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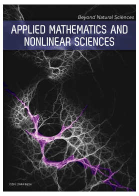

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On graphs with equal dominating and c-dominating energy

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Abstract

Graph energy and domination in graphs are most studied areas of graph theory. In this paper we try to connect these two areas of graph theory by introducing c-dominating energy of a graph G. First, we show the chemical applications of c-dominating energy with the help of well known statistical tools. Next, we obtain mathematical properties of c-dominating energy. Finally, we characterize trees, unicyclic graphs, cubic and block graphs with equal dominating and c-dominating energy.

Keywords: Dominating set, Connected dominating set, Energy, Dominating energy, c-Dominating energy.
AMS 2010 codes: 05C69; 05C90; 05C35; 05C12.

1 Introduction

All graphs considered in this paper are finite, simple and undirected. In particular, these graphs do not have loops. Let $G = (V,E)$ be a graph with the vertex set $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$ and the edge set $E(G) = \{e_1, e_2, e_3, \cdots, e_m\}$, that is $|V(G)| = n$ and $|E(G)| = m$. The vertex u and v are adjacent if $uv \in E(G)$. The open(closed) neighborhood of a vertex $v \in V(G)$ is $N(v) = \{u : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$ respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ and is defined as $d_G(v) = |N(v)|$. A vertex $v \in V(G)$ is pendant if $|N(v)| = 1$ and is called supporting vertex if it is adjacent to pendant vertex. Any vertex $v \in V(G)$ with $|N(v)| > 1$ is called internal vertex. If $d_G(v) = r$ for every vertex $v \in V(G)$, where $r \in \mathbb{Z}^+$ then G is called r-regular. If $r = 2$ then it is called cycle graph C_n and for $r = 3$ it is called the cubic graph. A graph G is unicyclic if $|V| = |E|$. A graph G is called a block graph, if every block in G is a complete graph. For undefined terminologies we refer the reader to [16].

A subset $D \subseteq V(G)$ is called dominating set if $N[D] = V(G)$. The minimum cardinality of such a set D is called the domination number $\gamma(G)$ of G. A dominating set D is connected if the subgraph induced by D is connected. The minimum cardinality of connected dominating set D is called the connected dominating number $\gamma_c(G)$ of G [27].

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The energy $E(G)$ of a graph G is equal to the sum of the absolute values of the eigenvalues of the adjacency matrix of G. This quantity, introduced almost 30 years ago [13] and having a clear connection to chemical problems [15], has in newer times attracted much attention of mathematicians and mathematical chemists [3, 8–12, 20, 22–24, 28, 30, 31].

In connection with energy (that is defined in terms of the eigenvalues of the adjacency matrix), energy-like quantities were considered also for the other matrices: Laplacian [15], distance [17], incidence [18], minimum covering energy [1] etc. Recall that a great variety of matrices has so far been associated with graphs [4, 5, 10, 29].

Recently in [25] the authors have studied the dominating matrix which is defined as :
Let $G = (V,E)$ be a graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$ and let $D \subseteq V(G)$ be a minimum dominating set of G. The minimum dominating matrix of G is the $n \times n$ matrix defined by $A_D(G) = (a_{ij})$, where $a_{ij} = 1$ if $v_iv_j \in E(G)$ or $v_i = v_j \in D$, and $a_{ij} = 0$ if $v_iv_j \notin E(G)$.

The characteristic polynomial of $A_D(G)$ is denoted by $f_n(G,\mu) := det(\mu I - A_D(G))$.

The minimum dominating eigenvalues of a graph G are the eigenvalues of $A_D(G)$. Since $A_D(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. The minimum dominating energy of G is then defined as

$$E_D(G) = \sum_{i=1}^{n} |\mu_i|.$$

Motivated by dominating matrix, here we define the minimum connected dominating matrix abbreviated as (c-dominating matrix). The c-dominating matrix of G is the $n \times n$ matrix defined by $A_{D_c}(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, \text{ if } v_iv_j \in E; \\ 1, \text{ if } i = j \text{ and } v_i \in D_c; \\ 0, \text{ otherwise.} \end{cases}$$

The characteristic polynomial of $A_{D_c}(G)$ is denoted by $f_n(G,\lambda) := det(\lambda I - A_{D_c}(G))$.

The c-dominating eigenvalues of a graph G are the eigenvalues of $A_{D_c}(G)$. Since $A_{D_c}(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The c-dominating energy of G is then defined as

$$E_{D_c}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

To illustrate this, consider the following examples:

Figure 1.

Example 1. Let G be the 5-vertex path P_5, with vertices v_1, v_2, v_3, v_4, v_5 and let its minimum connected dominating set be $D_c = \{v_2, v_3, v_4\}$. Then

$$A_{D_c}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $A_{D_c}(G)$ is $\lambda^5 - 3\lambda^4 - \lambda^3 + 5\lambda^2 + \lambda - 1 = 0$. The minimum connected dominating eigenvalues are $\lambda_1 = 2.618$, $\lambda_2 = 1.618$, $\lambda_3 = 0.382$, $\lambda_4 = -1.000$ and $\lambda_5 = -0.618$.
Therefore, the minimum connected dominating energy is $E_{D_c}(P_5) = 6.236$.

Example 2. Consider the following graph

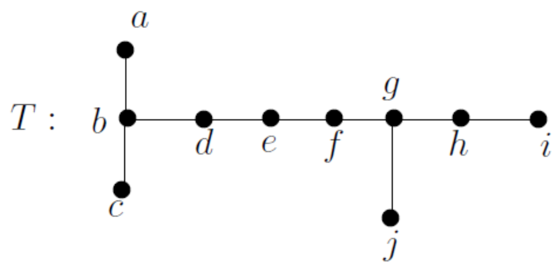


Figure 2.

Let G be a tree T as shown above and let its minimum connected dominating set be $D_c = \{b,d,e,f,g,h\}$. Then

$$A_{D_c}(G) = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&1&0&1&1&0&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0&0 \\ 0&0&0&0&1&1&1&0&0&0 \\ 0&0&0&0&0&1&1&1&0&1 \\ 0&0&0&0&0&0&1&1&1&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \end{pmatrix}$$

By direct calculation, we get the minimum connected dominating eigenvalues are $\lambda_1 = 2.945$, $\lambda_2 = 2.596$, $\lambda_3 = 1.896$, $\lambda_4 = 1.183$, $\lambda_5 = -1.263$, $\lambda_6 = -1.152$, $\lambda_7 = 0.579$, $\lambda_8 = 0.000$, $\lambda_9 = -0.268$ and $\lambda_{10} = -0.516$.
Therefore, the minimum connected dominating energy is $E_{D_c}(T) = 12.398$.

Example 3. The c-dominating energy of the following graphs can be calculated easily:

1. $E_{D_c}(K_n) = (n-2) + \sqrt{n(n-2)+5}$, where K_n is the complete graph of order n.
2. $E_{D_c}(K_{1,n-1}) = \sqrt{4n-3}$ where $K_{1,n-1}$ is the star graph.
3. $E_{D_c}(K_{n\times 2}) = (2n-3) + \sqrt{4n(n-1)-9}$, where $K_{n\times 2}$ is the coctail party graph.

In this paper, we are interested in studying the mathematical aspects of the c-dominating energy of a graph. This paper has organized as follows: The section 1, contains the basic definitions and background of the current topic. In section 2, we show the chemical applicability of c-dominating energy for molecular graphs G. The section 3, contains the mathematical properties of c-dominating energy. In the last section, we have characterized, trees, unicyclic graphs and cubic graphs and block graphs with equal minimum dominating energy and c-dominating energy. Finally, we conclude this paper by posing an open problem.

2 Chemical Applicability of $E_{D_c}(G)$

We have used the c-dominating energy for modeling eight representative physical properties like boiling points(bp), molar volumes(mv) at 20°C, molar refractions(mr) at 20°C, heats of vaporization (hv) at 25°C,

critical temperatures(ct), critical pressure(cp) and surface tension (st) at 20°C of the 74 alkanes from ethane to nonanes. Values for these properties were taken from http://www.moleculardescriptors.eu/dataset.htm. The c-dominating energy $E_{D_c}(G)$ was correlated with each of these properties and surprisingly, we can see that the E_{D_c} has a good correlation with the heats of vaporization of alkanes with correlation coefficient $r = 0.995$.
The following structure-property relationship model has been developed for the c-dominating energy $E_{D_c}(G)$.

$$hv = 10E_{D_c}(G) \pm 5. \tag{1}$$

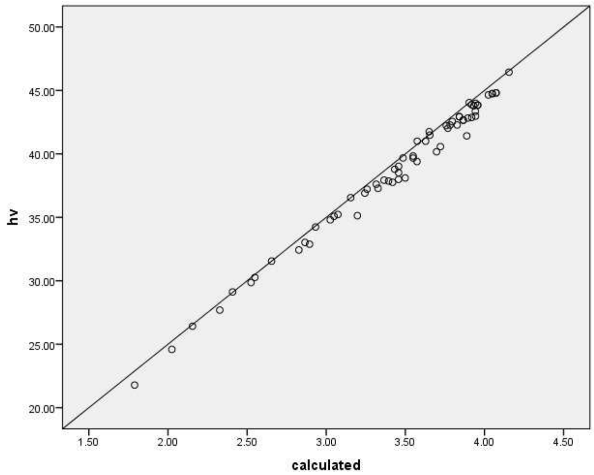


Figure 3: Correlation of $E_{D_c}(G)$ with heats of vaporization of alkanes.

3 Mathematical Properties of c-Dominating Energy of Graph

We begin with the following straightforward observations.

Observation 1. *Note that the trace of* $A_{D_c}(G) = \gamma_c(G)$.

Observation 2. *Let* $G = (V,E)$ *be a graph with* γ_c*-set* D_c. *Let* $f_n(G,\lambda) = c_0\lambda^n + c_1\lambda^{n-1} + \cdots + c_n$ *be the characteristic polynomial of* G. *Then*

1. $c_0 = 1$,
2. $c_1 = -|D_c| = -\gamma_c(G)$.

Theorem 3. *If* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *are the eigenvalues of* $A_{D_c}(G)$, *then*

1. $\sum_{i=1}^{n} \lambda_i = \gamma_c(G)$
2. $\sum_{i=1}^{n} \lambda_i^2 = 2m + \gamma_c(G)$.

Proof.

1. Follows from Observation 1.
2. The sum of squares of the eigenvalues of $A_{D_c}(G)$ is just the trace of $A_{D_c}(G)^2$. Therefore

$$\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji}$$

$$= 2\sum_{i<j}(a_{ij})^2 + \sum_{i=1}^{n}(a_{ii})^2$$
$$= 2m + \gamma_c(G).$$

□

We now obtain bounds for $E_{D_c}(G)$ of G, similar to McClelland's inequalities [21] for graph energy.

Theorem 4. *Let G be a graph of order n and size m with $\gamma_c(G) = k$. Then*

$$E_{D_c}(G) \leq \sqrt{n(2m+k)}. \tag{2}$$

Proof. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $A_{D_c}(G)$. Bearing in mind the Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i\right)^2 \left(\sum_{i=1}^{n} b_i\right)^2$$

we choose $a_i = 1$ and $b_i = |\lambda_i|$, which by Theorem 3 implies

$$E_{D_c}^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2$$
$$\leq n\left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$
$$= n\sum_{i=1}^{n} \lambda_i^2$$
$$= 2(2m+k).$$

□

Theorem 5. *Let G be a graph of order n and size m with $\gamma_c(G) = k$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be a non-increasing arrangement of eigenvalues of $A_{D_c}(G)$. Then*

$$E_{D_c}(G) \geq \sqrt{2mn + nk - \alpha(n)(|\lambda_1| - |\lambda_n|)^2} \tag{3}$$

where $\alpha(n) = n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}])$, where $[x]$ denotes the integer part of a real number k.

Proof. Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be real numbers for which there exist real constants a, b, A and B, so that for each i, $i = 1, 2, \cdots, n, a \leq a_i \leq A$ and $b \leq b_i \leq B$. Then the following inequality is valid (see [6]).

$$|\, n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \,| \leq \alpha(n)(A-a)(B-b), \tag{4}$$

where $\alpha(n) = n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}])$. Equality holds if and only if $a_1 = a_2 = \cdots = a_n$ and $b_1 = b_2 = \cdots = b_n$. We choose $a_i := |\lambda_i|, b_i := |\lambda_i|$, $a = b := |\lambda_n|$ and $A = B := |\lambda_1|, i = 1, 2, \cdots, n$, inequality (4) becomes

$$|n\sum_{i=1}^{n} |\lambda_i|^2 - \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 | \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2. \tag{5}$$

Since $E_{G_c}(G) = \sum\limits_{i=1}^{n} |\lambda_i|$, $\sum\limits_{i=1}^{n} |\lambda_i|^2 = \sum\limits_{i=1}^{n} |\lambda_i|^2 = 2m+k$ and $E_{D_c}(G) \leq \sqrt{n(2m+k)}$, the inequality (5) becomes

$$\begin{aligned} n(2m+k) - (E_{D_c})^2 &\leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2 \\ (E_{D_c})^2 &\geq 2mn + nk - \alpha(n)(|\lambda_1| - |\lambda_n|)^2. \end{aligned}$$

Hence equality holds if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. □

Corollary 6. *Let G be a graph of order n and size m with $\gamma_c(G) = k$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be a non-increasing arrangement of eigenvalues of $A_{D_c}(G)$. Then*

$$E_{D_c}(G) \geq \sqrt{2mn + nk - \frac{n^2}{4}(|\lambda_1| - |\lambda_n|)^2}. \tag{6}$$

Proof. Since $\alpha(n) = n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}]) \leq \frac{n^2}{4}$, therefore by (3), result follows. □

Theorem 7. *Let G be a graph of order n and size m with $\gamma_c(G) = k$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be a non-increasing arrangement of eigenvalues of $A_{D_c}(G)$. Then*

$$E_{G_c}(G) \geq \frac{|\lambda_1||\lambda_2|n + 2m + k}{|\lambda_1| + |\lambda_n|}. \tag{7}$$

Proof. Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be real numbers for which there exist real constants r and R so that for each i, $i = 1, 2, \cdots, n$ holds $ra_i \leq b_i \leq Ra_i$. Then the following inequality is valid (see [11]).

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i^2 \leq (r+R)\sum_{i=1}^{n} a_i b_i. \tag{8}$$

Equality of (8) holds if and only if, for at least one i, $1 \leq i \leq n$ holds $ra_i = b_i = Ra_i$.
For $b_i := |\lambda_i|$, $a_i := 1$ $r := |\lambda_n|$ and $R := |\lambda_1|$, $i = 1, 2, \cdots, n$ inequality (8) becomes

$$\sum_{i=n}^{n} |\lambda_i|^2 + |\lambda_1||\lambda_n|\sum_{i=1}^{n} 1 \leq (|\lambda_1| + |\lambda_n|)\sum_{i=1}^{n} |\lambda_i|. \tag{9}$$

Since $\sum\limits_{i=1}^{n} |\lambda_i|^2 = \sum\limits_{i=1}^{n} \lambda_i^2 = 2m+k$, $\sum\limits_{i=1}^{n} |\lambda_i| = E_{D_c}(G)$, from inequality (9),

$$2m + k + |\lambda_1||\lambda_n|n \leq (\lambda_1 + \lambda_n)E_{D_c}(G)$$

Hence the result. □

Theorem 8. *Let G be a graph of order n and size m with $\gamma_c(G) = k$. If $\xi = |detA_{D_c}(G)|$, then*

$$E_{D_c}(G) \geq \sqrt{2m + k + n(n-1)\xi^{\frac{2}{n}}}. \tag{10}$$

Proof.

$$\begin{aligned} (E_{D_c}(G))^2 &= \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \\ &= \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i \neq j} |\lambda_i||\lambda_j|. \end{aligned}$$

Employing the inequality between the arithmetic and geometric means, we obtain

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}.$$

Thus,

$$\begin{aligned}(E_{D_G})^2 &\geq \sum_{i=1}^{n}|\lambda_i|^2+n(n-1)\left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}\\ &\geq \sum_{i=1}^{n}|\lambda_i|^2+n(n-1)\left(\prod_{i\neq j}|\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}\\ &= 2m+k+n(n-1)\xi^{\frac{2}{n}}.\end{aligned}$$

□

Lemma 9. *If $\lambda_1(G)$ is the largest minimum connected dominating eigenvalue of $A_{D_c}(G)$, then $\lambda_1 \geq \frac{2m+\gamma_c(G)}{n}$.*

Proof. Let X be any non-zero vector. Then we have $\lambda_1(A) = max_{X\neq 0}\{\frac{X'AX}{X'X}\}$, see [16]. Therefore, $\lambda_1(A_{D_c}(G)) \geq \frac{J'AJ}{J'J} = \frac{2m+\gamma_c(G)}{n}$. □

Next, we obtain Koolen and Moulton's [19] type inequality for $E_{D_c}(G)$.

Theorem 10. *If G is a graph of order n and size m and $2m+\gamma_C(G) \geq n$, then*

$$E_{D_c}(G) \leq \frac{2m+\gamma_c(G)}{n}+\sqrt{(n-1)\left[(2m+\gamma_c(G))-\left(\frac{2m+\gamma_c(G)}{n}\right)^2\right]}. \tag{11}$$

Proof. Bearing in mind the Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n}a_i\right)^2\left(\sum_{i=1}^{n}b_i\right)^2.$$

Put $a_i = 1$ and $b_i = |\lambda_i|$ then

$$\begin{aligned}\left(\sum_{i=2}^{n}a_ib_i\right)^2 &\leq (n-1)\left(\sum_{i=2}^{n}b_i\right)^2\\ (E_{D_c}(G)-\lambda_1)^2 &\leq (n-1)(2m+\gamma_c(G)-\lambda_1^2)\\ E_{D_c}(G) &\leq \lambda_1+\sqrt{(n-1)(2m+\gamma_c(G)-\lambda_1^2)}.\end{aligned}$$

Let

$$f(x) = x+\sqrt{(n-1)(2m+\gamma_c(G)-x^2)}. \tag{12}$$

For decreasing function

$$\begin{aligned}f'(x) &\leq 0\\ \Rightarrow 1-\frac{x(n-1)}{\sqrt{(n-1)(2m+\gamma_c(G)-x^2)}} &\leq 0\\ x &\geq \sqrt{\frac{2m+\gamma_c(G)}{n}}.\end{aligned}$$

Since $(2m+k) \geq n$, we have $\sqrt{\frac{2m+\gamma_c(G)}{n}} \leq \frac{2m+\gamma_c(G)}{n} \leq \lambda_1$. Also $f(\lambda_1) \leq f\left(\frac{2m+\gamma_c(G)}{n}\right)$.

$$\text{i.e } E_{D_c}(G) \leq f(\lambda_1) \leq f\left(\frac{2m+\gamma_c(G)}{n}\right).$$
$$\text{i.e } E_{D_c}(G) \leq f\left(\frac{2m+\gamma_c(G)}{n}\right)$$

Hence by (12), the result follows. □

4 Graphs with equal Dominating and c-Dominating Energy

Its a natural question to ask that for which graphs the dominating energy and c-dominating energy are equal. To answer this question, we characterize graphs with equal dominating energy and c-dominating energy. The graphs considered in this section are trees, cubic graphs, unicyclic graphs, block graphs and cactus graphs.

Theorem 11. *Let $G = T$ be a tree with at least three vertices, then $E_D(G) = E_{D_c}(G)$ if and only if every internal vertex of T is a support vertex.*

Proof. Let $G = T$ be a tree of order at least 3. Let $F = \{u_1, u_2, \cdots, u_k\}$ be the set of internal vertices of T. Then clearly F is the minimal dominating set of G. Therefore in $A_D(G)$ the values of $u_i = 1$ in the diagonal entries. Further, observe that $\langle F \rangle$ is connected. Hence F is the minimal connected dominating set. Therefore, $A_D(G) = A_{D_c}(G)$. In general, $A_D(G) = A_{D_c}(G)$ is true if every minimum dominating set is connected. In other words, $A_D(G) = A_{D_c}(G)$ if $\gamma(G) = \gamma_c(G)$. Therefore, the result follows from Theorem 2.1 in [2]. □

In the next three theorems we characterize unicyclic graphs with $A_D(G) = A_{D_c}(G)$. Since, $A_D(G) = A_{D_c}(G)$ if $\gamma(G) = \gamma_c(G)$. Therefore, the proof of our next three results follows from Theorem 2.2, Theorem 2.4 and Theorem 2.5 in [2].

Theorem 12. *Let G be a unicyclic graph with cycle $C = u_1u_2\cdots, u_nu_1$ $n \geq 5$ and let $X = \{v \in C : d_G(v) \geq 2\}$. Then $E_D(G) = E_{D_c}(G)$ if the following conditions hold:*

1. *(a). Every $v \in V - N[X]$ with $d_G(V) \geq 2$ is a support vertex.*
2. *(b). $\langle X \rangle$ is connected and $|X| \leq 3$.*
3. *(c). If $\langle X \rangle = P_1 or P_3$, both vertices in $N(X)$ of degree at least 3 are supports and if $\langle X \rangle = P_2$, at least one vertex in $N(X)$ of degree at least three is a support.*

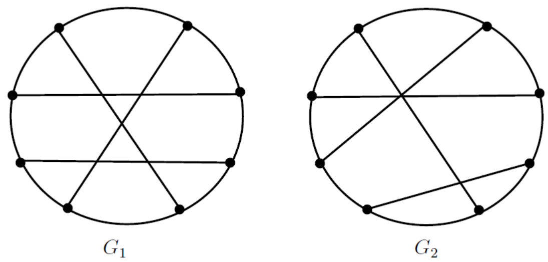
Theorem 13. *Let G be unicyclic graph with $|V(G)| \geq 4$ containing a cycle $C = C_3$, and let $X = \{v \in C : d_G(v) = 2\}$. Then $E_D(G) = E_{D_c}(G)$ if the following conditions hold:*

1. *(a). Every $v \in V - N[X]$ with $d_G(V) \geq 2$ is a support vertex.*
2. *(b). There exists some unique $v \in C$ with $d_G(v) \geq 3$ or for every $v \in C$ of $d_G(v) \geq 3$ is a support.*

Theorem 14. *Let G be unicyclic graph with $|V(G)| \geq 5$ containing a cycle $C = C_4$, and let $X = \{v \in C : d_G(v) = 2\}$. Then $E_D(G) = E_{D_c}(G)$ if the following conditions hold:*

1. *(a). Every $v \in V - N[X]$ with $d_G(V) \geq 2$ is a support vertex.*
2. *(b). If $|X| = 1$, all the three remaining vertices of C are supports and if $|X| \geq 2$, C contains at least one support.*

Theorem 15. *Let G be a connected cubic graph of order n, Then $E_D(G) = E_{D_c}(G)$ if $G \cong K_4, \overline{C_6, K_{3,3}}$, G_1 or G_2 where G_1 and G_2 are given in Fig. 4.*



Theorem 16. *Let G be a block graph of with $l \geq 2$. Then $E_D(G) = E_{D_c}(G)$ if every cutvertex of G is an end block cutvertex.*

Proof. Since $E_D(G) = E_{D_c}(G)$ if $\gamma(G) = \gamma_c(G)$. Therefore, the result follows from Theorem 2 in [7]. □

We conclude this paper by posing the following open problem for the researchers:
Open Problem: Construct non- cospectral graphs with unequal domination and connected domination numbers having equal dominating energy and c-dominating energy.

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