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Analytical and approximate solutions of Fractional Partial Differential-Algebraic Equations

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Abstract

In this paper, we have extended the Fractional Differential Transform method for the numerical solution of the system of fractional partial differential-algebraic equations. The system of partial differential-algebraic equations of fractional order is solved by the Fractional Differential Transform method. The results exhibit that the proposed method is very effective.

Keywords: Fractional Differential Transform Method, Fractional Partial Differential-Algebraic Equations, Caputo fractional derivative, Approximate solution

1 Introduction

In the past several years ago, various methods have been proposed to obtain the numerical solution of partial differential-algebraic equations [2], [7], [11]- [16]. In this study, we consider the following system of partial differential-algebraic equations of fractional order

$$AD_t^{\alpha} v(t,x) + BL_x v(t,x) + Cv(t,x) = f(t,x),$$

Where $\alpha$is a parameter describing the fractional derivative and $t \in (0,t_e), 0 < \alpha \leq 1$ and $x \in (-l,l) \subset R, A, B, C \in R^{n\times n}$, are constant matrices, $u, f : [0,t_e] \times [-l,l] \to R^n$. The purpose of this paper is to consider the numerical solution of FPDAEs by using Fractional Differential Transform Method.

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2 Basic Definitions

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 1. A real function $f(x), x > 0$ is said to be in the space $C_\mu, \mu \varepsilon R$ if there exists a real number $P > \mu$ such that $f(x) = x^p f_1(x)$, where$f_1(x) \varepsilon C[(0,\infty)$. Clearly $C_\mu < C_\beta$ if $\mu < \beta$.

Definition 2. A function $f(x), x > 0$ is said to be in the spaceC_μ^m, $m \varepsilon N \cup \{0\}$ if $f^{(m)} \in C_\mu$.

Definition 3. The Riemann-Liouville fractional integral operator of the order $\alpha > 0$ of a function, $f \in C_\mu, \mu \geq -1$ is defined as:

$$(J_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau) d\tau, x > a, \tag{1}$$

$$(J_a^0 f)(x) = f(x). \tag{2}$$

Properties of the operator J^αcan be found in (Caputo, 1967), we mention only the following:

For$fC_\mu, \mu \geq -1, \alpha, \beta \geq 0$,and $\gamma > -1$.

$$(J_a^\alpha J_a^\beta f)(x) = (J_a^{\alpha+\beta} f)(x), \tag{3}$$

$$(J_a^\alpha J_a^\beta f)(x) = (J_a^\beta J_a^\alpha f)(x) \tag{4}$$

$$J_a^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}. \tag{5}$$

Definition 4. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$(D_a^\alpha f)(x) = (J_a^{m-\alpha} D^m f)(x) = \frac{1}{\Gamma(m-a)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \tag{6}$$

for $m-1 < \alpha < m$, $m \in N$, $x > 0$,.

Lemma 1. *If $-1 < \alpha < m$, $m \in N$ and $\mu \geq -1$, then*

$$(J_a^\alpha D_a^\alpha f)(x) = f(x) - \sum_{k=0}^{m-1} f^k(a)(\frac{(x-a)^k}{k!}), a \geq 0 \tag{7}$$

$$(D_a^\alpha J_a^\alpha f)(x) = f(x) \tag{8}$$

3 fractional Two-Dimensional Differential Transform Method

Differential Transform Method (DTM) is an analytic method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the FDTM obtains a polynomial series solution using an iterative procedure. The proposed method is based on the combination of the classical two-dimensional FDTM and generalized Taylor's Table 1 formula. Consider a function of two variables $u(x,y)$ and suppose that it can be represented as a product of two single-variable functions, that is, $u(x,y) = f(x)g(y)$ based on the properties of fractional two-dimensional differential transform [1], [3]- [6], [8]- [10], the function$u(x,y)$ can be represented as:

$$u(x,y) = \sum_{k=0}^{\infty} F_\alpha(k)(x-x_0)^{k\alpha} \sum_{h=0}^{\infty} G_\beta(h)(y-y_0)^{h\beta} = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta}, \tag{9}$$

Where$0 < \alpha,\beta \leq 1$,$U_{\alpha,\beta}(k,h) = F_{\alpha}(k)G_{\beta}(h)$, is called the spectrum of $u(x,y)$. The fractional two-dimensional differential transform of the function $u(x,y)$ is given by

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}[(D_{x_0}^{\alpha})^k (D_{y_0}^{\beta})^h u(x,y)]_{(x_0,y_0)}. \tag{10}$$

Where$(D_{x_0}^{\alpha})^k = \underbrace{D_{x_0}^{\alpha} \cdot D_{x_0}^{\alpha} \cdots D_{x_0}^{\alpha}}_{k}$

In the case of $\alpha = 1$ and $\beta = 1$ the Fractional two-dimensional differential transform (9) reduces to the classical two-dimensional differential transform. Let $U_{\alpha,\beta}(k,h), w_{\alpha,\beta}(k,h)$ and $V_{\alpha,\beta}(k,h)$ are the differential transformations of the functions $u(x,y), w(x,y)$ and $v(x,y)$, from Equations(9) and (10), some basic properties of the two-dimensional differential transform are introduced in Table 1.

Table 1. The operations for the two-dimensional differential transform method

Transformed function	Original function
$U_{\alpha,\beta} = V_{\alpha,\beta} + W_{\alpha,\beta}$	$u(x,y) = v(x,y)w(x,y)$
$U_{\alpha,\beta} = \lambda V_{\alpha,\beta}$	$u(x,y) = \lambda v(x,y)$
$U_{\alpha,\beta}(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} V_{\alpha,\beta}(r,h-s)W_{\alpha,\beta}(k-r,s)$	$u(x,y) = v(x,y)w(x,y)$
$U_{\alpha,\beta}(k,h) = \delta(k-n)\delta(h-m) = \begin{cases} 1, k = n, h = m \\ 0, k \neq n, h \neq m \end{cases}$	$u(x,y) = (x-x_0)^{m\alpha}(y-y_0)^{n\beta}$
$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} V_{\alpha,\beta}(k+1,h)$	$u(x,y) = D_{x_0}^{\alpha} v(x,y)$
$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\beta(h+1)+1)}{\Gamma(\beta h+1)} V_{\alpha,\beta}(k,h+1)$	$u(x,y) = D_{y_0}^{\beta} v(x,y)$
$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} \frac{\Gamma(\beta(h+1)+1)}{\Gamma(\beta h+1)} \cdot V_{\alpha,\beta}(k+1,h+1), 0 < \alpha,\beta \leq 1$	$u(x,y) = D_{x_0}^{\alpha} D_{y_0}^{\beta} v(x,y)$

Then, the fractional differential transform (10) becomes;

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}[D_{x_0}^{\alpha k}(D_{y_0}^{\beta})^h u(x,y)]_{(x_0,y_0)}, \tag{11}$$

4 Numerical example

Here, the fractional differential transform method will be applied for solving the fractional partial differential-algebraic equation.

Example 2. *Consider the fractional partial differential-algebraic equation*

$$\begin{pmatrix} 0 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} D_t^{\alpha} v + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} v_{xx} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} v = f, \tag{12}$$

$x \in [-0.5, 0.5], t \in [0, 1]$.
With initial condition

$$v_1(x,0) = x^2, v_{1t}(x,0) = -x^2, v_2(x,0) = x^2, v_{2t}(x,0) = \frac{-x^2}{2}, v_3(x,0) = 0, v_{3t}(x,0) = x^2, \tag{13}$$

where

$$f_1(x,t) = \frac{-1}{2}x^2 e^{\frac{-1}{2}t} + x^2\cos(t), f_2(x,t) = 2x^2 e^t \frac{-1}{2}x^2 e^{\frac{-1}{2}t} - x^2\cos(t), f_3(x,t) = -2\sin(t) + x^2\sin(t), \tag{14}$$

with the exact solution

$$v(x,t) = \begin{pmatrix} x^2 e^{-t} \\ x^2 e^{\frac{-t}{2}} \\ x^2\sin(t) \end{pmatrix}, \tag{15}$$

Equivalently, equation (12) *can be written as*

$$\begin{pmatrix} 0 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} D_t^\alpha v_1 \\ D_t^\alpha v_2 \\ D_t^\alpha v_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_{1xx} \\ v_{2xx} \\ v_{3xx} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \tag{16}$$

$$D_t^\alpha v_2 + D_t^\alpha v_3 = f_1, 2D_t^\alpha v_1 - D_t^\alpha v_2 - D_t^\alpha v_3 - v_2 = f_2, -v_{3xx} + v_3 = f_3. \tag{17}$$

By using the DTM in equation (17)*, we obtain*

$$\begin{gathered} V_2(k,h+1)\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} + V_3(k,h+1)\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} = F_1(k,h), \\ 2V_1(k,h+1)\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} - V_2(k,h+1)\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} + V_3(k,h+1)\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} - V_2(k,h) = F_2(k,h), \\ -(k+2)(k+1)V_3(k+2,h) + V_3(k,h) = F_3(k,h), \end{gathered} \tag{18}$$

from the initial condition (13)*, we have*

$$V_1(k,1) = -\delta(k-2) = \{ \begin{matrix} -1 & k=2, \\ 0 & k \neq 2, \end{matrix} V_2(k,1) = -\frac{1}{2}\delta(k-2) = \{ \begin{matrix} -\frac{1}{2} & k=2, \\ 0 & k \neq 2, \end{matrix}$$

$$V_3(k,0) = 0, k = 0,1,\ldots, V_1(k,0) = V_2(k,0) = V_3(k,1) = \delta(k-2) = \{ \begin{matrix} 1 & k=2, \\ 0 & k \neq 2, \end{matrix}. \tag{19}$$

By using the differential inverse reduced transform of $V_1(k,h), V_2(k,h)$ *and* $V_3(k,h)$*, we get*

$$\begin{aligned} v_1(x,t) = [1 &- \frac{1}{\Gamma(\alpha+1)}t^\alpha + \frac{6\Gamma(\alpha+1)+2}{8\Gamma(2\alpha+1)}t^{2\alpha} + \frac{-54\Gamma(2\alpha+1)+12\Gamma(\alpha+1)}{96\Gamma(3\alpha+1)}t^{3\alpha} \\ &+ \frac{50\Gamma(3\alpha+1)-54\Gamma(2\alpha+1)}{192\Gamma(4\alpha+1)}t^{4\alpha} + \frac{-330\Gamma(4\alpha+1)+40\Gamma(3\alpha+1)}{7680\Gamma(5\alpha+1)}t^{5\alpha} \\ &+ \frac{66\Gamma(5\alpha+1)+310\Gamma(4\alpha+1)}{15360\Gamma(6\alpha+1)}t^{6\alpha} + \frac{-1806\Gamma(6\alpha+1)+84\Gamma(5\alpha+1)}{1290240\Gamma(7\alpha+1)}t^{7\alpha} \end{aligned}$$

$$+\frac{770\Gamma(7\alpha+1)-1806\Gamma(6\alpha+1)}{2580480\Gamma(8\alpha+1)}t^{8\alpha}+\frac{-9234\Gamma(8\alpha+1)+144\Gamma(7\alpha+1)}{371589120\Gamma(9\alpha+1)}t^{9\alpha}$$

$$+\frac{1026\Gamma(9\alpha+1)+9198\Gamma(8\alpha+1)}{743178240\Gamma(10\alpha+1)}t^{10\alpha}+\frac{-45078\Gamma(10\alpha+1)+220\Gamma(9\alpha+1)}{163499212800\Gamma(11\alpha+1)}t^{11\alpha}(+\ldots]x^2,$$

$$v_2(x,t)=[1+\frac{1-2\Gamma(\alpha+1)}{2\Gamma(\alpha+1)}t^{\alpha}+\frac{\Gamma(\alpha+1)}{4\Gamma(2\alpha+1)}t^{2\alpha}+\frac{-27\Gamma(2\alpha+1)+8\Gamma(3\alpha+1)}{48\Gamma(3\alpha+1)}t^{3\alpha}$$

$$+\frac{\Gamma(3\alpha+1)}{96\Gamma(4\alpha+1)}t^{4\alpha}+\frac{155\Gamma(4\alpha+1)-32\Gamma(5\alpha+1)}{3840\Gamma(5\alpha+1)}t^{5\alpha}+\frac{\Gamma(5\alpha+1)}{7680\Gamma(6\alpha+1)}t^{6\alpha}$$

$$+\frac{-903\Gamma(6\alpha+1)+128\Gamma(7\alpha+1)}{645120\Gamma(7\alpha+1)}t^{7\alpha}+\frac{\Gamma(7\alpha+1)}{1290240\Gamma(8\alpha+1)}t^{8\alpha}$$

$$+\frac{4599\Gamma(8\alpha+1)-512\Gamma(9\alpha+1)}{185794560\Gamma(9\alpha+1)}t^{9\alpha}+\frac{\Gamma(9\alpha+1)}{371589120\Gamma(10\alpha+1)}t^{10\alpha}$$

$$+\frac{-22539\Gamma(10\alpha+1)+2048\Gamma(11\alpha+1)}{81749606400\Gamma(11\alpha+1)}t^{11\alpha}+\ldots]x^2,$$

$$v_3(x,t)=[t^{\alpha}-\frac{1}{6}t^{3\alpha}+\frac{1}{120}t^{5\alpha}-\frac{1}{5040}t^{7\alpha}+\frac{1}{362880}t^{9\alpha}-\frac{1}{39916800}t^{11\alpha}$$

$$+\frac{1}{6227020800}t^{13\alpha}-\frac{1}{1307674368000}t^{15\alpha}+\frac{1}{355687428096000}t^{17\alpha}$$

$$+\frac{1}{355687428096000}t^{17\alpha}-\frac{1}{121645100408832000}t^{19\alpha}+\ldots]x^2. \tag{20}$$

For special case $\alpha=1$,the solution will be as follows:

$$v_1(x,t)=(1-t+\frac{1}{2}t^2-\frac{1}{6}t^3+\frac{1}{24}t^4-\frac{1}{120}t^5+\frac{1}{720}t^6-\frac{1}{5040}t^7+\frac{1}{40320}t^8$$

$$-\frac{1}{362880}t^9+\frac{1}{3628800}t^{10}-\frac{1}{39916800}t^{11}+\ldots)x^2=x^2e^{-t},$$

$$v_2(x,t)=(1-\frac{1}{2}t+\frac{1}{8}t^2-\frac{1}{48}t^3+\frac{1}{384}t^4-\frac{1}{3840}t^5+\frac{1}{46080}t^6-\frac{1}{645120}t^7$$

$$+\frac{1}{10321920}t^8-\frac{1}{185794560}t^9+\frac{1}{3715891200}t^{10}-\frac{1}{81749606400}t^{11}+\ldots)x^2=x^2e^{\frac{-t}{2}},$$

$$v_3(x,t)=(t-\frac{1}{6}t^3+\frac{1}{120}t^5-\frac{1}{5040}t^7+\frac{1}{362880}t^9-\frac{1}{39916800}t^{11}+\frac{1}{6227020800}t^{13}$$

$$-\frac{1}{1307674368000}t^{15}+\frac{1}{355687428096000}t^{17}-\frac{1}{121645100408832000}t^{19}+\ldots)x^2=x^2sin(t). \tag{21}$$

Which is the exact solution. $v_1(x,t),v_2(x,t)$ and $v_3(x,t)$are calculated for different values of α Numerical comparisons are given in Tables 2, 3, 4. It is obvious that this is a numerical solution in Fig. 1, we plot the numerical solutions given in Eq. (12) for $\alpha=0.5,\alpha=0.75$ and $\alpha=1$.

Table 2: *Numerical solution of* $v_1(x,t)$

x	t	$v_{1FDTM} for \alpha = 0.5$	$v_{1FDTM} for \alpha = 0.75$	$v_{1FDTM} for \alpha = 1$	v_{1Exact}
0.01	0.01	0.00008959719941	0.00009662911415	0.0000990049833	0.00009900498337
0.02	0.02	0.0003431068324	0.0003776380166	0.0003920794694	0.0003920794693
0.03	0.03	0.0007472332426	0.0008326326273	0.0008734009802	0.0008734009802
0.04	0.04	0.001293179864	0.001453053708	0.001537263103	0.001537263103
0.05	0.05	0.001974225181	0.002231408442	0.002378073561	0.002378073561
0.06	0.06	0.002784918905	0.003160962963	0.003390352321	0.003390352321
0.07	0.07	0.003720686720	0.004235572371	0.004568729718	0.004568729718
0.08	0.08	0.004777600641	0.005449572303	0.005907944618	0.005907944617
0.09	0.09	0.005952231651	0.006797703573	0.007402842601	0.007402842601
0.1	0.1	0.007241548560	0.008275056651	0.009048374181	0.009048374180

Table 3: *Numerical solution of* $v_2(x,t)$

x	t	$v_{2FDTM} for \alpha = 0.5$	$v_{2FDTM} for \alpha = 0.75$	$v_{2FDTM} for \alpha = 1$	v_{2Exact}
0.01	0.01	0.0000958378902	0.0000985749886	0.00009950124792	0.0000995012479
0.02	0.02	0.0003768305937	0.0003904880649	0.0003960199335	0.0003960199335
0.03	0.03	0.0008368543404	0.0008711844260	0.0008866007456	0.0008866007456
0.04	0.04	0.001471463303	0.001536815144	0.001568317877	0.001568317877
0.05	0.05	0.001974225181	0.002231408442	0.002378073561	0.002378073561
0.06	0.06	0.003250122011	0.003409338065	0.003493603921	0.003493603921
0.07	0.07	0.004387993961	0.004610074086	0.004731466540	0.004731466540
0.08	0.08	0.005687862525	0.005983322378	0.006149052410	0.006149052411
0.09	0.09	0.007147172212	0.007526404232	0.004443579603	0.004443579603
0.1	0.1	0.008763494580	0.009236753030	0.0095412294245	0.0095412294245

Table 4: *Numerical solution of* $v_3(x,t)$

x	t	$v_{3FDTM} for \alpha = 0.5$	$v_{3FDTM} for \alpha = 0.75$	$v_{3FDTM} for \alpha = 1$	v_{3Exact}
0.01	0.01	0.0000099833416	0.00000316175064	$9.99983333 \cdot 10^{-7}$	$9.999833334 \cdot 10^{-7}$
0.02	0.02	0.0000563801691	0.00002126315673	0.00000799946668	0.00000799946667
0.03	0.03	0.0001551063182	0.00006481973859	0.00002699595018	0.00002699595018
0.04	0.04	0.0003178709294	0.0001429176157	0.00006398293469	0.00006398293470
0.05	0.05	0.0005543701518	0.0002638505172	0.0001249479232	0.0001249479232
0.06	0.06	0.0008730245614	0.0004353631002	0.0002158704233	0.0002158704233
0.07	0.07	0.001281346113	0.0006647804520	0.0003427199520	0.0003427199520
0.08	0.08	0.001786153808	0.0009590878739	0.0005114540415	0.0005114540414
0.09	0.09	0.002393713674	0.001324984549	0.0007280162485	0.0007280162485
0.1	0.1	0.003109835929	0.001768921863	0.0009983341664	0.0009983341665

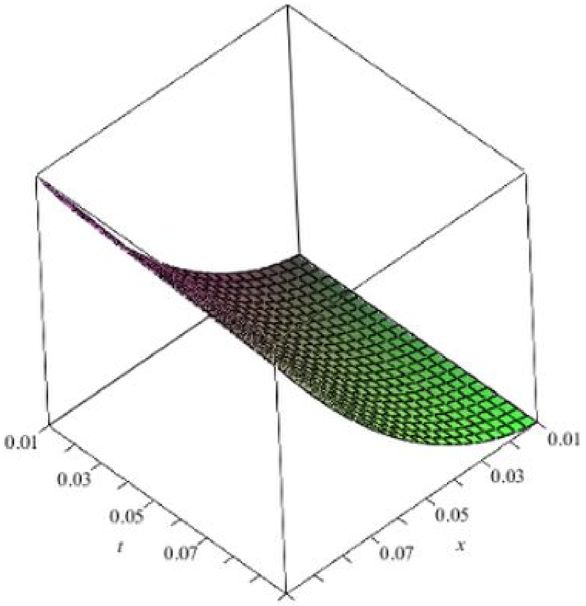

Fig. 1 Exact solution of $v_1(x,t)$.

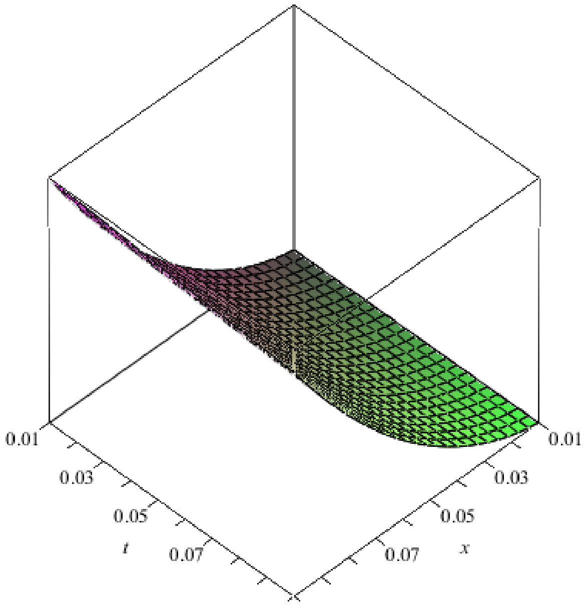

Fig. 2 Values of $v_1(x,t)$ for $\alpha = 1$.

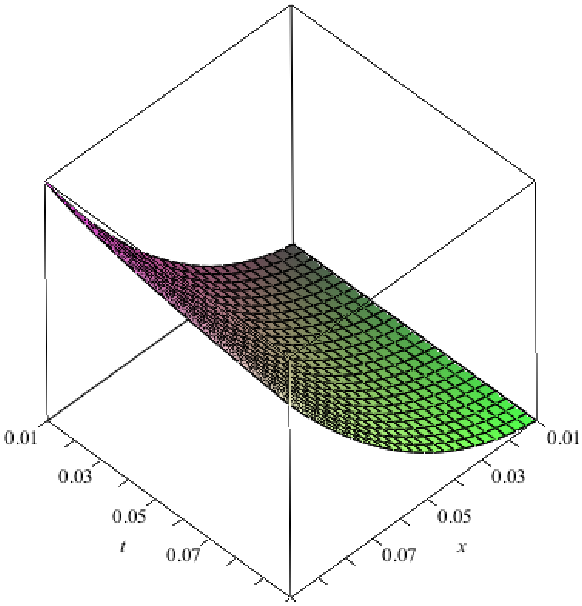

Fig. 3 Values of $v_1(x,t)$ for $\alpha = 0.5$.

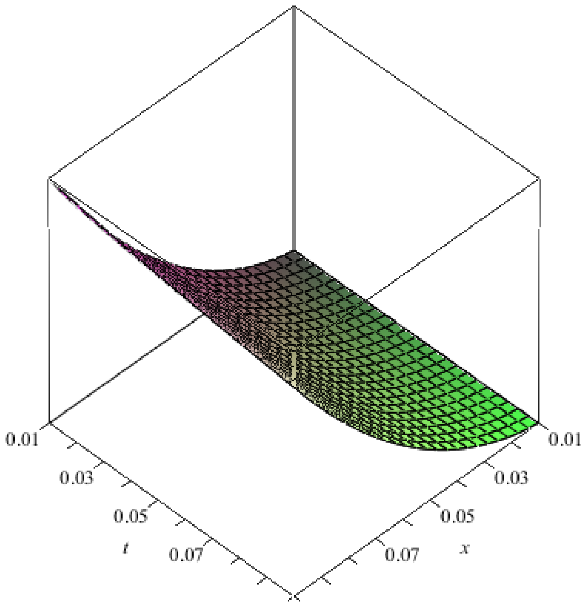

Fig. 4 Values of $v_1(x,t)$ for $\alpha = 0.75$.

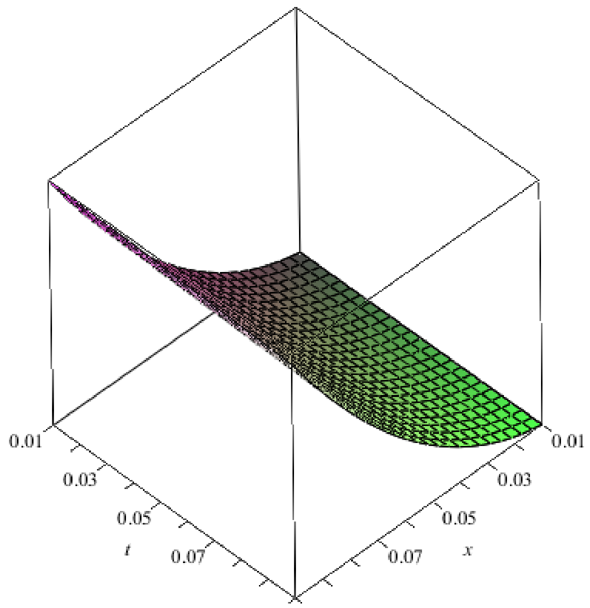

Fig. 5 Exact solution of $v_2(x,t)$.

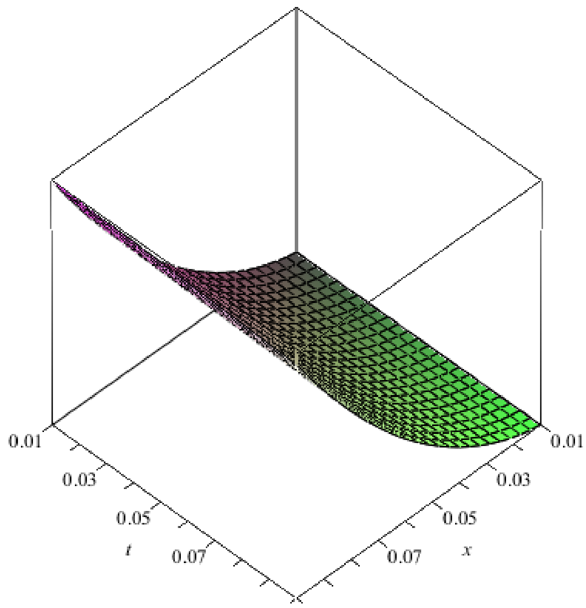

Fig. 6 Values of $v_2(x,t)$ for $\alpha = 1$.

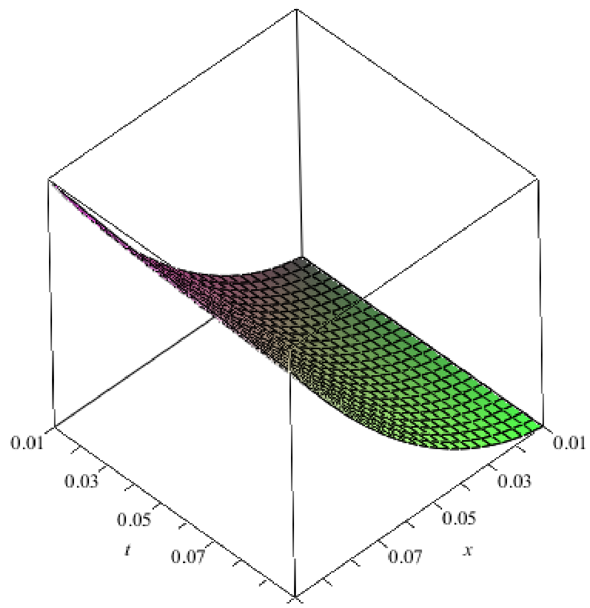

Fig. 7 Values of $v_2(x,t)$ for $\alpha = 0.5$

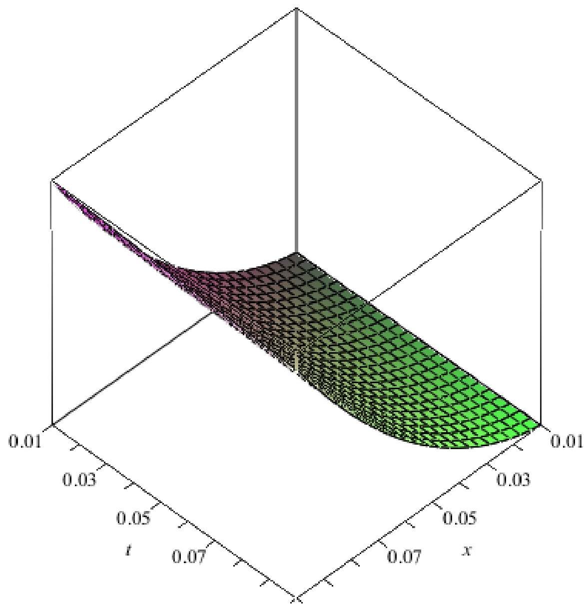

Fig. 8 Values of $v_2(x,t)$ for $\alpha = 0.75$.

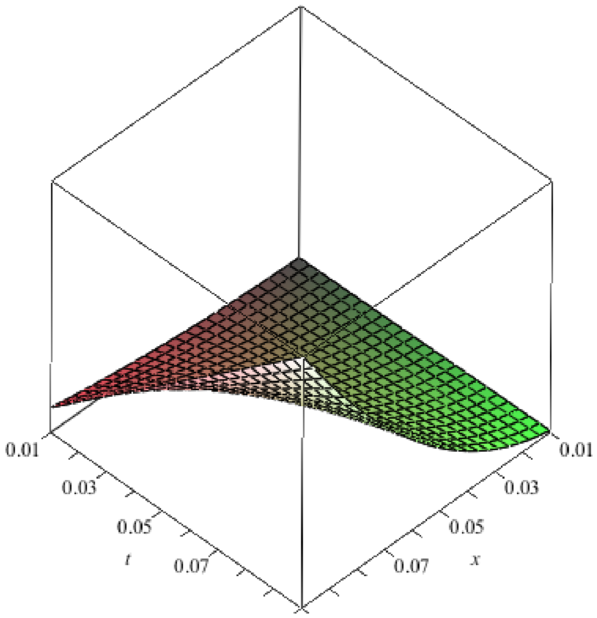

Fig. 9 Exact solution of $v_3(x,t)$.

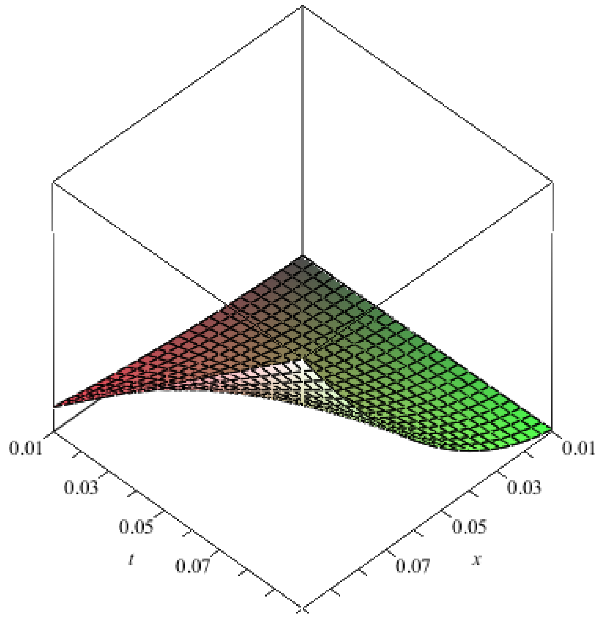

Fig. 10 Values of $v_3(x,t)$ for $\alpha = 1$.

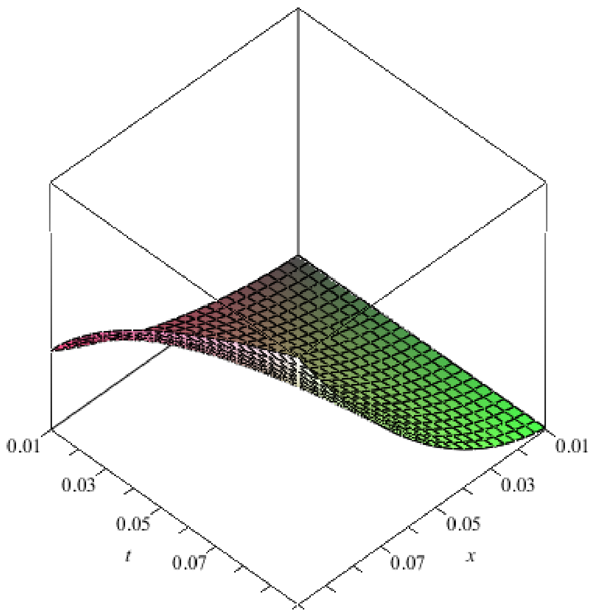

Fig. 11 Values of $v_3(x,t)$ for $\alpha = 0.5$.

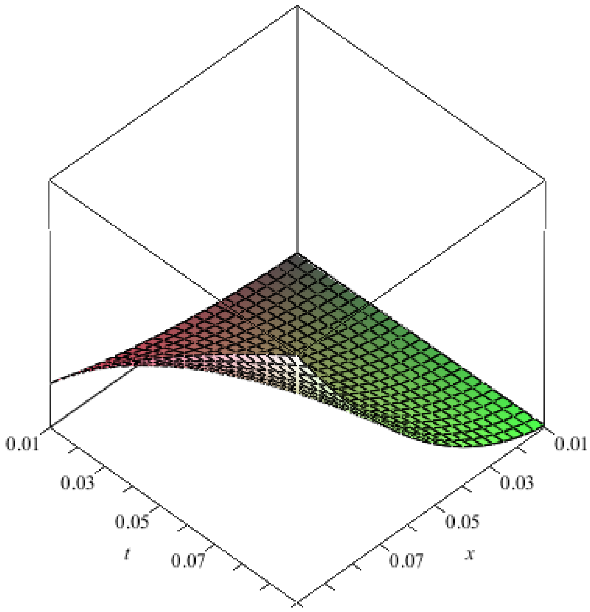

Fig. 12 Values of $v_3(x,t)$ for $\alpha = 0.75$.

5 Conclusions

The generalized differential transformation method displayed in this work is an effective method for the numerical solution of a fractional partial differential-algebraic equation system. With full solutions, approximate solutions collected by the GDTM were compared to shapes and charts. On the other hand, the results are quite reliable for solving this problem. The exact closed-form solution was obtained for all the examples presented in this paper. FDTM offers an excellent opportunity for future research. As a result of this comparison, it is seen that the solutions obtained by the generalized differential transformation method are harmonious with the full solutions.

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