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An Asymptotic Result for neutral differential equations

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Abstract

We obtain asymptotic result for the solutions of neutral differential equations. Our technique depends on characteristic equations.

Keywords: neutral differential equation; characteristic equation; asymptotic behaviour.

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1 Introduction

Neutral differential equations (NDEs) describe a certain form of delay differential equations. Recently, these equations have received much interest, because of they play important part in the mathematical modeling of natural phenomena. (NDEs) emerge in many fields of engineering and mathematical science. Most of solutions of (NDEs) can not be obtained in closed form. For this reason, researching the qualitative behaviour of solutions is an effective option. Until today, many authors have investigated qualitative behaviour of solutions of (NDEs). Particularly, for more results on the qualitative behaviour of solutions of (NDEs) see [1]-[14] and references therein.

There are several methods to investigate asymptotic behaviour of solutions, such as characteristic equations, fixed point methods, and Lyapunov functionals. While studying (NDEs), each of these methods has its advantages and disadvantages. In the present paper, we use characteristic equations to investigate asymptotic properties of solutions of a (NDE).

In [1], Ardjouni and Djoudi deal with the first order (NDE),

$$x'(t) = - \sum_{j=1}^N b_j(t)x(t - \tau_j(t)) + \sum_{j=1}^N c_j(t)x'(t - \tau_j(t)).$$

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Applying the fixed point methods, they obtained asymptotic results to solutions of above (NDE).

In [8], Dix et al. obtained asymptotic results of solutions to first order linear (NDEs).

Motivated by the results of references therein, we investigate asymptotic properties of solutions to first order (NDE)

$$y'(z) - a(z)y(z) = \sum_{i=1}^m b_i(z)y(z - g_i(z)) + \sum_{j=1}^n c_j(z)y'(z - h_j(z)) = 0, \quad z \geq z_0 \quad (1)$$

with initial condition (IC)

$$y(z) = \theta(z), \quad \text{for } \inf\{z - g_i(z), z - h_j(z) : i = 1, 2, \dots, m, j = 1, 2, \dots, n\}, \quad (2)$$

where $a, b_i, c_j \in C(\mathbb{R}^+, \mathbb{R})$ and $g_i, h_j \in C(\mathbb{R}^+, \mathbb{R}^+)$.

2 Main results

We denote

$$h = \sup\{h_j(z) : j = 1, 2, \dots, n\} \\ l = \sup\{g_i(z), h_j(z) : i = 1, 2, \dots, m, j = 1, 2, \dots, n\}.$$

Let $C([z_0 - l, z_0], \mathbb{R})$ represent the set of continuous real-valued functions on $[z_0 - l, z_0]$.

By the considered (NDE) (1), we combine the following equation:

$$\lambda(z) - a(z) = \sum_{i=1}^m b_i(z) e^{\int_z^{z-g_i(z)} \lambda(t) dt} + \sum_{j=1}^n c_j(z) \lambda(z - h_j(z)) e^{\int_z^{z-h_j(z)} \lambda(t) dt}, \quad \text{for } z \geq z_0 \\ \lambda(z) = \lambda(z_0), \quad \text{for } z \in [z_0 - l, z_0]. \quad (3)$$

Lemma. For each (IC) (2), there exists a solution of (NDE) (1).

Proof. Firstly, we will show that the characteristic equation has a unique solution. Let $l_1 = \{\inf g_i(z), h_j(z), \text{ for } i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n\}$. Let

$$\zeta(z) = e^{\int_{z_0}^z \lambda(t) dt}, \quad \text{for } z \geq z_0 - l$$

and

$$\beta(z) = e^{\int_{z_0}^z \lambda(t) dt}, \quad \text{for } z_0 - l \leq z \leq z_0.$$

So, from the characteristic equation (3), we get

$$\zeta'(z) = \lambda(z)\zeta(z) = a(z)\zeta(z) + \sum_{i=1}^m b_i(z)\beta(z - g_i(z)) + \sum_{j=1}^n c_j(z)\lambda(z - h_j(z))\beta(z - h_j(z))$$

for $z_0 \leq z \leq z_0 + l_1$. Then, we obtain the solution of above equation as follows:

$$\zeta(z) = \{\zeta(z_0) + \int_{z_0}^z (\sum_{i=1}^m b_i(t) e^{\int_{z_0}^{t-g_i(t)} \lambda_{z_0}(s) ds} \\ + \sum_{j=1}^n c_j(t) \lambda_{z_0}(t - h_j(t)) e^{\int_{z_0}^{t-h_j(t)} \lambda_{z_0}(s) ds} - \int_{z_0}^t a(s) ds) e^{\int_{z_0}^t a(s) ds} dt\} e^{\int_{z_0}^z a(s) ds}.$$

From this, we can define $\lambda(z)$ as follows:

$$\lambda(z) = \frac{\zeta'(z)}{\zeta(z)}$$

on $[z_0, z_0 + l_1]$. Now, let $\beta(z) = \zeta(z)$ on $[z_0 - l, z_0 + l_1]$. Then, for $z \in [z_0 + l_1, z_0 + 2l_1]$, we get

$$\zeta'(z) = \lambda(z)\zeta(z) = a(z)\zeta(z) + \sum_{i=1}^m b_i(z)\beta(z - g_i(z)) + \sum_{j=1}^n c_j(z)\lambda(z - h_j(z))\beta(z - h_j(z)).$$

Similarly, from the solution of above equation, we can define $\zeta(z)$ on $[z_0 + l_1, z_0 + 2l_1]$. So, we define $\lambda(z)$ for all $z \geq z_0 - l$.

For existing of solution of (NDE) (1)-(2), we will consider two case:

Case 1. Let, $\theta(z)$ does not have zeros on $[z_0 - l, z_0]$. Let

$$\theta(z) = \theta(z_0)e^{\int_{z_0}^z \lambda_{z_0}(t)dt}.$$

That is

$$\lambda_{z_0}(z) = \frac{\theta'(z)}{\theta(z)}.$$

So, the characteristic equation has solution such that

$$y(z) = \theta(z_0)e^{\int_{z_0}^z \lambda_{z_0}(t)dt}$$

is a solution of (1)-(2) on $[z_0 - l, \infty)$. Especially, equation (1) by constant $\theta(z) = c \neq 0$ has a solution.

Case 2. Let, $\theta(z)$ has zeros on $[z_0 - l, z_0]$. Since θ is continuous on $[z_0 - l, z_0]$, there is a constant $c \neq 0$. So, we can write

$$\omega(z) = \theta(z) + c > 0$$

for $[z_0 - l, z_0]$. Then, equation (1) has a solution u by initial condition ω . Thus,

$$y(z) = u(z) - y_c(z)$$

is a solution of (1)-(2). Here y_c is a solution of (1) with $\theta(z) = c$.

Theorem 2. Suppose that

$$\sup_{z \geq z_0 + l - h} \left\{ \sum_{i=1}^m |b_i(z)| |g_i(z)| e^{\int_z^{z-g_i(z)} \lambda(t)dt} + \sum_{j=1}^n |c_j(z)| |1 - h_j(z)| |\lambda(z - h_j(z))| e^{\int_z^{z-h_j(z)} \lambda(t)dt} \right\} < 1 \quad (4)$$

Then for each solution y of (NDE) (1)-(2), there exists a K (constant), such that

$$y(z)e^{-\int_{z_0}^z \lambda(t)dt} \rightarrow K, \text{ for } z \rightarrow \infty$$

and

$$\left\{ y(z)e^{-\int_{z_0}^z \lambda(t)dt} \right\}' \rightarrow 0, \text{ for } z \rightarrow \infty.$$

Proof. For solutions y of (1)-(2) and λ of (3), we set

$$\gamma(z) = y(z)e^{-\int_{z_0}^z \lambda(t)dt}, z \geq z_0 - l.$$

Applying (1) and (3), we obtain

$$\begin{aligned}\gamma'(z) &= (y'(z) - y(z)\lambda(z))e^{-\int_{z_0}^z \lambda(t)dt} \\ &= \left(\sum_{i=1}^m b_i(z)y(z - g_i(z)) - y(z)\sum_{i=1}^m b_i(z)e^{\int_z^{z-g_i(z)} \lambda(t)dt}\right. \\ &\quad \left.+ \sum_{j=1}^n c_j(z)y'(z - h_j(z)) - y(z)\sum_{j=1}^n c_j(z)\lambda(z - h_j(z))e^{\int_z^{z-h_j(z)} \lambda(t)dt}\right)e^{-\int_{z_0}^z \lambda(t)dt}.\end{aligned}$$

Since $y(z) = \gamma(z)e^{\int_{z_0}^z \lambda(t)dt}$, and $y'(z) = (\gamma'(z) + \gamma(z)\lambda(z))e^{\int_{z_0}^z \lambda(t)dt}$, from above equality, we obtain

$$\begin{aligned}\gamma'(z) &= \sum_{i=1}^m b_i(z)[\gamma(z - g_i(z)) - \gamma(z)]e^{\int_z^{z-g_i(z)} \lambda(t)dt} \\ &\quad + \sum_{j=1}^n c_j(z)[\gamma'(z - h_j(z)) + (\gamma(z - h_j(z)) - \gamma(z))\lambda(z - h_j(z))]e^{\int_z^{z-h_j(z)} \lambda(t)dt},\end{aligned}$$

for $z \geq z_0$. From this, we get

$$\begin{aligned}\gamma'(z) &= \sum_{i=1}^m b_i(z) \int_z^{z-g_i(z)} e^{\int_z^{z-g_i(z)} \lambda(t)dt} \gamma'(s)ds \\ &\quad + \sum_{j=1}^n c_j(z)[\gamma'(z - h_j(z)) + \lambda(z - h_j(z)) \int_z^{z-h_j(z)} \gamma'(s)ds]e^{\int_z^{z-h_j(z)} \lambda(t)dt}\end{aligned}\tag{5}$$

for $z \geq z_0 + l - h$.

If all b_i 's and c_j 's are equal zero on $[z_0 + l - h, \infty)$, from (5), γ is constant and $\gamma' = 0$ on $[z_0 + l - h, \infty)$ which would complete the proof. Thus, we suppose that $b_i \neq 0$ or $c_j \neq 0$ on $[z_0 + l - h, \infty)$. Let

$$\begin{aligned}\sigma &= \sup_{z \geq z_0 + l - h} \left\{ \sum_{i=1}^m |b_i(z)| |g_i(z)| e^{\int_z^{z-g_i(z)} \lambda(t)dt} \right. \\ &\quad \left. + \sum_{j=1}^n |c_j(z)| |1 - h_j(z)| |\lambda(z - h_j(z))| e^{\int_z^{z-h_j(z)} \lambda(t)dt} \right\}.\end{aligned}$$

From (4),

$$0 < \sigma < 1.\tag{6}$$

Let

$$M = \max \{ |\gamma'(z)| : z \in [z_0 - h, z_0 + l - h] \}.$$

We shall prove that

$$|\gamma'(z)| \leq M \text{ for all } z \geq z_0 - h,\tag{7}$$

on $[z_0 - h, \infty)$.

Conversely, suppose that there exist $\varepsilon > 0$ and $z \geq z_0 - h$ such that $|\gamma'(z)| > M + \varepsilon$. Since $|\gamma'(z)| \leq M$ and from the continuity of γ' , there exists $z^* > z_0 + l - h$ such that

$$|\gamma'(z)| < M + \varepsilon, \text{ for } z \in [z_0 - h, z^*)$$

and

$$|\gamma'(z)| = M + \varepsilon,$$

for $z_0 - h \leq z \leq z_0 + l - h$.

From the definition of σ , (5) and (6), we get

$$\begin{aligned} M + \varepsilon &= |\gamma'(z^*)| \\ &\leq \sum_{i=1}^m |b_i(z^*)| \int_{z^*}^{z^*-g_i(z^*)} e^{\int_{z^*}^{z^*-g_i(z^*)} \lambda(t) dt} |\gamma'(s)| ds \\ &\quad + \sum_{j=1}^n |c_j(z^*)| \left[|\gamma'(z^* - h_j(z^*))| + |\lambda(z^* - h_j(z^*))| \int_{z^*}^{z^*-h_j(z^*)} |\gamma'(s)| ds \right] e^{\int_{z^*}^{z^*-h_j(z^*)} \lambda(t) dt} \\ &\leq (M + \varepsilon) \left\{ \sum_{i=1}^m |b_i(z^*)| |g_i(z^*)| e^{\int_{z^*}^{z^*-g_i(z^*)} \lambda(t) dt} \right. \\ &\quad \left. + \sum_{j=1}^n |c_j(z^*)| |1 - h_j(z^*)| |\lambda(z^* - h_j(z^*))| e^{\int_{z^*}^{z^*-h_j(z^*)} \lambda(t) dt} \right\} \\ &\leq (M + \varepsilon) \sigma < M + \varepsilon. \end{aligned}$$

So, we obtain a contradiction. Thus, inequality (7) holds. If $M = 0$, from (7), γ is constant and $\gamma' = 0$ on $[z_0 - h, \infty)$. Thus, we suppose that $M > 0$.

In view of, (5) and (7),

$$\begin{aligned} |\gamma'(z)| &\leq \sum_{i=1}^m |b_i(z)| \int_z^{z-g_i(z)} |\gamma'(s)| ds e^{\int_z^{z-g_i(z)} \lambda(t) dt} \\ &\quad + \sum_{j=1}^n |c_j(z)| \left[|\gamma'(z - h_j(z))| + |\lambda(z - h_j(z))| \int_z^{z-h_j(z)} |\gamma'(s)| ds \right] e^{\int_z^{z-h_j(z)} \lambda(t) dt} \\ &\leq M \left\{ \sum_{i=1}^m |b_i(z)| |g_i(z)| e^{\int_z^{z-g_i(z)} \lambda(t) dt} \right. \\ &\quad \left. + \sum_{j=1}^n |c_j(z)| |1 - h_j(z)| |\lambda(z - h_j(z))| e^{\int_z^{z-h_j(z)} \lambda(t) dt} \right\} \\ &\leq M \sigma, \text{ for } z \geq z_0 + l - h. \end{aligned}$$

Applying above inequality, we can show from induction,

$$|\gamma'(z)| \leq M \sigma^n, \text{ for } z \geq z_0 + nl - h \ (n = 0, 1, \dots). \quad (8)$$

For an arbitrary $z \geq z_0 - h$, we define $n = \frac{z - z_0 + h}{l}$. So, $z \geq z_0 + nl - h$ and $\frac{z - z_0 + h}{l} - 1 < n$. Therefore, from (6) and (8),

$$|\gamma'(z)| \leq M\sigma^n \leq M\sigma^{\frac{z - z_0 + h}{l} - 1}. \quad (9)$$

We get $n \rightarrow \infty$, as $z \rightarrow \infty$, and from (6), $\sigma^n \rightarrow 0$. Thus, from (9),

$$\lim_{z \rightarrow \infty} \gamma'(z) = 0.$$

To prove that $\lim \gamma(z)$ exists, as $z \rightarrow \infty$, we benefit from the Cauchy convergence criterion. For $z > Z \geq z_0 - h$, by (9), we get

$$\begin{aligned} |\gamma(z) - \gamma(Z)| &\leq \int_Z^z |\gamma'(t)| dt = \int_Z^z M\sigma^{\frac{s - z_0 + h}{l} - 1} ds \\ &= M \frac{l}{\ln \sigma} \left[\mu^{\frac{s - z_0 + h}{l}} - 1 \right]_{s=Z}^{s=z} \\ &= M \frac{l}{\ln \sigma} \left[\mu^{\frac{z - z_0 + h}{l} - 1} - \mu^{\frac{Z - z_0 + h}{l} - 1} \right]. \end{aligned}$$

We have $z \rightarrow \infty$, as $Z \rightarrow \infty$, and from (6), the right sides of above inequality tends zero. Thus,

$$\lim |\gamma(z) - \gamma(Z)| = 0$$

which implies that the existence of $\lim \gamma(z)$, as $z \rightarrow \infty$. The proof is completed.

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