,...,k, \tag{3.1}$$

with the initial conditions

$$d_{jk}(0)=(u_{0k},\omega_j),\ d'_{jk}(0)=(u_{1k},\omega_j) \tag{3.2}$$

where u_{0k} and u_{1k} are chosen in V_k so that

$$u_{0k}\to u_0,\ \ u_{1k}\to u_1 \text{ in } H^{2m}(\Omega)\cap H_0^m(\Omega) \text{ as } k\to\infty. \tag{3.3}$$

Here $(.,.)$ shows the inner product in $L^2(\Omega)$. Then, Sobolev imbedding theorem means that $\exists c_0>0$, such that

$$\|u_k(0)\|_{H_0^m}^2\le c_0\left\|P^{\frac12}u_0\right\|_2^2,\ \ \|u'_k(0)\|_{H_0^m}^2\le c_0\left\|P^{\frac12}u_1\right\|_2^2\ \ \forall k=1,2,..., \tag{3.4}$$

and (3.1) shows that for any $v\in V_k$,

$$(u''_k,v)-(Pu_k,v)-(Pu'_k,v)-\left(Pu''_k,v\right)+(g,v)=(f,v),\ \ \forall v\in V_k. \tag{3.5}$$

We know, the system (3.1) and (3.2) accept a unique solution $u_k(t)$ on the interval $[0,T]$ for any $T>0$. Such a solution can be expanded to the overall interval $[0,\infty)$. We show by $C_i\,(i=1,2,...)$ the constants that are independent of k and $t\ge 0$, by C_0 the constant depending on k_1, k_2 in (A_2) and Sobolev imbedding constant c_0 in (3.4). These constants may be different from line to line.

Multiplying (3.1) by $d'_{jk}(t)$ and summing the resulting equations over j, we obtain

$$E'_1(t)+\left\|P^{\frac12}u'_k(t)\right\|_2^2=0,\ \forall t\ge 0 \tag{3.6}$$

where

$$E_1(t)=\frac12\left(\|u'_k(t)\|_2^2+\left\|P^{\frac12}u_k(t)\right\|_2^2+\left\|P^{\frac12}u'_k(t)\right\|_2^2\right)+\int_\Omega G(x,u_k(t))\,dx-\int_\Omega f(x)\,u_k(t)\,dx. \tag{3.7}$$

Also, multiplying (3.1) by $d_{jk}(t)$, we get

$$E'_2(t)+\left\|P^{\frac12}u_k(t)\right\|_2^2+\int_\Omega g(x,u_k)\,u_k(t)\,dx=\|u'_k(t)\|_2^2+\left\|P^{\frac12}u'_k(t)\right\|_2^2+\int_\Omega f(x)\,u_k(t)\,dx \tag{3.8}$$

where

$$E_2(t)=\frac12\left\|P^{\frac12}u_k(t)\right\|_2^2+\int_\Omega u_k(t)\,u'_k(t)\,dx+\int_\Omega P^{\frac12}u_k(t)\,P^{\frac12}u'_k(t)\,dx. \tag{3.9}$$

If we take sufficient large $k_1>0$ and use the assumption (A_2), we get

$$\psi'_k(t)+\lambda_1\psi_k(t)\le C_0\left(F^2+H_1^{2m}\right),\ \psi_k(t)=k_1E_1(t)+E_2(t) \tag{3.10}$$

with some positive λ_1, relating to the indicated constants in (A_2).

We note that

$$\begin{aligned}\psi_k(t)\le{}& c_0(k_1+c_0)\left(\left\|P^{\frac12}u'_k(t)\right\|_2^2+\left\|P^{\frac12}u_k(t)\right\|_2^2\right)\\ &+k_1^2\left(F^2+H_1^{2m}\right)+k_1\int_\Omega (G+h_1|u_k(t)|)\,dx\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}2\psi_k(t)\ge{}&(k_1-1)\left(\|u'_k(t)\|_2^2+\left\|P^{\frac12}u'_k(t)\right\|_2^2\right)+(k_1-5c_0)\left\|P^{\frac12}u_k(t)\right\|_2^2\\ &+k_1\int_\Omega (G+h_1|u_k(t)|)\,dx-k_1^2\left(F^2+H_1^{2m}\right)\end{aligned} \tag{3.12}$$

with $k_1 \geq \max\{3, 2+5c_0\}$.

The application of Gronwall lemma to (3.10) holds

$$\left\|P^{\frac{1}{2}}u_k(t)\right\|_2^2 + \left\|P^{\frac{1}{2}}u_k'(t)\right\|_2^2 + \int_\Omega \left(G(x,u_k(t)) + h_1(x)\,|u_k(t)|\right)dx \leq C_1 e^{-\lambda_1 t} + C_2,\ t \geq 0$$

which shows

$$\left\|P^{\frac{1}{2}}u_k(t)\right\|_2^2 + \left\|P^{\frac{1}{2}}u_k'(t)\right\|_2^2 \leq C_1 e^{-\lambda_1 t} + C_2,\ t \geq 0 \tag{3.13}$$

where $C_1 = C_1\left(\left\|P^{\frac{1}{2}}u_0\right\|_2, \left\|P^{\frac{1}{2}}u_1\right\|_2\right)$, $C_2 = C_2(F, H_1)$.

Also, we differentiate (3.1) with respect to t and get

$$\left(u_k''', \omega_j\right) - \left(Pu_k', \omega_j\right) - \left(Pu_k'', \omega_j\right) - \left(Pu_k''', \omega_j\right) + \left(g_u u_k', \omega_j\right) = 0,\ \ j = 1, 2, \ldots, k. \tag{3.14}$$

Multiplying (3.14) by $d_{jk}''(t)$ and summing the resulting equations over j, we obtain

$$E_3'(t) + \left\|P^{\frac{1}{2}}u_k''(t)\right\|_2^2 + \int_\Omega g_u u_k' u_k'' dx = 0 \tag{3.15}$$

with

$$\begin{aligned} E_3(t) &= \frac{1}{2}\left(\left\|u_k''(t)\right\|_2^2 + \left\|P^{\frac{1}{2}}u_k'(t)\right\|_2^2 + \left\|P^{\frac{1}{2}}u_k''(t)\right\|_2^2\right) \\ &\leq C_0\left(\left\|P^{\frac{1}{2}}u_k''(t)\right\|_2^2 + \left\|P^{\frac{1}{2}}u_k'(t)\right\|_2^2\right),\ t \geq 0 \end{aligned} \tag{3.16}$$

in which the Sobolev embedding theorem has been used.

Furthermore, the growth condition (2.2) and the Hölder inequality mean that

$$\begin{aligned} \int_\Omega \left|g_u u_k' u_k''\right| dx &\leq k_2 \int_\Omega \left(|h_2|\,|u_k'|\,|u_k''| + |u_k|^{\alpha-1}\,|u_k'|\,|u_k''|\right) dx \\ &\leq C_0\left(\|h_2\|_{n/2} + \left\|P^{\frac{1}{2}}u_m\right\|_2^{\alpha-1}\right)\left\|P^{\frac{1}{2}}u_k'\right\|_2 \left\|P^{\frac{1}{2}}u_k''\right\|_2. \end{aligned}$$

Therefore, we get

$$\int_\Omega \left|g_u u_k' u_k''\right| dx \leq \frac{1}{2}\left\|P^{\frac{1}{2}}u_k''(t)\right\|_2^2 + C_0\left\|P^{\frac{1}{2}}u_k'(t)\right\|_2^2\left(\left\|P^{\frac{1}{2}}u_k(t)\right\|_2^{2(\alpha-1)} + H_2^{2m}\right) \tag{3.17}$$

and

$$E_3'(t) + \frac{1}{2}\left\|P^{\frac{1}{2}}u_k''(t)\right\|_2^2 \leq C_0\left\|P^{\frac{1}{2}}u_k'(t)\right\|_2^2\left(\left\|P^{\frac{1}{2}}u_k(t)\right\|_2^{2(\alpha-1)} + H_2^{2m}\right). \tag{3.18}$$

Then, the applications of the estimates (3.13) and (3.15)-(3.18) give that $\exists \lambda_1 \geq \lambda_2 > 0$, depending on C_0, such that

$$\begin{aligned} E_3'(t) + \lambda_2 E_3(t) &\leq C_0\left\|P^{\frac{1}{2}}u_k'(t)\right\|_2^2\left(1 + \left\|P^{\frac{1}{2}}u_k(t)\right\|_2^{2(\alpha-1)} + H_2^2\right) \\ &\leq C_3 e^{-\lambda_1 t} + C_4. \end{aligned} \tag{3.19}$$

Here, assume $C_3 = C_3\left(\left\|P^{\frac{1}{2}}u_0\right\|_2, \left\|P^{\frac{1}{2}}u_1\right\|_2, F, H_1, H_2\right)$, $C_4 = C_4(F, H_1, H_2)$. Then (3.19) means that

$$E_3(t) \leq E_3(0)\,e^{-\lambda_2 t} + C_3 e^{-\lambda_2 t} + \lambda_2^{-1} c_{4,}\ t \geq 0. \tag{3.20}$$

We show that $E_3(0)$ is uniformly bounded for k under the conditions in Theorem 3 now. It follows by (3.1) that

$$\left(u_k''(t) - Pu_k(t) - Pu_k'(t) - Pu_k''(t), u_k''(t)\right) = \left(f, u_k''(t)\right) - \left(g, u_k''(t)\right). \tag{3.21}$$

Especially, suppose $t = 0$, we get

$$\begin{aligned} &\left\|u_k''(0)\right\|_2^2 + \left\|P^{\frac{1}{2}}u_k''(0)\right\|_2^2 + \int_\Omega P^{\frac{1}{2}}u_k''(0).\left(P^{\frac{1}{2}}u_k(0) + P^{\frac{1}{2}}u_k'(0)\right)dx \\ = &\int_\Omega f(x)\, u_k''(0)\, dx - \int_\Omega (g, u_k(0))\, u_k''(0)\, dx. \end{aligned} \tag{3.22}$$

By Young inequality with ε,

$$\begin{aligned} \int_\Omega \left|P^{\frac{1}{2}}u_k''(0).P^{\frac{1}{2}}u_k(0)\right| dx &\leq \varepsilon \left\|P^{\frac{1}{2}}u_k''(0)\right\|_2^2 + C_\varepsilon \left\|P^{\frac{1}{2}}u_k(0)\right\|_2^2, \\ \int_\Omega \left|P^{\frac{1}{2}}u_k''(0).P^{\frac{1}{2}}u_k'(0)\right| dx &\leq \varepsilon \left\|P^{\frac{1}{2}}u_k''(0)\right\|_2^2 + C_\varepsilon \left\|P^{\frac{1}{2}}u_k'(0)\right\|_2^2, \\ \int_\Omega \left|g(x, u_k(0))\, u_k''(0)\, dx\right| \leq \left\|u_k''(0)\right\|_{\frac{2n}{n-2}} \|g\|_{\mu_1} &\leq \varepsilon \left\|P^{\frac{1}{2}}u_k''(0)\right\|_2^2 + C_\varepsilon \|g\|_{\mu_1}^2, \end{aligned} \tag{3.23}$$

and

$$\int_\Omega \left|f(x)\, u_k''(0)\right| dx \leq \varepsilon \left\|P^{\frac{1}{2}}u_k''(0)\right\|_2^2 + C_\varepsilon \|f\|_2^2$$

with $\mu_1 = 2n/(n+2)$. Since $\mu_1\alpha = 2n\alpha/(n+2) \leq 2n/(n-2)$, we obtain by (2.2) that

$$\int_\Omega |g|^{\mu_1} dx \leq C_0 \int_\Omega \left(|u_k(0)|^{\mu_1\alpha} + |h_2|^{\mu_1}\right) dx \leq C_0 \left(\left\|P^{\frac{1}{2}}u_0\right\|_2^{\mu_1\alpha} + \|h_2\|_2^{\mu_1}\right). \tag{3.24}$$

Suppose $0 < \varepsilon \leq 1/6$. Then, from (3.22) to (3.24) that

$$\begin{aligned} E_3(0) &\leq \left\|P^{\frac{1}{2}}u_k''(0)\right\|_2^2 + \left\|u_k''(0)\right\|_2^2 + \left\|P^{\frac{1}{2}}u_k'(0)\right\|_2^2 \\ &\leq C_0\left(\left\|P^{\frac{1}{2}}u_k'(0)\right\|_2^2 + \left\|P^{\frac{1}{2}}u_k(0)\right\|_2^2 + F^2 + \|g\|_{\mu_1}^2\right) \\ &\leq C_0\left(\left\|P^{\frac{1}{2}}u_1\right\|_2^2 + \left\|P^{\frac{1}{2}}u_0\right\|_2^2 + F^2 + \left\|P^{\frac{1}{2}}u_0\right\|_2^{2\alpha} + \|h_2\|_2^2\right) \equiv C_3. \end{aligned} \tag{3.25}$$

Therefore, the inequality (3.20) shows

$$\left\|P^{\frac{1}{2}}u_k''(t)\right\|_2^2 + \left\|P^{\frac{1}{2}}u_k'(t)\right\|_2^2 + \left\|P^{\frac{1}{2}}u_k''(t)\right\|_2^2 \leq C_3 e^{-\lambda_2 t} + \lambda_2^{-1} C_4,\ t \geq 0 \tag{3.26}$$

and the estimates (3.13) and (3.26) give that

$$\begin{cases} \{u_k(t)\} \text{ is bounded in } L^\infty([0,\infty); H_0^m(\Omega)), \\ \{u_k'(t)\} \text{ is bounded in } L^\infty([0,\infty); H_0^m(\Omega)), \\ \{u_k''(t)\} \text{ is bounded in } L^\infty([0,\infty); H_0^m(\Omega)). \end{cases} \tag{3.27}$$

So, there exists a subsequences in $\{u_k\}$ (still showed by $\{u_k\}$) such that

$$\begin{cases} u_k \to u \text{ weakly star in } L^\infty([0,\infty); H_0^m(\Omega)), \\ u_k' \to u' \text{ weakly star in } L^\infty\left([0,\infty); L^2(\Omega)\right), \\ u_k'' \to u'' \text{ weakly star in } L^2([0,\infty); H_0^m(\Omega)). \end{cases} \tag{3.28}$$

From applying the fact that $L^{\infty}([0,\infty);H_0^m(\Omega)) \hookrightarrow L^2([0,\infty);H_0^m(\Omega))$ and the Lions-Aubin compactness Lemma in [20], we obtain from (3.27) and (3.28) that

$$u_k \to u,\ u_k' \to u' \text{ strongly in } L^2\left([0,\infty);L^2(\Omega)\right) \tag{3.29}$$

and then $u_k \to u$ a.e in $\Omega \times [0,\infty)$.

Using the growth condition (2.2), for any $T > 0$, the integral

$$\int_0^T \int_\Omega |g(x,u_k(x,t))|^{\frac{\alpha+1}{\alpha}} dxdt$$

is bounded. Accordingly, by Lemma 2 in Chap. 1 [17], we conclude

$$g(x,u_k) \to g(x,u) \text{ weakly in } L^{\frac{\alpha+1}{\alpha}}\left([0,T];L^{\frac{\alpha+1}{\alpha}}(\Omega)\right) \tag{3.30}$$

with these convergences, by using the limit in the approximate equation (3.5), we get

$$\left(u''(t),v\right) - (Pu,v) - \left(Pu',v\right) - \left(Pu'',v\right) + (g(x,u),v) = (f,v),\ \forall v \in H_0^m(\Omega), \tag{3.31}$$

So, $u(t)$ is a weak solution of (1.1) and supplies (2.5) and (2.6), and the proof of existence for the solution $u(t)$ of (1.1) is completed.

We derive the estimates for $\|Pu(t)\|_2$ and $\|Pu_t(t)\|_2$ now. Also, we write u instead of u_k for convenience and view the estimates for u as a limit of u_k. Supposing $v = -Pu$ in (3.31), we obtain

$$E_4'(t) + \|Pu(t)\|_2^2 \le \left\|P^{\frac{1}{2}}u_t(t)\right\|_2^2 + \|Pu_t(t)\|_2^2 + C_0\left(F^2 + \|g\|_2^2\right) \tag{3.32}$$

with some $C_0 > 0$ and

$$E_4(t) = \frac{1}{2}\|Pu(t)\|_2^2 + \int_\Omega P^{\frac{1}{2}}u_t(t)P^{\frac{1}{2}}u(t)\,dx + \int_\Omega Pu_t(t)Pu(t)\,dx. \tag{3.33}$$

Also, assuming $v = -Pu_t$ in (3.31), we get

$$\begin{aligned}\int_\Omega Pu_t(-u_{tt} + Pu + Pu_{tt})\,dx + \|Pu_t\|_2^2 &= \int_\Omega gP^{\frac{1}{2}}u_t dx - \int_\Omega fPu_t dx \\ &\le \frac{1}{2}\|Pu_t\|_2^2 + C_0\left(F^2 + \|g\|_2^2\right).\end{aligned} \tag{3.34}$$

This means that

$$E_5'(t) + \frac{1}{2}\|Pu_t(t)\|_2^2 \le C_0\left(F^2 + \|g\|_2^2\right) \tag{3.35}$$

with

$$E_5(t) = \frac{1}{2}\left(\left\|P^{\frac{1}{2}}u_t(t)\right\|_2^2 + \|Pu_t(t)\|_2^2 + \|Pu(t)\|_2^2\right). \tag{3.36}$$

We note that

$$\|u\|_{2\alpha}^{2\alpha} \le C_0\left\|P^{\frac{1}{2}}u\right\|_2^{2\alpha\theta} + \|Pu\|_2^{2\alpha(1-\theta)} \le \eta\|Pu\|_2^2 + C_\eta\left\|P^{\frac{1}{2}}u\right\|_2^{2\beta} \tag{3.37}$$

with small $\eta > 0$ and $2\alpha\theta = (n-2)\alpha - n < 2$, $\beta = \alpha(1-\theta)/(1-\alpha\theta) > 0$. Then, (3.37) shows

$$\|g\|_2^2 \le C_0\left(\|u\|_{2\alpha}^{2\alpha} + H_2^{2m}\right) \le \eta\|Pu\|_2^2 + C_\eta\left\|P^{\frac{1}{2}}u\right\|_2^{2\beta} + C_0H_2^{2m}. \tag{3.38}$$

Then, by (2.5), (3.35) and (3.38) that

$$\begin{aligned}E_5'(t)+\frac{1}{2}\|Pu_t(t)\|_2^2 &\leq \eta\|Pu(t)\|_2^2+C_\eta\left\|P^{\frac{1}{2}}u(t)\right\|_2^{2\beta}+C_0\left(F^2+H_2^{2m}\right)\\ &\leq C_1e^{-\lambda_1\beta t}+\eta\|Pu(t)\|_2^2+C_2.\end{aligned}\tag{3.39}$$

Assume $\phi(t)=k_1E_5(t)+E_4(t)$. We get from (3.32) and (3.39) that

$$\phi'(t)+\frac{k_1-1}{2}\|Pu_t(t)\|_2^2+(1-(1+k_1/2)\eta)\|Pu(t)\|_2^2\leq C_1e^{-\lambda_1\beta t}+C_2.\tag{3.40}$$

Suppose $k_1\geq 3$ and η is small that $1-\eta(1+k_1/2)\geq 4/5$. Then, (3.40) shows

$$\phi'(t)+\|Pu_t(t)\|_2^2+\frac{1}{2}\|Pu(t)\|_2^2\leq C_1e^{-\lambda_1\beta t}+C_2.\tag{3.41}$$

We note that

$$E_4(t)\leq\frac{3}{5}\|Pu\|_2^2+3\|Pu_t(t)\|_2^2+\frac{1}{2}\left(\left\|P^{\frac{1}{2}}u\right\|_2^2+\left\|P^{\frac{1}{2}}u_t\right\|_2^2\right)\tag{3.42}$$

and

$$\begin{aligned}\phi(t) &\leq\left(\frac{3}{5}+\frac{k_1}{2}\right)\|Pu\|_2^2+\left(3+\frac{k_1}{2}\right)\|Pu_t\|_2^2+\frac{1}{2}\left(\left\|P^{\frac{1}{2}}u\right\|_2^2+\left\|P^{\frac{1}{2}}u_t\right\|_2^2\right)\\ &\leq C_0\left(\|Pu(t)\|_2^2+\|Pu_t(t)\|_2^2\right)+C_1e^{-\lambda_1\beta t}+C_2.\end{aligned}\tag{3.43}$$

Also (3.41) and (3.43) give that $\exists\lambda_1\beta\geq\lambda_3>0$, depending on C_0, such that

$$\phi'(t)+\lambda_3\phi(t)\leq C_1e^{-\lambda_1\beta t}+C_2,\ t\geq 0.\tag{3.44}$$

So,

$$\phi(t)\leq\phi(0)e^{-\lambda_3 t}+C_1e^{-\lambda_3 t}+C_2\lambda_3^{-1},\ t\geq 0.\tag{3.45}$$

Otherwise, we get

$$\begin{aligned}\phi(t)=k_1E_4(t)+E_3(t) &\geq\frac{k_1}{2}\left(\left\|P^{\frac{1}{2}}u_t\right\|_2^2+\|Pu_t\|_2^2+\|Pu\|_2^2\right)\\ &\quad-\frac{1}{2}\left(\left\|P^{\frac{1}{2}}u_t\right\|_2^2+\|Pu_t\|_2^2+\left\|P^{\frac{1}{2}}u\right\|_2^2\right)\\ &\geq\frac{k_1-1}{2}\left\|P^{\frac{1}{2}}u_t\right\|_2^2+\left(\frac{k_1}{2}-1\right)\|Pu_t\|_2^2+\frac{k_1-c_0}{2}\|Pu\|_2^2\\ &\geq\left\|P^{\frac{1}{2}}u_t\right\|_2^2+\|Pu_t\|_2^2+\|Pu\|_2^2,\end{aligned}\tag{3.46}$$

where the facts $k_1\geq\{4,2+c_0\}$ and Sobolev imbedding theorem (see [17])

$$\left\|P^{\frac{1}{2}}u\right\|_2^2\leq c_0\|Pu\|_2^2\quad\forall u\in H^{2m}(\Omega)\cap H_0^m(\Omega)$$

have been used. So, by the estimates (3.45) and (3.46) that

$$\left\|P^{\frac{1}{2}}u_t(t)\right\|_2^2+\|Pu(t)\|_2^2+\|Pu_t(t)\|_2^2\leq C_5e^{-\lambda_3 t}+C_4\lambda_3^{-1},\ t\geq 0\tag{3.47}$$

with $C_4=C_4(F,H_1,H_2)$, $C_5=C_5(\|Pu_0\|_2,\|Pu_1\|_2,F,H_1,H_2)$.

To establish the uniqueness, we suppose that $u(t)$ and $v(t)$ are two solutions of (1.1), which supply the estimates (2.5)-(2.7) and $u(0)=v(0)$, $u'(0)=v'(0)$. Taking $U(t)=u_t(t)$, $V(t)=v_t(t)$ and $W(t)=U(t)-V(t)$, then we see from (1.1) that

$$W_t - PW - PW_t - P(u-v) = g(x,v) - g(x,u),\ \ x\in\Omega,\ t\geq 0. \tag{3.48}$$

Multiplying (3.48) by W, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|W(t)\|_2^2+\left\|P^{\frac{1}{2}}W(t)\right\|_2^2\right)+\left\|P^{\frac{1}{2}}W(t)\right\|_2^2+\int_\Omega P^{\frac{1}{2}}(u-v)P^{\frac{1}{2}}W\,dx\\&=\int_\Omega (g(x,v)-g(x,u))W\,dx\end{aligned} \tag{3.49}$$

and

$$\begin{aligned}&\|W(t)\|_2^2+\left\|P^{\frac{1}{2}}W(t)\right\|_2^2+2\int_0^t\left\|P^{\frac{1}{2}}W(s)\right\|_2^2ds+2\int_0^t\int_\Omega P^{\frac{1}{2}}(u(s)-v(s))P^{\frac{1}{2}}W(s)\,dxds\\&=2\int_0^t\int_\Omega (g(x,v(s))-g(x,u(s)))W(s)\,dxds.\end{aligned} \tag{3.50}$$

Since

$$\left|P^{\frac{1}{2}}(u(s)-v(s))\right|\leq\int_0^s\left|P^{\frac{1}{2}}(u_\tau(\tau)-v_\tau(\tau))\right|d\tau=\int_0^s\left|P^{\frac{1}{2}}W(\tau)\right|d\tau$$

then

$$\left\|P^{\frac{1}{2}}(u(s)-v(s))\right\|_2\leq s^{1/2}\left(\int_0^s\left\|P^{\frac{1}{2}}W(\tau)\right\|_2^2d\tau\right)^{1/2}$$

and

$$\begin{aligned}\int_0^t\int_\Omega\left|P^{\frac{1}{2}}(u(s)-v(s))P^{\frac{1}{2}}W(s)\right|dxds&\leq\int_0^t\int_\Omega\int_0^s\left|P^{\frac{1}{2}}W(s)\right|\left|P^{\frac{1}{2}}W(\tau)\right|dxd\tau ds\\&\leq\int_0^t\int_0^s\left\|P^{\frac{1}{2}}W(s)\right\|_2\left\|P^{\frac{1}{2}}W(\tau)\right\|_2d\tau ds\\&\leq t\int_0^t\left\|P^{\frac{1}{2}}W(s)\right\|_2^2ds.\end{aligned} \tag{3.51}$$

Now, taking $U_\varepsilon(s)=\varepsilon u(s)+(1-\varepsilon)v(s)$, $0\leq\varepsilon\leq 1$, we get

$$\begin{aligned}G&=\int_0^t\int_\Omega|g(x,u(s))-g(x,v(s))||W(s)|\,dxds=\int_0^t\int_\Omega\left|\int_0^1\frac{d}{d\varepsilon}g(x,U_\varepsilon)\,d\varepsilon\right||W(s)|\,dxds\\&\leq\int_0^t\int_\Omega\int_0^1|g_u(x,U_\varepsilon)||u(s)-v(s)||W(s)|\,d\varepsilon dxds\\&\leq k_2\int_0^t\int_\Omega\left(|u|^{\alpha-1}+|v|^{\alpha-1}+h_2(x)\right)|u(s)-v(s)||W(s)|\,dxds\\&\leq c_0\int_0^t\left(\|u(s)\|_{\sigma_1}^{\sigma_1}+\|v(s)\|_{\sigma_1}^{\sigma_1}+\|h_2\|_{\sigma_2}^{\sigma_2}\right)\left\|P^{\frac{1}{2}}(u(s)-v(s))\right\|_2\left\|P^{\frac{1}{2}}W(s)\right\|_2ds\end{aligned}$$

where $\sigma_1=n(\alpha-1)/2\leq 2n/(n-2)$, $\sigma_2=n/2$.

From (2.5) and Sobolev imbedding theorem, there is $C_3>0$ such that

$$\|u(s)\|_{\sigma_1}^{\sigma_1}+\|v(s)\|_{\sigma_1}^{\sigma_1}+\|h_2\|_{\sigma_2}^{\sigma_2}\leq C_0\left(\left\|P^{\frac{1}{2}}u(s)\right\|_2^{\sigma_1}+\left\|P^{\frac{1}{2}}v(s)\right\|_2^{\sigma_1}+\|h_2\|_{\sigma_2}^{\sigma_2}\right)\leq C_3\ \forall s\geq 0.$$

Then,

$$G \leq C_3 \int_0^t s^{1/2} \left(\int_0^s \left\| P^{\frac{1}{2}} W(\tau) \right\|_2^2 d\tau \right)^{1/2} \left\| P^{\frac{1}{2}} W(s) \right\|_2 ds \leq C_3 t \int_0^t \left\| P^{\frac{1}{2}} W(\tau) \right\|_2^2 d\tau. \tag{3.52}$$

Then, the estimates (3.50)-(3.52) indicate that

$$\|W(t)\|_2^2 + \left\| P^{\frac{1}{2}} W(t) \right\|_2^2 + 2 \int_0^t \left\| P^{\frac{1}{2}} W(s) \right\|_2^2 \leq (C_3+1)\, t \int_0^t \left\| P^{\frac{1}{2}} W(s) \right\|_2^2 ds. \tag{3.53}$$

The integral inequality (3.53) represents that there exists $T_1 > 0$, such that $W(t) = 0$ in $[0, T_1]$. As a result, $u(t) - v(t) = u(0) - v(0) = 0$ in $[0, T_1]$.

Then, we conduce that $u(t) = v(t)$ on $[T_1, 2T_1]$, $[2T_1, 3T_1], \ldots$, and $u(t) = v(t)$ on $[0, \infty)$. This shows the proof of uniqueness.

Now, we establish $u \in C([0,\infty); H_0^m(\Omega))$. Assume $t > s \geq 0$. Then,

$$\begin{aligned} \left\| P^{\frac{1}{2}} (u(t) - u(s)) \right\|_2^2 &= \int_\Omega \left| \int_s^t P^{\frac{1}{2}} u_\tau(\tau)\, d\tau \right|^2 dx \\ &\leq (t-s) \int_s^t \left\| P^{\frac{1}{2}} u_\tau(\tau)\, d\tau \right\|_2^2 d\tau \to 0 \text{ as } t \to s. \end{aligned} \tag{3.54}$$

This indicates $u(t) \in C([0,\infty); H_0^m(\Omega))$. Also, we get

$$\begin{aligned} \|P(u(t) - u(s))\|_2^2 &= \int_\Omega \left| \int_s^t P u_\tau(\tau)\, d\tau \right|^2 dx \\ &\leq (t-s) \int_s^t \|P u_\tau(\tau)\|_2^2 d\tau \to 0 \text{ as } t \to s. \end{aligned} \tag{3.55}$$

and $u(t) \in C\left([0,\infty); H^{2m}(\Omega) \cap H_0^m(\Omega)\right)$.

Moreover, we get

$$\left\| P^{\frac{1}{2}} (u_t(t) - u_t(s)) \right\|_2^2 \leq (t-s) \int_s^t \left\| P^{\frac{1}{2}} u_{tt}(\tau) \right\|_2^2 d\tau \to 0 \text{ as } t \to s. \tag{3.56}$$

This shows that $u(t) \in C^1([0,\infty); H_0^m)$ and the proof of Theorem 3 is completed.

4 Global attractor for the problem (1)

By Theorem 3, we see that the solution operatör $S(t)(u_0, u_1) = (u(t), u_t(t))$, $t \geq 0$ of the problem (1.1) creates a semigroup on $X = \left(H^{2m}(\Omega) \cap H_0^m(\Omega)\right) \times \left(H^{2m}(\Omega) \cap H_0^m(\Omega)\right)$, which supplies these properties:

(1) $S(t) : X \to X$ for all $t \geq 0$;

(2) $S(t+s) = S(t)S(s)$ for $t, s \geq 0$;

(3) $S(t)(u_0, u_1) \to S(s)(u_0, u_1)$ in X as $t \to s$ for any $(u_0, u_1) \in X$.

For establishing the existence of the (X,X)-global attractor for the problem (1.1), firstly, we show the continuity of $S(t)$ relating to the initial data (u_0, u_1).

The proof of Theorem 4

Suppose $u_k(t)$, $u(t)$ is corresponding solution of the problem (1.1) with the initial data $(u_{0k}, u_1 k)$ and (u_0, u_1) respectively, $k = 1, 2, \ldots$.

Since $(u_{0k}, u_{1k}) \to (u_0, u_1)$ in X, $\{(u_{0k}, u_{1k})\}$ is bounded in X. Set $w_k(t) = u_k(t) - u(t)$. Then, w_k holds

$$\begin{cases} w_k'' - P w_k - P w_k' - P w_k'' = g(x,u) - g(x,u_k) = G_k, \ (x,t) \in \Omega \times (0,\infty), \\ w_k(x,0) = u_{0k}(x) - u_0(x), \ w_k'(x,0) = u_{1k}(x) - u_1(x), \ \ x \in \Omega, \\ w_k(x,t) = 0, \ (x,t) \in \partial\Omega \times [0,\infty). \end{cases} \tag{4.1}$$

Multiplying the equation in (4.1) by $w_k', -Pw_k$ and $-Pw_k'$, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|w_k'\|_2^2+\left\|P^{\frac{1}{2}}w_k\right\|_2^2+\left\|P^{\frac{1}{2}}w_k'\right\|_2^2\right)+(1-\eta)\left\|P^{\frac{1}{2}}w_k'\right\|_2^2\leq C_\eta\|G_k\|_2^2 \tag{4.2}$$

and

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|Pw_k\|_2^2+\int_\Omega\left(Pw_k'Pw_k+P^{\frac{1}{2}}w_kP^{\frac{1}{2}}w_k'\right)dx\right)+(1-\eta)\|Pw_k\|_2^2\\&\leq\|Pw_k'\|_2^2+\left\|P^{\frac{1}{2}}w_k'\right\|_2^2+C_\eta\|G_k\|_2^2\leq c_0\|Pw_k'\|_2^2+C_\eta\|G_k\|_2^2\end{aligned} \tag{4.3}$$

and

$$\frac{1}{2}\frac{d}{dt}\left(\left\|P^{\frac{1}{2}}w_k'\right\|_2^2+\|Pw_k\|_2^2+\|Pw_k'\|_2^2\right)+(1-\eta)\|Pw_k'\|_2^2\leq C_\eta\|G_k\|_2^2 \tag{4.4}$$

with small $\eta>0$. Then, by (4.2) and (4.4) we obtain

$$\begin{aligned}&y_k'(t)+(k(1-\eta)-c_0)\|Pw_k'\|_2^2+(1-\eta)\left\|P^{\frac{1}{2}}w_k'(t)\right\|_2^2+(1-\eta)\|Pw_k(t)\|_2^2\\&\leq kC_\eta\|G_k\|_2^2\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}y_k(t)=&\frac{k_1+1}{2}\left(\|Pw_k(t)\|_2^2+\left\|P^{\frac{1}{2}}w_k'(t)\right\|_2^2\right)\\&+\frac{1}{2}\left(\left\|P^{\frac{1}{2}}w_k(t)\right\|_2^2+\|w_k'(t)\|_2^2\right)+\frac{k_1}{2}\left\|P^{\frac{1}{2}}w_k'(t)\right\|_2^2\\&+\int_\Omega\left(Pw_k'(t)Pw_k(t)+P^{\frac{1}{2}}w_k(t)P^{\frac{1}{2}}w_k'(t)\right)dx\\\leq&\frac{k_1+2}{2}\left(\|Pw_k(t)\|_2^2+\left\|P^{\frac{1}{2}}w_k'(t)\right\|_2^2\right)\\&+\frac{k_1+1}{2}\|Pw_k'(t)\|_2^2+\|w_k'(t)\|_2^2+\left\|P^{\frac{1}{2}}w_k(t)\right\|_2^2\\\leq&C_0\left(\|Pw_k'(t)\|_2^2+\|Pw_k(t)\|_2^2+\left\|P^{\frac{1}{2}}w_k'(t)\right\|_2^2\right).\end{aligned} \tag{4.6}$$

By taking $k_1\geq3$

$$\begin{aligned}y_k(t)=&\frac{k_1+1}{2}\left(\|Pw_k\|_2^2+\left\|P^{\frac{1}{2}}w_k'\right\|_2^2\right)\\&+\frac{1}{2}\left(\left\|P^{\frac{1}{2}}w_k\right\|_2^2+\|w_k'\|_2^2\right)+\frac{k_1}{2}\|Pw_k'\|_2^2\\&-\frac{1}{2}\left(\|Pw_k'\|_2^2+\|Pw_k'\|_2^2\right)-\frac{1}{2}\left(\left\|P^{\frac{1}{2}}w_k'\right\|_2^2+\left\|P^{\frac{1}{2}}w_k'\right\|_2^2\right)\\\geq&\|Pw_k(t)\|_2^2+\left\|P^{\frac{1}{2}}w_k'(t)\right\|_2^2+\|Pw_k'(t)\|_2^2,\ \ t\geq0.\end{aligned} \tag{4.7}$$

Otherwise, we obtain from assumption (A_2),

$$\begin{aligned}\|G_k\|_2^2=&\int_\Omega|g(x,u_k)-g(x,u)|^2dx=\int_\Omega g_u^2w_k^2dx\\\leq&c_0\int_\Omega\left(|u_k|^{2(\alpha-1)}+|u|^{2(\alpha-1)}+h_2^2\right)w_k^2dx.\end{aligned} \tag{4.8}$$

The application of Sobolev imbedding theorem and the estimate (2.7) gives

$$\int_{\Omega}|u_k|^{2(\alpha-1)}w_k^2dx\leq\|w_k\|_{2\mu_2}^2\|u_k\|_{2(\alpha-1)\mu_3}^{2(\alpha-1)}\leq C_3\|w_k\|_{2\mu_2}^2\leq C_3\|w_k\|_2^2 \tag{4.9}$$

with $\mu_2=n/(n-4)^+$ and $\mu_3=\mu_2/(\mu_2-1)$. Similarly,

$$\int_{\Omega}|u|^{2(\alpha-1)}w_k^2dx\leq\|w_k\|_{2\mu_2}^2\|u\|_{2(\alpha-1)\mu_3}^{2(\alpha-1)}\leq C_3\|w_k\|_{2\mu_2}^2\leq C_3\|Pw_k\|_2^2 \tag{4.10}$$

and

$$\int_{\Omega}h_2^2w_k^2dx\leq\|w_k\|_{2\mu_2}^2\|h_2\|_{N/2}^2\leq C_3\|w_k\|_{2\mu_2}^2\leq C_3\|Pw_k\|_2^2. \tag{4.11}$$

Then, we get from (4.5) to (4.11) that $\lambda_4>0$, such that

$$y_k'(t)+\lambda_4y_k(t)\leq C_3\|G_k\|_2^2\leq C_3\|w_k\|_{2\mu_2}^2\leq C_3\|Pw_k\|_{2\mu_2}^2\leq C_3y_k(t) \tag{4.12}$$

where C_3 is as in (2.6), independent of k. The differential inequality (4.12) means

$$y_k(t)\leq y_k(0)e^{(C_3-\lambda_4)t},\ \ t\geq 0. \tag{4.13}$$

Then, from (4.6) and (4.7), we obtain

$$y_k(0)\leq C_0\left(\left\|P^{\frac{1}{2}}(u_{1k}-u_1)\right\|_2^2+\|P(u_{0k}-u_0)\|_2^2+\|P(u_{1k}-u_1)\|_2^2\right)\to 0\text{ as }k\to\infty \tag{4.14}$$

and

$$\|Pw_k(t)\|_2^2+\left\|P^{\frac{1}{2}}w_k'(t)\right\|_2^2+\left\|Pw_k'(t)\right\|_2^2\leq y_k(t)\leq y_k(0)e^{(C_3-\lambda_4)t}\to 0\text{ as }k\to\infty. \tag{4.15}$$

This indicates that $S(t):X\to X$ is continuous. Now we show that $\{S(t)\}_{t\geq 0}$ is asymptotically compact in X from the method in [9].

Assume $\{(u_{0k},u_{1k})\}$ is a bounded sequence and $\{u_k(t)\}$ be the corresponding solutions of the problem (1.1) in $C\left([0,\infty);H^{2m}(\Omega)\cap H_0^m(\Omega)\right)$. We suppose $t_k\to\infty$ as $k\to\infty$. For any $T>0$, assume $t_n,t_k>T$. Then, the application of (4.12) to $w_{kn}(t)=u_n(t+t_n-T)-u_k(t+t_n-T)$, we get

$$Y_{kn}(t)\leq Y_{kn}(0)e^{-\lambda_4t}+C_3\int_0^te^{-\lambda_4(t-s)}\|w_{kn}(s)\|_{2\mu_2}^2ds,\ t\geq 0 \tag{4.16}$$

with

$$Y_{kn}(t)=\|Pw_{kn}(t)\|_2^2+\left\|P^{\frac{1}{2}}w_{kn}'(t)\right\|_2^2+\left\|Pw_{kn}'(t)\right\|_2^2. \tag{4.17}$$

Especially, we take $t=T$ and obtain

$$\begin{aligned}&\|P(u_n(t_n)-u_k(t_k))\|_2^2+\left\|P^{\frac{1}{2}}\left(u_n'(t_n)-u_k'(t_k)\right)\right\|_2^2+\left\|P\left(u_n'(t_n)-u_k'(t_k)\right)\right\|_2^2\\ &\leq Y_{kn}(0)e^{-\lambda_4T}+C_3\sup_{0\leq s\leq T}\|u_k(t_k-T+s)-u_n(t_n-T+s)\|_{2\mu_2}^2.\end{aligned} \tag{4.18}$$

Since the embedding $\left(H^{2m}(\Omega)\cap H_0^m(\Omega)\right)\hookrightarrow L^{2\mu_2}(\Omega)$ is compact, we can remove a subsequence $\left\{u_{k_{k_1}}\left(t_{k_{k_1}}-T+s\right)\right\}$ which converges in $L^{2\mu_2}(\Omega)$. Therefore, for any $\varepsilon>0$, firstly we fix $T>0$, such that

$$Y_{kn}(0)e^{-\lambda_4T}<\frac{\varepsilon}{2}. \tag{4.19}$$

Supposing $n_0 > 0$ and $k_1, j > n_0$, we get

$$C_3 \sup_{0 \le s \le T} \left\| u_{k_{k_1}} \left(t_{k_{k_1}} - T + s \right) - u_{k_j} \left(t_{k_j} - T + s \right) \right\|_{2\mu_2}^2 < \frac{\varepsilon}{2}. \tag{4.20}$$

Then, it follows by (4.18) to (4.20) that $\left\{ u_{k_{k_1}} \left(t_{k_{k_1}} \right) \right\}$ is a Cauchy sequence in X and we finalize that $\{S(t)\}_{t \ge 0}$ is asyptotically compact on X and now Theorem 4 is established.

Proof of Theorem 5

From Lemma 2, it is sufficient to indicate that there exists a continuous operator semigroup $\{S(t)\}$ on X such that $S(t)(u_0, u_1) = (u(t), u_t(t))$ for each $t \ge 0$. By the estimates (2.7), we accomplish that

$$\beta_0 = \left\{ (u, v) \in X \middle| \left\| P^{\frac{1}{2}} v \right\|_2^2 + \|Pu\|_2^2 + \|Pv\|_2^2 \le C_4 \right\} \tag{4.21}$$

is an absorbing set of $\{S(t)\}_{t \ge 0}$ and for any $(u_0, u_1) \in X$,

$$dist_X \left(S(t)(u_0, u_1), \beta_0 \right) \le C_5 e^{-\lambda_3 t}, \; t \ge 0 \tag{4.22}$$

where the constants C_4, C_5 are in (2.7). By Theorem 2, $S(t) : X \to X$ is continuous and asymptotically compact on X. From a general theory (see [1, 11]), we conclude that $S(t)$ admits a global attractor A on X defined by

$$A = \omega(\beta_0) = \bigcap_{\tau \ge 0} \left[\bigcup_{t \ge \tau} S(t) \beta_0 \right]_X \tag{4.23}$$

where $[D]_X$ is the closure of the set D in X. Then we prove the Theorem 5.

5 Decay property of solution for (1)

In this section, we search the decay property of solution to (1.1) with $f \equiv 0$. Firstly, we present a well-known Lemma that will be needed.

Lemma 7. *([18]) Assume $E : [0, \infty) \to [0, \infty)$ is a non-increasing function and suppose that there are constants $q \ge 0$ and $\gamma > 0$ such that*

$$\int_S^\infty E^{q+1}(t)\, dt \le \gamma^{-1} E(0)^q E(s), \; \forall S \ge 0. \tag{5.1}$$

Then, we get

$$E(t) \le E(0) \left(\frac{1+q}{1+q\gamma t} \right)^{1/q} \; \forall t \ge 0 \; if \; q > 0 \tag{5.2}$$

and

$$E(t) \le E(0) e^{1-\gamma t} \; \forall t \ge 0 \; if \; q = 0. \tag{5.3}$$

Proof of Theorem 7

Suppose $u(t)$ is a weak solution in Theorem 3 with $f = 0$. Show

$$E(t) = \frac{1}{2} \left(\|u(t)\|_2^2 + \left\| P^{\frac{1}{2}} u(t) \right\|_2^2 + \left\| P^{\frac{1}{2}} u_t(t) \right\|_2^2 \right) + \int_\Omega G(u(t))\, dx, \; t \ge 0. \tag{5.4}$$

Then, we obtain by (1.1) that

$$E'(t) + \left\| P^{\frac{1}{2}} u_t(t) \right\|^2 = 0, \; \forall t \ge 0. \tag{5.5}$$

This indicates that $E(t)$ is non-increasing in $[0,\infty)$.

Multiplying the equation in (1.1) by $E^q(t)\,u(t)$, $q>0$, we obtain

$$\int_S^T E^q(t)\int_\Omega u\left(u_{tt}-Pu-Pu_{tt}+g(u)\right)dxdt=0,\ \forall T>S\geq 0. \tag{5.6}$$

We note that

$$\begin{aligned}
\int_S^T E^q(t)(u,u_{tt})\,dt &= E^q(t)(u,u_t)\,|_S^T-\int_S^T\left(qE(t)^{q-1}E'(t)(u,u_t)+E^q(t)\|u_t(t)\|_2^2\right)dt;\\
-\int_S^T E^q(t)(u,Pu)\,dt &= \int_S^T E^q(t)\left\|P^{\frac12}u\right\|_2^2\\
-\int_S^T E^q(t)(u,Pu_t)\,dt &= \int_S^T E^q(t)\left(P^{\frac12}u,P^{\frac12}u_t\right)dt
\end{aligned}$$

and

$$\begin{aligned}
-\int_S^T E^q(t)(u,Pu_{tt})\,dt = &-\int_S^T\left(qE(t)^{q-1}E'(t)\left(P^{\frac12}u,P^{\frac12}u_t\right)+E^q(t)\left\|P^{\frac12}u_t(t)\right\|_2^2\right)dt\\
&+E^q(t)\left(P^{\frac12}u,P^{\frac12}u_t\right)|_S^T.
\end{aligned}$$

Then, we get by (5.6) that

$$\begin{aligned}
2\int_S^T E^{q+1}(t)\,dt = &-E^q(t)\left[(u,u_t)+\left(P^{\frac12}u,P^{\frac12}u_t\right)|_S^T\right]\\
&+q\int_S^T E(t)^{q-1}E'(t)\left[(u,u_t)+\left(P^{\frac12}u,P^{\frac12}u_t\right)\right]dt\\
&+2\int_S^T E^q(t)\left(\|u_t(t)\|_2^2+\left\|P^{\frac12}u_t(t)\right\|_2^2\right)dt\\
&+\int_S^T E^q(t)\left(P^{\frac12}u,P^{\frac12}u_t\right)dt+\int_S^T E^q(t)\left(2G(u)-ug(u)\right)dt.
\end{aligned} \tag{5.7}$$

Since $G(u)\geq 0$, $E(t)\geq 0$. Moreover, we get the following estimates from (5.5):

$$\left\|P^{\frac12}u_t(t)\right\|_2\leq\left(-E'(t)\right)^{1/2},\quad \left\|P^{\frac12}u(t)\right\|_2^2\leq 2\left(E(t)\right)^{1/2},\quad \left\|P^{\frac12}u_t(t)\right\|_2\leq 2\left(E(t)\right)^{1/2},\ \forall t\geq 0, \tag{5.8}$$

$$\left|E^q(t)\left((u,u_t)+\left(P^{\frac12}u,P^{\frac12}u_t\right)\right)\right|\leq C_0E^q(t)\left\|P^{\frac12}u\right\|_2\left\|P^{\frac12}u_t(t)\right\|_2\leq C_0E^{q+1}(t), \tag{5.9}$$

$$\begin{aligned}
&\int_S^T\left|E(t)^{q-1}E'(t)\left[(u,u_t)+\left(P^{\frac12}u,P^{\frac12}u_t\right)\right]\right|dt\\
&\leq C_0\int_S^T E(t)^{q-1}\left|E'(t)\right|\left\|P^{\frac12}u\right\|_2\left\|P^{\frac12}u_t\right\|_2dt\leq C_0E^{q+1}(S),
\end{aligned} \tag{5.10}$$

$$2\int_S^T E^q(t)\left(\|u_t(t)\|_2^2+\left\|P^{\frac12}u_t(t)\right\|_2^2\right)dt\leq C_0\int_S^T E^q(t)\left(-E'(t)\right)^{1/2}\leq C_0E^{q+1}(S), \tag{5.11}$$

$$\begin{aligned}
\int_S^T E^q(t)\left(P^{\frac12}u,P^{\frac12}u_t\right)dt &\leq\int_S^T E^q(t)\left\|P^{\frac12}u\right\|_2\left\|P^{\frac12}u_t\right\|_2\\
&\leq\int_S^T E^{q+1}(t)\,dt+C_1E^{q+1}(S).
\end{aligned} \tag{5.12}$$

Then we obtain from (5.8) to (5.12) that

$$\int_S^T E^{q+1}(t)\,dt \le C_0 E^{q+1}(S) \le C_0 E^q(0)\,E(S) \equiv \gamma^{-1} E^q(0)\,E(S)\,. \tag{5.13}$$

From Lemma 10, we get

$$\begin{aligned} E(t) &= \frac{1}{2}\left(\|u(t)\|_2^2 + \left\|P^{\frac{1}{2}}u(t)\right\|_2^2 + \left\|P^{\frac{1}{2}}u_t(t)\right\|_2^2\right) + \int_\Omega G(u(t))\,dx \\ &\le E(0)\left(\frac{1+q}{1+q\gamma t}\right)^{1/q} \le C_1(1+t)^{-1/q}\,. \end{aligned}$$

This is the estimates (2.8) and the proof of Theorem 7 is completed.

Conclusion 8. *In this paper, we obtained the global attractor and the asymptotic behavior of global solution for the higher-order evolution equation with damping term. This improves and extends many results in the literature such as (Xie and Zhong (2007); Chen et al. (2011)).*

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