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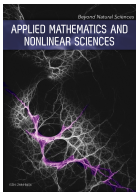

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On Solutions of Fractional order Telegraph Partial Differential Equation by Crank-Nicholson Finite Difference Method

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Abstract

The exact solution is calculated for fractional telegraph partial differential equation depend on initial boundary value problem. Stability estimates are obtained for this equation. Crank-Nicholson difference schemes are constructed for this problem. The stability of difference schemes for this problem is presented. This technique has been applied to deal with fractional telegraph differential equation defined by Caputo fractional derivative for fractional orders $\alpha = 1.1, 1.5, 1.9$. Numerical results confirm the accuracy and effectiveness of the technique.

Keywords: Fractional order Telegraph Partial Differential equations, Finite Difference Method, Stability
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Introduction and Preliminaries

Fractional differential equations have many implementations in finance, engineering, physics and seismology [1–3]. These type equations are solvable with restpect to variables time and space. Some difference schemes are given for the space-fractional heat equations in [4–7, 18–22]. A new difference scheme for time fractional heat equation based on the Crank-Nicholson method has been presented in [5]. Orsingher and Beghin [14] have presented the Fourier transform of the fundamental solutions to time-fractional telegraph equations of order 2α. In [15], the time-fractional advection dispersion equations have been presented. In [16], Liu has studied fractional difference approximations for time-fractional telegraph equation. Modanli and Akgül [12] have worked the second-order partial differential equations by two accurate methods. Finally, Modanli and Akgul [13] have solved the fractional telegraph differential equations by theta-method. For more details see [23-27].

In this study, the Crank-Nicholson difference schemes method has been applied to fractional derivatives to get numerical results.

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Now, we examine the following fractional telegraph equations

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial^{\alpha-1} u(t,x)}{\partial t^{\alpha-1}} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = f(t,x), \\ 0 < x < L,\ 0 < t < T,\ 1 < \alpha < 2, \\ u(0,x) = r_1(x),\ u_t(0,x) = r_2(x),\ 0 \le t \le T, \\ u(t,X_L) = u(t,X_R) = 0,\ X_L < x < X_R. \end{cases} \tag{1}$$

Here, $r_1(x)$, $r_2(x)$ are smooth function defined with the space $[0,T]$, $f(t,x)$ is smooth function defined with the space $(0,L)\times(0,t)$ and $u(t,x)$ is unknown function with the domain $[0,L]\times[0,T]$. For the equation (1), the Crank-Nicholson finite difference scheme method is applied. With using this method, obtained numerical results are very good and efficient for given examples.

Definition 1. The Caputo fractional derivative $D_t^\alpha u(t,x)$ of order α with respect to time is defined as:

$$\begin{aligned} \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} &= D_t^\alpha u(t,x) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{1}{(t-p)^{\alpha-n+1}} \frac{\partial^\alpha u(p,x)}{\partial p^\alpha} dp,\ (n-1 < \alpha < n) \end{aligned} \tag{2}$$

and for $\alpha = n \in N$ defined as:

$$D_t^\alpha u(t,x) = \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{\partial^n u(t,x)}{\partial t^n}.$$

Definition 2. First-order approach difference method for the computation of the problem (1) has been presented as:

$$D_t^\alpha U_n^k \cong g_{\alpha,\tau} \sum_{j=0}^{k-1} b_j^{(\alpha)} (U_n^{k-j} - U_n^{k-j-1}), \tag{3}$$

where $g_{\alpha,\tau} = \frac{\tau^{2-\alpha}}{\Gamma(3-\alpha)}$ and $b_j^{(\alpha)} = (j+1)^{2-\alpha} - j^{2-\alpha}$.

Next section, we shall give Crank-Nicholson difference scheme for fractional order telegraph differential equation.

1 Crank-Nicolson Difference Scheme and its Stabilty

Using the formula (3) and definition of Crank-Nicholson first order difference schemes, we can construct the following difference scheme formula for (1) as:

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+g_{\alpha,\tau}\sum_{j=0}^{k-1}b_j^{(\alpha)}(u_n^{k-j}-u_n^{k-j-1})+\frac{1}{2}(u_n^{k+1}+u_n^k) \\ -\frac{1}{h^2}\left(\left(u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}\right)+\left(u_{n+1}^k-2u_n^k+u_{n-1}^k\right)\right)=f_n^k, \\ f_n^k=f(t_k,x_n),\ 1<\alpha<2, \\ u_n^0=r_1(x_n),\ \frac{u_n^1-u_n^0}{\tau}=r_2((x_n),\ 0\le n\le M, \\ u_0^k=u_M^k=0,\ 0\le k\le N. \end{cases} \tag{4}$$

The formula (4) can be rewriten as:

$$\begin{cases} (-\frac{1}{2h^2})u_{n+1}^k+(-\frac{1}{2h^2})u_{n+1}^{k+1}+(\frac{1}{\tau^2})u_n^{k-1}+(-\frac{2}{\tau^2}+\frac{1}{h^2}+\frac{1}{2})u_n^k \\ +(\frac{1}{\tau^2}+\frac{1}{h^2}+\frac{1}{2})u_n^{k+1}+(-\frac{1}{2h^2})u_{n-1}^k+(-\frac{1}{2h^2})u_{n-1}^{k+1} \\ +\frac{1}{2}(u_n^{k+1}+u_n^k)+g_{\alpha,\tau}\sum_{j=0}^{k}b_j^{(\alpha)}(u_n^{k-j}-u_n^{k-j-1}) \\ -\tau g_{\alpha,\tau}b_k r_2(x_n)=f_n^k, 1<\alpha<2, \\ f_n^k=f(t_k,x_n), \\ u_n^0=r_1(x_n),\ \frac{u_n^1-u_n^0}{\tau}=r_2((x_n),\ 0\le n\le M, \\ u_0^k=u_M^k=0,\ 0\le k\le N. \end{cases} \tag{5}$$

We can write the above system in matrix form as

$$Au_{n+1}+Bu_n+Cu_{n-1}=\varphi_n, \tag{6}$$

where $u_n=[u_n^0,u_n^1,...,u_n^N]$, $\varphi_n=[\varphi_n^0,\varphi_n^1,...,\varphi_n^N]^T$ and $\varphi_n^k=f(t_k,x_n)+\tau g_{\alpha,\tau}b_k r_2(x_n)$. Here $A_{(N+1)\times(N+1)}$ and $B_{(N+1)\times(N+1)}$ are the matrices of the following form,

$$A=-\frac{1}{2h^2}\begin{bmatrix} 0&0&0&...&0&0&0 \\ 0&1&1&...&0&0&0 \\ .&.&.&...&.&.&. \\ .&.&.&...&.&.&. \\ .&.&.&...&.&.&. \\ 0&0&0&...&0&0&1 \\ 0&0&0&...&0&0&0 \end{bmatrix},$$

$$B=\begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ a-b_0g_{\alpha,\tau} & b+g_{\alpha,\tau} & c & ... & 0 \\ -b_1g_{\alpha,\tau} & a+g_{\alpha,\tau}(b_1-b_0) & b+g_{\alpha,\tau} & ... & 0 \\ . & . & . & ... & . \\ . & . & . & ... & . \\ . & . & . & . & . \\ -b_{k-1}g_{\alpha,\tau} & (b_{k-2}-b_{k-3})g_{\alpha,\tau} & (b_{k-3}-b_{k-4})g_{\alpha,\tau} & ... & c \\ -b_kg_{\alpha,\tau} & (b_k-b_{k-1})g_{\alpha,\tau} & (b_{k-1}-b_{k-2})g_{\alpha,\tau} & ... & b+b_0g_{\alpha,\tau} \end{bmatrix},$$

where $a=\frac{1}{\tau^2}$, $b=-\frac{2}{\tau^2}+\frac{1}{h^2}+\frac{1}{2}$ and $c=\frac{1}{\tau^2}+\frac{1}{h^2}+\frac{1}{2}$.

Then we have

$$u_n=\alpha_{n+1}u_{n+1}+\beta_{n+1},\ 1\le k\le M. \tag{7}$$

Next we should determine the matrices α_{n+1} and β_{n+1} above. Using the Dirichlet boundary condititon

$$u(0,t)=u(0,L)=0,\ 0\le t\le T,$$

we obtain $u_0=\alpha_1u_1+\beta_1$. From that, we can choose $\alpha_1=O_{(N+1)\times(N+1)}$ and $\beta_1=O_{N+1)}$. Substitute $u_n=\alpha_{n+1}u_{n+1}+\beta_{n+1}$ and $u_{n-1}=\alpha_nu_n+\beta_n$ into the equation (7), then

$$(A+B\alpha_{n+1}+A\alpha_n\alpha_{n+1})u_{n+1}+(B\beta_{n+1}+A\alpha_n\beta_{n+1}+A\beta n)=\varphi_n.$$

Thus, we get

$$A+B\alpha_{n+1}+A\alpha_n\alpha_{n+1}=0,$$
$$B\beta_{n+1}+A\alpha_n\beta_{n+1}+A\beta n=\varphi_n.$$

Thus, we obtain the following equalities

$$\alpha_{n+1}=-(B+A\alpha_n)^{-1}A,$$
$$\beta_{n+1}=(B+A\alpha_n)^{-1}(\varphi_n-A\beta n),$$

where $1\le n\le M$.

For the stability, implementing the technique of analyzing the eigenvalues of the iteration matrices of the schemes.

Let $\rho(A)$ be the spectral radius of a matrix A, which indicates the maximum of the absolute value of the eigenvalues of the matrix A. We can write the following results.

Theorem 1. *The difference scheme (5) is stable.*

Proof. From the method [15], we should prove that $\rho(\alpha_n)<1$, $1\le n\le M$.

$\rho(\alpha_1)=0<1$ is clearly.

$$\rho(\alpha_2)=\left\|-B^{-1}A\right\|\le\left\|-B^{-1}\right\|\|A\|=\frac{1}{\min\limits_{1\le k\le N-1}\left\{|a_{kk}|-\sum\limits_{\substack{m\ne k,\\ m=1}}^{N-1}|a_{km}|\right\}}\|A\|$$

$$=\frac{\left|-\frac{1}{2h^2}\right|+\left|-\frac{1}{2h^2}\right|}{\left|-\frac{2}{\tau^2}+\frac{1}{h^2}+\frac{1}{2}+\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\right|}=\frac{\frac{1}{h^2}}{-\frac{2}{\tau^2}+\frac{1}{h^2}+\frac{1}{2}+\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}}\le 1,\ \text{for}\ \frac{1}{2}+\frac{1}{h^2}\frac{\tau^{2-\alpha}}{\Gamma(3-\alpha)}-\frac{2}{\tau^2}\ge 0.$$

If $\rho(\alpha_n) < 1$, let us calculate $\rho(\alpha_{n+1})$.

$$A = -\begin{bmatrix} 0 & 0 & 0 \dots 0\,0 & 0 \\ 0 & \frac{\frac{1}{h^2}}{-\frac{2}{\tau^2}+\frac{1}{h^2}+\frac{1}{2}+\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}-\frac{1}{2h^2}\alpha_{n_{2,2}}} & 1 \dots 0\,0 & 0 \\ . & . & . \dots . \; . & . \\ . & . & . \dots . \; . & . \\ . & . & . \dots . \; . & . \\ 0 & 0 & 0 \dots 0\,0 & 0 \\ 0 & 0 & 0 \dots 0\,0 & \frac{\frac{1}{h^2}}{-\frac{2}{\tau^2}+\frac{1}{h^2}+\frac{1}{2}+\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}-\frac{1}{2h^2}\alpha_{n_{N+1,N+1}}} \end{bmatrix}.$$

We know that $\alpha_{ni} = \rho(\alpha_n)$ and $0 \le \rho(\alpha_n) < 1$ for $2 \le i \le N+1$. Then, we can obtain that $\rho(\alpha_{n+1}) < 1$. As a result, we obtain the desired result with induction. □

Remark 2. *Using Matlab programming for* $N = M = 10$, $\alpha = 1.5, 0 \le t \le 1$, $0 \le x \le \pi$ *and* $h = \frac{\pi}{M}$, $tau = \frac{1}{N}$, *we obtain the following spectral radius of a matrix as:*

$\rho(\alpha_1) = 0$, $\rho(\alpha_2) = 0.0482$, $\rho(\alpha_3) = 0.0484$, $\rho(\alpha_4) = 0.0485$, $\rho(\alpha_5) = 0.0482$, $\rho(\alpha_6) = 0.0487$, $\rho(\alpha_7) = 0.0487$, $\rho(\alpha_8) = 0.0475$, $\rho(\alpha_9) = 0.0487$ *and* $\rho(\alpha_{10}) = 0.0486$.

These final results prove the stability estimation of the Theorem 1.

Remark 3. *Applying the method in [16, 17], we can get the convergence of the method from stability and consistency of the proposed method.*

Now, we give numerical applications for the fractional telegraph partial differential equation by Crank-Nicholson method.

2 Numerical implementation

Example . We take into consideration the following fractional telegraph partial differential equation:

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial^{\alpha-1} u(t,x)}{\partial t^{\alpha-1}} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = \sin x(6t + 6\frac{t^{4-\alpha}}{\Gamma(5-\alpha)} + 2(t^3+1)), \\ 0 < x < \pi,\ 0 < t < 1,\ 1 < \alpha < 2, \\ u(0,x) = \sin x,\ u_t(0,x) = 0,\ 0 \le t \le 1, \\ u(t,0) = u(t,\pi) = 0,\ 0 \le x \le \pi. \end{cases} \tag{8}$$

The exact solution is given as $u(t,x) = (t^3+1)\sin x$. We implement difference schemes method to solve the problem. We utilize a procedure of modified Gauss elimination method for difference equation (8). We obtain the maximum norm of the error of the numerical solution by:

$$\varepsilon = \max_{\substack{n=0,1,\dots,M \\ k=0,1,2\dots,N}} |u(t,x) - u(t_k,x_n)|,$$

where $u_n^k = u(t_k,x_n)$ is the approximate solution. The error analysis in Table 1 gives our error analysis for difference schemes method.

Table 1. Error Analysis

$\tau=\frac{1}{N}, h=\frac{pi}{M}$			
The difference scheme (8)			In method [13]
	α		
$N=M=40$	1.5	0.0040	0.0242
$N=M=80$	1.5	5.4707×10^{-4}	0.0118
$N=M=160$	1.5	0.0022	0.0058
$N=100, M=10$	1.1	7.0178×10^{-4}	
	1.5	0.0045	
	1.9	0.0093	
$N=225, M=15$	1.1	3.4496×10^{-4}	
	1.5	0.0040	
	1.9	0.0083	
$N=400, M=20$	1.1	2.0762×10^{-4}	
	1.5	0.0034	
	1.9	0.0079	

We have compared Crank-Nicholson finite difference scheme method by the theta method [13] for the variable values $N=M=40,80,160$. From these comparisons, we see that this method is more effective then the method used in [13].

3 Conclusion

In this work, stability estimates were presented for fractional telegraph differential equations. Stability inequalities were given for the difference schemes method. We applied the difference schemes-method for investigating fractional telegraph partial differential equations. Approximate solutions were obtained by this method. MATLAB software program was utilized for all results.

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