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Two Reliable Methods for The Solution of Fractional Coupled Burgers' Equation Arising as a Model of Polydispersive Sedimentation

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Abstract

In this article, we attain new analytical solution sets for nonlinear time-fractional coupled Burgers' equations which arise in polydispersive sedimentation in shallow water waves using exp-function method. Then we apply a semi-analytical method namely perturbation-iteration algorithm (PIA) to obtain some approximate solutions. These results are compared with obtained exact solutions by tables and surface plots. The fractional derivatives are evaluated in the conformable sense. The findings reveal that both methods are very effective and dependable for solving partial fractional differential equations.

Keywords: Fractional Coupled Burger's Equation, Conformable Fractional Derivative, Perturbation-Iteration Algorithm, Exp-Function Method.
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1 Introduction

Fractional calculus, which includes arbitrary order derivatives and integrals, is the generalized form of the classical calculus. In the last decades, it has been frequently researched by many scientists to model real world problems. Therefore, it offered a decent way of implementation for plenty of models in miscellaneous areas of engineering and physics such as, electrical networks [9], fluid flow [11], image and signal processing [17], mathematical physics [30], viscoelasticity [25], biology [20], control [5] and see references therein [31–44].

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Besides, seeking analytical and approximate solutions of fractional partial differential equations (FPDEs) become more popular. Therefore, achieving the solutions of FPDEs important for these areas and has a distinct place.

Up to now, various powerful numerical techniques have been proposed for solutions of the (FPDEs). Some of them are, Adomian decomposition method (ADM) [21, 23], homotopy perturbation method (HPM) [2, 14], variational iteration method (VIM) [22], Legendre wavelet operational matrix method (LWOMM) [26], homotopy analysis method (HAM) [18, 19] and residual power series method [6, 7].

In this article, we use exp-function method [15] and perturbation-iteration algorithm (PIA) [27–29] to present new analytical and numerical solutions of fractional coupled Burgers' equations given as [24]:

$$\begin{aligned} D_t^{\alpha}u + uD_xu + vD_yu - \frac{1}{\Re}(D_x^2u + D_y^2u) = 0, \\ D_t^{\alpha}v + uD_xv + vD_yv - \frac{1}{\Re}(D_x^2v + D_y^2v) = 0, \end{aligned} \tag{1}$$

The exp-function method is a robust technique for obtaining compacton-like, periodic and solitary solutions of FPDEs. It transforms the given system to an ordinary differential equation and yields to solve it efficiently. In addition, perturbation-iteration algorithm is established by using the perturbation expansion. With choosing proper initial and boundary conditions, it can be performed directly to the model without discretization or any other special conversions.

The methodology in the other sections can be described as follows. Some basic definitions are presented in Section 2. Analysis of the implemented methods are given in Section 3. In Section 4, both methods are used to obtain analytical and approximate solutions of coupled Burgers' equation. Finally, the paper ends with a conclusion in Section 5.

2 Preliminaries

There are different types of arbitrary order differentiation. The most widely used are the Riemann-Liouville(RLFD) and Caputo fractional derivatives (CFD).

Definition 1. *The RLFD operator $D^{\alpha}f(x)$ for $\alpha > 0$ and $q-1 < \alpha < q$ defined as [12, 13]:*

$$D^{\alpha}f(x) = \frac{d^q}{dx^q}\left[\frac{1}{\Gamma(q-\alpha)}\int_{\alpha}^{x}\frac{f(t)}{(x-t)^{\alpha+1-q}}dt\right] \tag{2}$$

Definition 2. *The CFD of order $\alpha > 0$ for $n \in \mathbb{N}$, $n-1 < \alpha < n$, D_*^{α}, defined as [10]:*

$$D_*^{\alpha}f(x) = J^{n-\alpha}D^n f(x) = \frac{1}{\Gamma(n-\alpha)}\int_{\alpha}^{x}(x-t)^{n-\alpha-1}\left(\frac{d}{dt}\right)^n f(t)dt \tag{3}$$

Along with these definitions, a new fractional derivative definition, namely the conformable fractional derivative, has been introduced by Khalil et al. [16].

Definition 3. *The conformable fractional derivative of an $\alpha-th$ order function $f:[0,\infty)\to R$ is defined by*

$$T_{\alpha}(f)(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-(f)(t)}{\varepsilon} \tag{4}$$

where $0 < \alpha \leq 1$ and $t > 0$.

Theorem 1. *Basic properties of conformable derivative of α-differantiable f and g functions for $0 < \alpha \leq 1$ at point $t > 0$ are*

1. $T_\alpha(mf+ng) = mT_\alpha(f)+nT_\alpha(g)$, $m,n \in \mathbb{R}$
2. $T_\alpha(t^p) = pt^{p-\alpha}$ for all p
3. $T_\alpha(f.g) = fT_\alpha(g)+gT_\alpha(f)$
4. $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$
5. Let $f(t) = c$ be a constant function. Then $T_\alpha(c) = 0$.
6. $T_\alpha(f)(t) = t^{1-\alpha}\frac{df(t)}{dt}$, if f is differentiable.

Definition 4. *The conformable partial derivatives of an $\alpha-th$ order f function with $x_1,...,x_n$ variables are [8]*

$$\frac{d^\alpha}{dx_i^\alpha} f(x_1,...,x_n) = \lim_{\varepsilon\to 0}\frac{f(x_1,...,x_{i-1},x_i+\varepsilon x_i^{1-\alpha},...,x_n)-f(x_1,...,x_n)}{\varepsilon}.$$

where $0 < \alpha \leq 1$.

Definition 5. *The conformable integral of an $\alpha-th$ order f function starting from $a \geqslant 0$ is defined by [19]*

$$I_\alpha^a(f)(s) = \int_a^s \frac{f(t)}{t^{1-\alpha}}dt. \tag{5}$$

3 Descriptions of the Implemented Methods

3.1 Exp-Function Method

Taking account into the following nonlinear time fractional equation in order to explain the basic idea of the implemented method [15]

$$F\left(u,\frac{\partial^\alpha u}{\partial t^\alpha},\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2},\dots\right)=0 \tag{6}$$

where the fractional derivatives are in conformable sense. We can introduce the wave variable as

$$u(x,y,t) = u(\eta), \eta = kx+wy+c\frac{t^\alpha}{\alpha} \tag{7}$$

where k, w, c are arbitrary constants that can be examined later. With the help of conformable chain rule [1], we have

$$\frac{\partial^\alpha(.)}{\partial t^\alpha} = c\frac{d(.)}{d\eta},\ \frac{\partial(.)}{\partial x} = k\frac{d(.)}{d\eta},\ \frac{\partial(.)}{\partial y} = w\frac{d(.)}{d\eta},\dots. \tag{8}$$

Hence Eq.(6) changes into differential equation with integer order as follows.

$$Q(u,u_\eta,u_{\eta\eta},u_{\eta\eta\eta},u_{\eta\eta\eta\eta},...) = 0. \tag{9}$$

Due to exp-function method , it is supposed that the wave solution can be regarded in the following form

$$u(\eta) = \frac{\sum_{n=-d}^{j} a_n e^{n\eta}}{\sum_{s=-q}^{p} b_m e^{m\eta}} = \frac{a_j e^{j\eta}+\dots+a_{-d}e^{-d\eta}}{b_p e^{p\eta}+\dots+a_{-q}e^{-q\eta}} \tag{10}$$

where p, q, j and d are positive integers that can be examined later, a_n and b_m are unrecognized constants. To calculate the values of j and p, highest order the linear term of Eq. (9) is equalized with the highest order nonlinear term. By using the same procedure, the values for q and d, can be calculated by balancing the lowest order linear term of Eq. (9) with lowest order nonlinear term. As a result we can acquire the traveling wave solutions of the considered Eq. (6)

3.2 Perturbation-Iteration Algorithm (PIA)

Formerly, a perturbation based algorithm has been introduced by Aksoy and Pakdemirli [3]. In the method, an iterative algorithm is proposed using the perturbation expansion. Previously, this method is implemented on ordinary FDEs [27], fractional-integro differential equations [28] and systems of FDEs [29].

In this article, the most basic PIA, PIA(1,1) is used to attain approximate solutions of FPDEs. For this purpose, one we consider the correction term in the perturbation expansion and correction terms of first derivatives in the Taylor series expansion [3, 4].

To describe the main idea of PIA, take the FPDE

$$F\left(u^{\alpha}(x,t),u(x,t),u_x(x,t),u_{xx}(x,t),\ldots,\varepsilon\right)=0, \tag{11}$$

where ε is assumed as an artificially small parameter. The perturbation expansions with one correction terms are

$$\begin{aligned} u_{n+1} &= u_n+\varepsilon(u_c)_n \\ T_\alpha(u_{n+1}) &= T_\alpha(u_n)+\varepsilon\left(u_c'\right)_n \end{aligned} \tag{12}$$

Subrogating (12) into (11) and expanding in the Taylor series form for only first order derivatives yields

$$\begin{aligned} F\left(u_n^{(\alpha)},u_n,(u_n)_x,(u_n)_{xx},\ldots,0\right)&+F_u\left(u_n^{(\alpha)},u_n,(u_n)_x,(u_n)_{xx},\ldots,0\right)\varepsilon(u_c)_n \\ &+F_{u^{(\alpha)}}\left(u_n^{(\alpha)},u_n,(u_n)_x,(u_n)_{xx},\ldots,0\right)\varepsilon\left(u_c^{(\alpha)}\right)_n \\ &+F_{u_x}\left(u_n^{(\alpha)},u_n,(u_n)_x,(u_n)_{xx},\ldots,0\right)\varepsilon((u_c)_x)_n \\ &+F_{u_{xx}}\left(u_n^{(\alpha)},u_n,(u_n)_x,(u_n)_{xx},\ldots,0\right)\varepsilon((u_c)_{xx})_n+\cdots \\ &+F_\varepsilon\left(u_n^{(\alpha)},u_n,(u_n)_x,(u_n)_{xx},\ldots,0\right)\varepsilon=0 \end{aligned} \tag{13}$$

or

$$\left(u_c^{(\alpha)}\right)_n\frac{\partial F}{\partial u^{(\alpha)}}+(u_c)_n\frac{\partial F}{\partial u}+((u_c)_x)_n\frac{\partial F}{\partial u_x}+((u_c)_{xx})_n\frac{\partial F}{\partial u_{xx}}+\cdots+\frac{\partial F}{\partial \varepsilon}+\frac{F}{\varepsilon}=0. \tag{14}$$

Rewriting (14) gives the subsequent PIA(1,1) iteration formula

$$u_t(x,t)+\frac{F_u}{F_{u_t}}u(x,t)=-\frac{F_\varepsilon+\frac{F}{\varepsilon}}{F_{u_t}}. \tag{15}$$

In this expansion, all of the derivatives are evaluated at $\varepsilon=0$. Using an initial function $u_0(x,t)$, firstly the correction term $(u_c)_0(x,t)$ is computed. Subrogating it into (12) gives the first approximate result $u_1(x,t)$. Similar procedure is applied until obtaining the other approximations.

4 Application of the Methods for $(2+1)$-dimensional Time-Fractional Coupled Burgers' Equation

4.1 Analytical Solution of Coupled Burgers' Equation

Think of the fractional coupled Burgers' equation [24] as

$$\begin{aligned} D_t^\alpha u+uD_xu+vD_yu-\frac{1}{\Re}(D_x^2u+D_y^2u)&=0, \\ D_t^\alpha v+uD_xv+vD_yv-\frac{1}{\Re}(D_x^2v+D_y^2v)&=0, \end{aligned} \tag{16}$$

where $0 < \alpha \leq 1$, $\Re$, is Reynolds number, $u = u(x,y,t)$ and $v = v(x,y,t)$. By the help of the chain rule [1] and the wave transform $\eta = kx + wy + c\frac{t^{\alpha}}{\alpha}$, we obtain

$$\begin{aligned} &cU'(\eta) + ku(\eta)U'(\eta) + wV(\eta)U'(\eta) - \frac{1}{\Re}(k^2+w^2)U''(\eta) = 0 \\ &cV'(\eta) + kuV'(\eta) + wvV'(\eta) - \frac{1}{\Re}(k^2+w^2)V''(\eta) = 0. \end{aligned} \tag{17}$$

Now assume that the solution of (17) can be described as

$$U(\eta) = \frac{a_c e^{c\eta} + \ldots + a_{-d}e^{-d\eta}}{b_p e^{p\eta} + \ldots + b_{-q}e^{-q\eta}}. \tag{18}$$

$$V(\eta) = \frac{d_s e^{s\eta} + \ldots + d_{-n}e^{-n\eta}}{f_l e^{l\eta} + \ldots + f_{-r}e^{-r\eta}}. \tag{19}$$

Using (18),(19) and (17) led to $c = p$, $d = q$, $s = l$ and $n = r$. For convenience lets assume all the coefficients $c = p = s = l = n = r = 1$. Now rewriting $u(\eta)$ and $v(\eta)$ due to above assumptions

$$\begin{aligned} U &= \frac{a_1 e^{\eta} + a_0 + a_{-1}e^{-\eta}}{b_1 e^{\eta} + b_0 + b_{-1}e^{-\eta}} \\ V &= \frac{d_1 e^{\eta} + d_0 + d_{-1}e^{-\eta}}{f_1 e^{\eta} + f_0 + f_{-1}e^{-\eta}}. \end{aligned} \tag{20}$$

Substituting the equations (20) into (17) and equalizing the coefficients of $e^{n\eta}$ yields a system of algebraic equations. Solving the system with respect to the constants expressed above we can handle the following solutions

$$\begin{aligned} c &= -\frac{\Re w d_0 b_0 + k^2 f_0 b_0 + w^2 f_0 b_0 + k a_0 f_0}{\Re b_0 f_0}, \\ a_{-1} &= \frac{f_{-1}\Re w d_0 b_0 + 2k^2 b_0 f_{-1} f_0 + 2\ w^2 b_0 f_{-1} f_0 + \Re k a_0 f_{-1} f_0 - \Re b_0 w d_{-1} f_0}{\Re k {f_0}^2}, \\ a_1 &= 0, b_{-1} = \frac{b_0 f_{-1}}{f_0}, \\ a_1 &= 0, b_1 = 0, d_1 = 0, f_1 = 0. \end{aligned} \tag{21}$$

So the solutions can be obtained as

$$u(x,y,t) = \frac{a_0 + Ae^B b_0 + b_0 f_{-1} e^B {f_0}^{-1}}{\Re k {f_0}^2} \tag{22}$$

and

$$v(x,y,t) = \frac{d_0 + d_{-1}e^B}{f_0 + f_{-1}e^B} \tag{23}$$

where

$$\begin{aligned} A &= \left(f_{-1}\Re w d_0 b_0 + 2k^2 b_0 f_{-1} f_0 + 2w^2 b_0 f_{-1} f_0 + \Re k a_0 f_{-1} f_0 - \Re b_0 w d_{-1} f_0\right) \\ B &= -kx - wy + \frac{\left(\Re w d_0 b_0 + k^2 f_0 b_0 + w^2 f_0 b_0 + \Re k a_0 f_0\right) t^{\alpha}}{\Re b_0 f_0 \alpha} \end{aligned} \tag{24}$$

4.2 Approximate Solution of Coupled Burgers' Equations

Regard the system (16) with the the conditions $u(x,y,0) = \frac{1+\frac{1}{4\Re}(\Re+8)e^{-x-y}}{1+\frac{1}{2}e^{-x-y}}$ and $v(x,y,0) = \frac{1+e^{-x-y}}{2+e^{-x-y}}$. For the values $k=1, w=1, f_0=2, d_0=1, b_0=1, d_{-1}=1, b_{-1}=1, f_{-1}=1$ and $a_0=1$, we can acquire the exact solutions as

$$u(x,y,t) = \frac{1+\frac{1}{4}(\Re+8)\,e^{-x-y+\frac{(3\Re+4)t^{\alpha}}{2\Re\alpha}}}{1+\frac{1}{2}\,e^{-x-y+\frac{(3\Re+4)t^{\alpha}}{2\Re\alpha}}},$$
$$v(x,y,t) = \frac{1+e^{-x-y+\frac{(3\Re+4)t^{\alpha}}{2\Re\alpha}}}{2+e^{-x-y+\frac{(3\Re+4)t^{\alpha}}{2\Re\alpha}}}. \tag{25}$$

Now we introduce a small perturbation parameter ε to the system and rewrite the equations as

$$D_t^{\alpha}u+\varepsilon uD_xu+\varepsilon vD_yu-\frac{1}{\Re}(D_x^2u+D_y^2u)=0,$$
$$D_t^{\alpha}v+\varepsilon uD_xv+\varepsilon vD_yv-\frac{1}{\Re}(D_x^2v+D_y^2v)=0, \tag{26}$$

Therefore, terms in formula (15)turn into

$$F=t^{1-\alpha}(u_n)_t(x,y,t)-\frac{1}{\Re}\left((u_n)_{xx}(x,y,t)+(u_n)_{yy}(x,y,t)\right),\ F_u=0,$$
$$F_{u_t}=t^{1-\alpha},\ F_{\varepsilon}=u_n(x,y,t)\,(u_n)_x\,(x,y,t)+v_n(x,y,t)(u_n)_y(x,y,t) \tag{27}$$

and

$$F=t^{1-\alpha}(v_n)_t(x,y,t)-\frac{1}{\Re}\left((v_n)_{xx}(x,y,t)+(v_n)_{yy}(x,y,t)\right),\ F_v=0,$$
$$F_{v_t}=t^{1-\alpha},\ F_{\varepsilon}=u_n(x,t)(v_n)_x(x,t)+v_n(x,t)(v_n)_y(x,t) \tag{28}$$

Subrogating above terms in the iteration formula (15) gives the subsequent partial differential equations

$$\varepsilon\Re\mathrm{t}\,(u_c)_t=-\Re t\,(u_n)_t+t^{\alpha}((u_n)_{xx}-\varepsilon\Re(u_n(u_n)_x+v_n(u_n)_y)+(u_n)_{yy}) \tag{29}$$

and

$$\varepsilon\Re\mathrm{t}\,(v_c)_t=-\Re t\,(v_n)_t+t^{\alpha}((v_n)_{xx}-\varepsilon\Re(u_n(v_n)_x+v_n(v_n)_y)+(v_n)_{yy}) \tag{30}$$

Beginning with the initial functions

$$u(x,y,0)=\frac{1+\frac{1}{4\Re}(\Re+8)\,e^{-x-y}}{1+\frac{1}{2}e^{-x-y}} \text{ and } v(x,y,0)=\frac{1+e^{-x-y}}{2+e^{-x-y}} \tag{31}$$

and using (15), the numerical results are obtained for $n=0,1,2,\dots$ respectively.

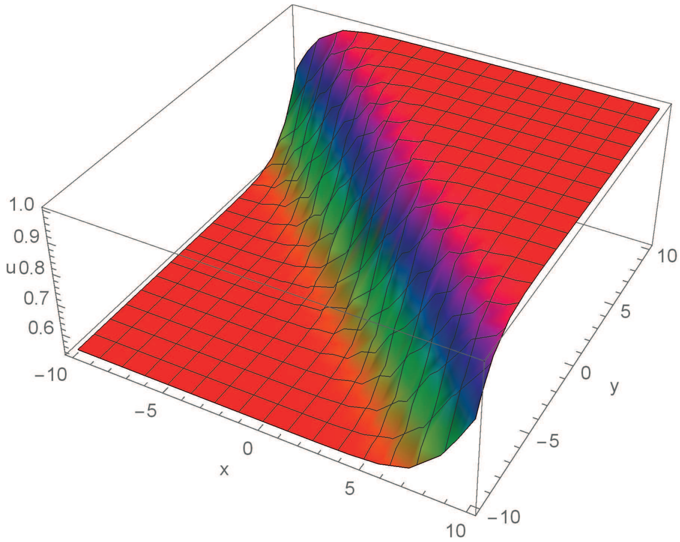

Fig. 1 The surface plot of the PIA solution $u_4(x,y,t)$ for $\Re = 100$ and $t = 0.1$ when $\alpha = 0.75$.

$$u_1(x,y,t) = \frac{4\Re e^{x+y} + \Re + 8}{2\Re\left(2e^{x+y}+1\right)} - \frac{(\Re-8)(3\Re+4)t^{\alpha}e^{x+y}}{2\alpha\Re^2\left(2e^{x+y}+1\right)^2} \tag{32}$$

$$v_1(x,y,t) = \frac{(3\Re+4)t^{\alpha}e^{x+y}}{2\alpha\Re\left(2e^{x+y}+1\right)^2} + \frac{e^{x+y}+1}{2e^{x+y}+1} \tag{33}$$

$$u_2(x,y,t) = -\frac{(\Re-8)(3\Re+4)^2t^{2\alpha}e^{x+y}\left(2e^{x+y}-1\right)\left(3\alpha\Re\left(2e^{x+y}+1\right)^2+16t^{\alpha}e^{x+y}\right)}{24\alpha^3\Re^4\left(2e^{x+y}+1\right)^5} - \frac{(\Re-8)(3\Re+4)t^{\alpha}e^{x+y}}{2\alpha\left(2\Re e^{x+y}+\Re\right)^2} + \frac{8-\Re}{2\Re\left(2e^{x+y}+1\right)} + 1 \tag{34}$$

$$v_2(x,y,t) = \frac{2(3\Re+4)^2t^{3\alpha}e^{2x+2y}\left(2e^{x+y}-1\right)}{3\alpha^3\Re^3\left(2e^{x+y}+1\right)^5} + \frac{(3\Re+4)^2t^{2\alpha}e^{x+y}\left(2e^{x+y}-1\right)}{8\alpha^2\Re^2\left(2e^{x+y}+1\right)^3} + \frac{(3\Re+4)t^{\alpha}e^{x+y}}{2\alpha\Re\left(2e^{x+y}+1\right)^2} + \frac{e^{x+y}+1}{2e^{x+y}+1} \tag{35}$$

Similarly, the fourth order solutions $u_4(x,y,t)$ and $v_4(x,y,t)$ are calculated. In Table 1 and 2, the fourth order PIA numerical approximate solutions are compared to exact solutions. Also the absolute errors are calculated for changing values of α and x. The results indicate the reliability of PIA. Besides using Figures $1-6$, figures for the solutions of PIA are illustrated for changing α values. They exhibit that PIA produces highly approximate results. It is also obvious that further iterations would generate convenient solutions.

5 Conclusion

In this study, initially exp-function method is employed to acquire a new exact solution set for fractional coupled Burgers' system of equations comes with polydispersive sedimentation. Then using PIA, some approximate solutions of the system are presented. It is observed that the exp-function method appears to be a robust and adequate tool for handling of FPDEs. Besides, comparison of the approximate solutions obtained by PIA for $\alpha = 0.75$, $\alpha = 0.85$ and $\alpha = 0.95$ reveals the power and fast convergence rate of the method even after a few approximations. The main advantage of the method is it does not require any special assumptions or transfor-

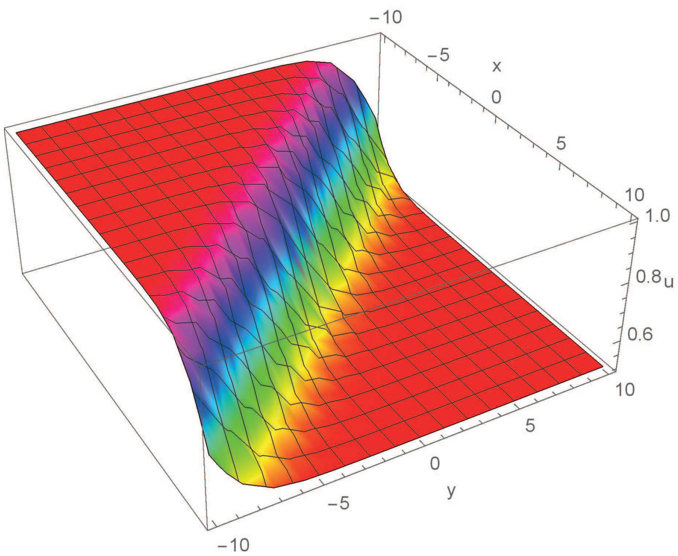

Fig. 2 The surface plot of the PIA solution $v_4(x,y,t)$ for $\Re = 100$ and $t = 0.1$ when $\alpha = 0.75$.

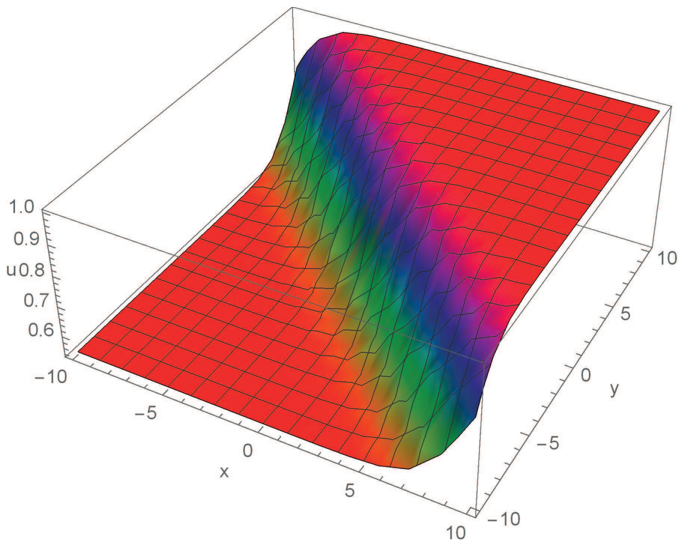

Fig. 3 The surface plot of the PIA solution $u_4(x,y,t)$ for $\Re = 100$ and $t = 0.1$ when $\alpha = 0.85$.

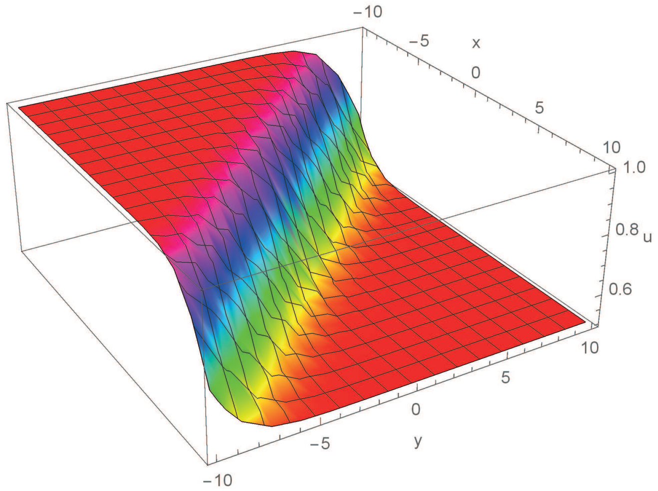

Fig. 4 The surface plot of the PIA solution $v_4(x,y,t)$ for $\Re = 100$ and $t = 0.1$ when $\alpha = 0.85$.

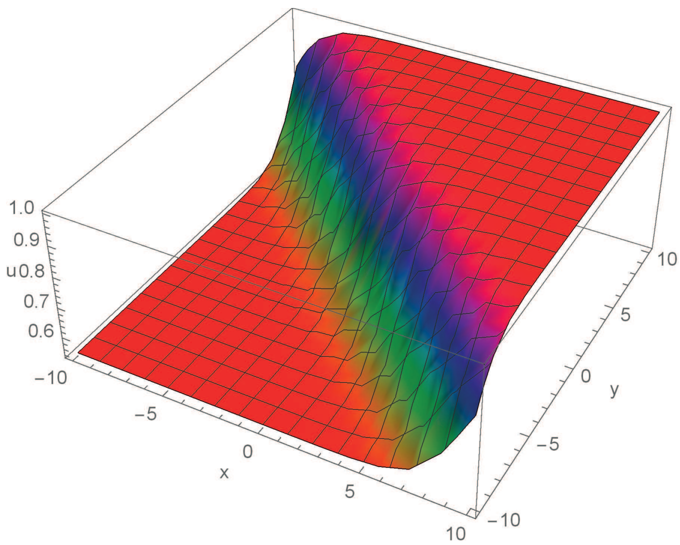

Fig. 5 The surface plot of the PIA solution $u_4(x,y,t)$ for $\Re = 100$ and $t = 0.1$ when $\alpha = 0.95$.

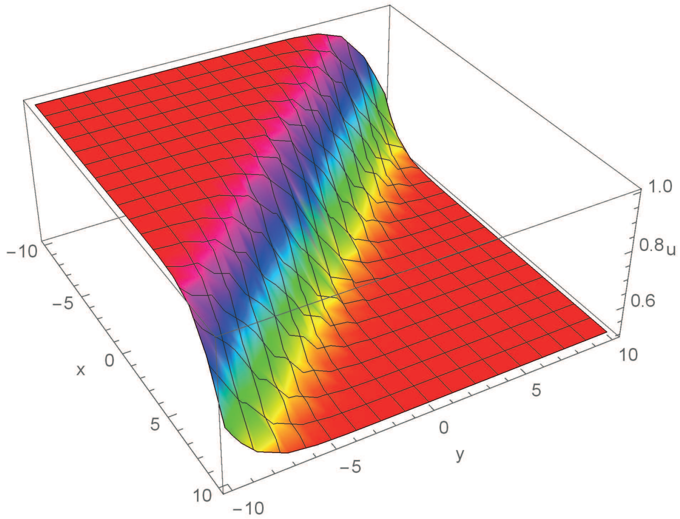

Fig. 6 The surface plot of the PIA solution $v_4(x,y,t)$ for $\Re = 100$ and $t = 0.1$ when $\alpha = 0.95$.

Table 1 PIA ($u_4(x,y,t)$) and exact solution values with absolute errors for $y = 1$, $t = 0.1$ and $\Re = 100$.

	$\alpha = 0.75$			$\alpha = 0.85$			$\alpha = 0.95$		
x	*PIA*	*Exact*	*Error*	*PIA*	*Exact*	*Error*	*PIA*	*Exact*	*Error*
0.0	0.903993	0.903996	2.72276E-6	0.911928	0.911928	4.58229E-7	0.917015	0.917015	8.27357E-8
0.1	0.911369	0.911371	2.60627E-6	0.91883	0.918831	4.36394E-7	0.923600	0.923600	7.85330E-8
0.2	0.918305	0.918308	2.43193E-6	0.925300	0.925301	4.05356E-7	0.929761	0.929761	7.27317E-8
0.3	0.924809	0.924811	2.21778E-6	0.931348	0.931348	3.68115E-7	0.935508	0.935508	6.58693E-8
0.4	0.930889	0.930891	1.97981E-6	0.936986	0.936986	3.27310E-7	0.940856	0.940856	5.84150E-8
0.5	0.936559	0.936561	1.73155E-6	0.94223	0.942230	2.85150E-7	0.945821	0.945821	5.07597E-8
0.6	0.941833	0.9411865	1.48390E-6	0.947095	0.947095	2.43396E-7	0.950421	0.950421	4.32130E-8
0.7	0.946727	0.946729	1.24520E-6	0.951600	0.951600	2.03383E-7	0.954674	0.954674	3.60076E-8
0.8	0.951260	0.951261	1.02143E-6	0.955763	0.955763	1.66055E-7	0.958600	0.958600	2.93064E-8
0.9	0.955449	0.95545	8.16543E-7	0.959603	0.959603	1.32019E-7	0.962216	0.962216	2.32127E-8
1.0	0.959314	0.959314	6.32769E-7	0.963139	0.963139	1.01605E-7	0.965542	0.965542	1.77807E-8

Table 2 PIA ($v_4(x,y,t)$) and exact solution values with absolute errors for $y = 1$, $t = 0.1$ and $\Re = 100$.

	$\alpha = 0.75$			$\alpha = 0.85$			$\alpha = 0.95$		
x	*PIA*	*Exact*	*Error*	*PIA*	*Exact*	*Error*	*PIA*	*Exact*	*Error*
0.0	0.604355	0.604352	2.95952E-6	0.595730	0.595730	4.98075E-7	0.590201	0.590201	8.99301E-8
0.1	0.596338	0.596335	2.83291E-6	0.588228	0.588228	4.74342E-7	0.583043	0.583043	8.53620E-8
0.2	0.588799	0.588796	2.64341E-6	0.581195	0.581195	4.40604E-7	0.576347	0.576347	7.90562E-8
0.3	0.581729	0.581727	2.41063E-6	0.574622	0.574622	4.00125E-7	0.570100	0.570100	7.15971E-8
0.4	0.575120	0.575118	2.15196E-6	0.568493	0.568493	3.55772E-7	0.564287	0.564287	6.34946E-8
0.5	0.568957	0.568956	1.88212E-6	0.562794	0.562794	3.09945E-7	0.558890	0.558890	5.51736E-8
0.6	0.563225	0.563223	1.61293E-6	0.557506	0.557506	2.64561E-7	0.553890	0.553890	4.69707E-8
0.7	0.557905	0.557904	1.35347E-6	0.552609	0.552609	2.21069E-7	0.549267	0.549267	3.91387E-8
0.8	0.552978	0.552977	1.11025E-6	0.548084	0.548084	1.80495E-7	0.545000	0.545000	3.18548E-8
0.9	0.548425	0.548424	8.87547E-7	0.543910	0.543910	1.43499E-7	0.541070	0.541070	2.52312E-8
1.0	0.544224	0.544223	6.87793E-7	0.540066	0.540066	1.10441E-7	0.537454	0.537454	1.93268E-8

mations. Thus it is obvious that both methods are powerful tools for solution of FPDEs and they are ready to be applied to different types of FPDEs arising in different research areas.

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