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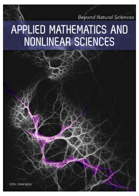

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Fully discrete convergence analysis of non-linear hyperbolic equations based on finite element analysis

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Abstract

With the development of modern partial differential equation (PDE) theory, the theory of linear PDE is becoming more and more perfect. Non-linear PDE has become a research hotspot of many mathematicians. In fact, when describing practical physical problems with PDEs, non-linear problems tend to be more general than linear problems, which are close to real problems and have practical physical significance. Hyperbolic PDEs are a kind of important PDEs describing the phenomena of vibration or wave motion. The solution of hyperbolic PDE can be decomposed into the form of multiplication of vibration and vibration or of exponential function and exponential function. Generally, the energy is infinite. A full discrete convergence analysis method for non-linear hyperbolic equation based on finite element analysis is proposed. Taking second-order and fourth-order non-linear hyperbolic equation as examples, the full discrete convergence of non-linear hyperbolic equation is analysed by finite element method and the super-convergence results are obtained.

Keywords: finite element analysis, nonlinearity, hyperbolic equation, fully discrete, convergence, error

1 Introduction

With the rapid development of science and technology, a variety of differential equation mathematical models have been pouring out [1,2]. The hyperbolic equation (group) model is one of the most important ones. It has a wide application background in natural science. It belongs to one-dimensional wave equation describing string vibration. Similarly, two-dimensional or three-dimensional wave equation can be derived from the vibration of elastic film or three-dimensional elastomer [4]. In addition, the three-dimensional wave equation can also be derived for the propagation of acoustic wave or electromagnetic wave. For example, the Maxwell equations describing electromagnetic fields are curled to simplify the standard vector wave equations [5]. When studying the propagation of high-frequency electromagnetic waves along transmission lines in time and space, the concepts of current intensity and voltage between coaxial and double lines of transmission lines can be introduced. They

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can be used as physical quantities to characterise the propagation process of such electromagnetic waves, and the concepts of resistance and inductance per unit transmission line can be used to describe the characteristics of dielectrics. According to the law, a set of telegraph equations can be established, which can be simplified to a standard wave equation without loss [6]. In addition, hydrodynamic problems in aviation, meteorology, ocean, petroleum exploration and other fields are reduced to solving non-linear hyperbolic partial differential equations (PDEs; known as conservation laws in foreign literature). The basic difficulty of this kind of equation is that the solution appears discontinuity. When the solution is solved by high-precision explicit scheme, the oscillation will occur at the discontinuity [7]. Hyperbolic equations (systems) are widely used in many fields of mathematical physics and have profound physical background, such as wave equation. Therefore, they have been paid more attention by mathematicians and engineering technicians. It is necessary to study them comprehensively and thoroughly both theoretically and numerically [8,9]. In this paper, the full discrete convergence analysis of the non-linear hyperbolic equation based on finite element analysis is presented. The full discrete convergence of the non-linear hyperbolic equation is analysed comprehensively [10].

2 Application Theory of Algorithm

2.1 Full Discrete and Convergence Analysis of Second-Order Non-linear Hyperbolic Equations

2.1.1 Question Description

The following mixed problems are considered:

$$\begin{cases} h(x,u)u_{tt} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\right) - \sum_{i=1}^{d} b_i(x.u)ux_i = f(x,u)\,(x,t)\in K\times[0,T] \\ u(x,0)=0, u_t(x,0)=0 \\ u(x,t)=0\,(x,t)\in\partial K\times[0,T] \end{cases} \tag{1}$$

where $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, $ux_i = \frac{\partial u}{\partial x_i}$; K is a fully smooth bounded open domain in R^d, and the boundary ∂K is smooth [11].

For the semi-discrete or fully discrete finite element method of the non-linear hyperbolic equation with only x or $h(x,u)\equiv 1$ in $h(x,u)$, there are some research results [12,13]. If u is included in $h(x,u)$, the error estimation will suffer or fail to reach the convergence order [14] when defining the non-linear or predictor–corrector scheme, and the error equation cannot be obtained by direct weighting method. In this paper, the finite element scheme of second-order nonsexual hyperbolic equation [15] is defined when h contains u. Question (1) is assumed as the following: for $(x,p)\in K\times R$,

(1) $a_{ij}(\cdot,\cdot)\in C^2(K\times R)$;$\left|a_{ij}(x,p)\right|\le C_1$,$[a_{ij}(x,p)]'_p$,$[a_{ij}(x,p)]_{p^2}''$, it is bounded to P. $a_{ij}(x,p)=a_{ji}(x,p)$, $\sum_{i,h=1}^{d} a_{ij}(x,p)r_ir_j \ge C_0\sum_{i=1}^{d}|r_i|^2$, among them, $\forall r=(r_1,r_2,...,r_d)\in R^d$.

(2) $C_2\le h(x,p)\le C_3$, $h(x,p)$ is Lipschitz continuous with respect to p.

(3) $b_i(x,p)$ and $[b_i(x,p)]'_p$ are bounded [16]. $(i=1,2,...,d)$, $b_i(\circ,\circ)\in C^1(K\times R)$.

(4) $f(x,p)$ is Lipschitz continuous with respect to p, $f(x,0)\in L^2(K)$.

(5)u, u_t,$u_{tt}\in L^\infty\left([0,T];H^{m+1}\cap W^{1,\infty}\right)\cap L^2\left([0,T];H^{m+1}\right)$, $u_{t^3}\in L^\infty\left([0,T];H^1\right)$, $u_{t^4}\in L^\infty\left([0,T];L^2\right)$, $m+1>\frac{d}{2}$, $m\ge 1$.

Let $w=u_t$, then the original question (1) becomes:

$$\begin{cases} h(x,u)w_t - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\right) - \sum_{i=1}^{d} b_i(x.u)ux_i = f(x,u)\,(x,t)\in K\times[0,T] \\ w(x,0)=0, x\in K \\ w(x,t)=0\,(x,t)\in\partial K\times[0,T] \end{cases} \tag{2}$$

The variational equations corresponding to question equation (1) and question equation (2) are:

$$\begin{cases} (h(u)u_{tt},V)+(a(u)5u,5V)=(b(u)5u,5V)+(f(u),V)\forall V\in H_0^1(K),t\in[0,T] \\ (u(0),V)=(u_t(0),V)=0\; x\in K \\ u(\cdot,t)\in H_0^1(K)\;[0,T] \end{cases} \tag{3}$$

$$\begin{cases} (h(u)w_{tt},V)+(a(u)5u,5V)=(b(u)5u,5V)+(f(u),V)\forall V\in H_0^1(K),t\in[0,T] \\ (w(0),V)=0\; x\in K \\ w(\cdot,t)\in H_0^1(K)\; t\in[0,T] \end{cases} \tag{4}$$

where $(f,g)=\int_K f(x)g(x)dx$; $(a(p)5f,5g)=\sum_{i,h=1}^{d}\int_K a_{ij}(x,p)\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}dx$, $(b(p)5f,g)=\sum_{i=1}^{d}\left(b_i(x,p)\frac{\partial f}{\partial x_i},g\right)$. For the convenience of calculation, x appearing in the function is omitted, and the intervals [0, T] and region K appearing in the space are also omitted. Also, $\|f\|^2=(f,f)$, $|f|_1^2=\sum_{i=1}^{d}\left\|\frac{\partial f}{\partial x_i}\right\|^2$, $\|f\|_1^2=\|f\|^2+|f|_1^2$ $\|\cdot\|_1$ and $|\cdot|_1$ on $H_0^1(K)$ are norms of the same order.

Let $S_h\subset H_0^1$ be a finite dimensional subspace with an approximation order of $m+1$. For $\forall V\cup S_h$, it satisfies the usual approximation properties and inverse estimates of $\|V\|L^\infty\le C_4h^{-\frac{d}{2}}\|V\|$ and $\|V\|_1\le Ch^{-1}\|V\|$. Elliptic projection is considered: $\tilde{u}(x,t)\in S_h$ and $t\in[0,T]$are solved to satisfy:

$$(a(u)5u,5V)=(a(u)5\tilde{u},5V)\;\forall V\in S_h \tag{5}$$

For the projection function $\tilde{u}(x,t)$, we assume that [3] $\|\tilde{u}\|L^\infty$, $\|5\tilde{u}\|L^\infty$ and $\left\|5\frac{\partial\tilde{u}}{\partial t}\right\|L^\infty$ are uniformly bounded. At the same time, the regularity results of elliptic equation and the properties of S_h can be obtained [4, 6, 7].

Lemma: if the above assumptions are satisfied, then for $p=2,\infty$, s = 0, 1 there are:

$$\|(u-\tilde{u})\|_{L^p(H^S)}+\|(u-\tilde{u})_t\|_{L^p(H^S)}+\|(u-\tilde{u})_{tt}\|_{L^p(H^S)}\le Ch^{m+1-s} \tag{6}$$

The interval $[0,T]$ is divided into N equal subintervals:$0=t_0<t_1<\ldots<t_{N-1}<t_N=T\cdot t_{n+1}-t_n=\Delta t$, $U^n=U(t_n)$, for the sake of simplicity of writing, the following marks are introduced:

$$U^{n+\frac{1}{2}}=\frac{1}{2}\left(U^{n+1}+U^n\right),(5f^n,5V)=\sum_{i=1}^{d}\left(\frac{\partial f^n}{\partial x_i},\frac{\partial V}{\partial x_i}\right) \tag{7}$$

$$d_tU^n=\frac{1}{\Delta t}\left(U^{n+1}-U^n\right),\partial U^n=U^{n+1}-U^n=\Delta t d_tU^n \tag{8}$$

$$\partial_t^2U^n=\frac{1}{(\Delta t)^2}\left(U^{n+1}-2U^n+U^{n+1}\right),h^n(U)=h(U^n) \tag{9}$$

$$h^{n+\frac{1}{2}}(U)=\frac{1}{2}\left(h^{n+1}(U)+h^n(U)\right),EU^{n+1}=2U^n-U^{n-1} \tag{10}$$

$$\overset{\frown}{h}^{n+1}(U)=h\left(EU^{n+1}\right),\overset{\frown}{h}^{n+\frac{1}{2}}(U)=\frac{1}{2}\left(\overset{\frown}{h}^{n+1}(U)+h^n(u)\right) \tag{11}$$

Also, let $U-u=U-\tilde{u}+\tilde{u}-u=a+Z$, $a=U-\tilde{u}$, $Z=\tilde{u}-u$, $w=u_t$, $\tilde{w}=\tilde{u}_t$, $W-w=W-\tilde{w}+\tilde{w}-w=\theta+d$, $\theta=W-\tilde{w}$, $d=\tilde{w}-w$.

Question (1) is defined as:

$$(h(U)^n)\partial_t^2+(a(U^n)5U^n,5V)+\lambda\left(5\left(U^{n+1}-2U^n+U^{n-1}\right),5V\right)=(b(U^n)5U^n,V)+(f(U^n),V)\forall V\in S_h \tag{12}$$

2.1.2 Fully Discrete Convergence Analysis of Second-Order Non-linear Hyperbolic Equations based on Finite Element Analysis

It is known that the solution of the equation (12) is unique. The error equation is obtained from equations (12), (10), (8) and (6):

$$\left(h\left(U^n\right)\partial_t^2 a^n, V\right) + \left(a\left(U^n\right)5a^n, 5V\right) + \lambda\left(5\left(a^{n+1} - 2a^n + a^{n-1}\right), 5V\right) = \left(A^n, V\right) - \left(B^n, 5V\right) - \left(C^n, 5V\right) + \left(D^n, V\right) \tag{13}$$

where

$$A^n = h\left(U^n\right)\left(u_{tt}^n 0 - \partial_t^2 u^n\right) + u_{tt}^n\left[h\left(u^n\right) - h\left(U^n\right)\partial_t^2 Z_n + f\left(U^n\right) - f\left(u^n\right)\right] \tag{14}$$

$$B^n = \left[a\left(U^n\right) - a\left(u^n\right)\right]5\tilde{u}^n \tag{15}$$

$$C^n = \lambda 5\left(Z^{n+1} - 2Z^n + Z^{n-1}\right) + \lambda 5\left(u^{n+1} - 2u^n + u^{n-1}\right) \tag{16}$$

$$D^n = \left[b\left(U^n\right)5a^n + \left(b\left(U^n\right) - b\left(u^n\right)\right)5\tilde{u}^n\right] + b\left(u^n\right)5Z^n = D_1^n + D_2^n \tag{17}$$

Let $Q = \left\|\frac{\partial\tilde{u}}{\partial t}\right\| L^\infty\left(L^\infty\right) + 1$, $\Delta t, h$ are taken to satisfy $C_4C_5(\Delta t)^2 h^{-\frac{d}{2}} \le Q$. According to (1.10) and inverse estimates, it can be seen that $\left\|d_t a^0\right\|_{L^\infty} \le Q$, $\left\|d_t U^0\right\|_{L^\infty} \le 2Q$.

If inductive assumption $\max\limits_{0\le n\le M-2}\|dta^n\|_{L^\infty} \le Q$, then $\max\limits_{0\le n\le M-2}\|dtU^n\|_{L^\infty} \le 2Q$.

Taking the test function $V = a^{n+1} - a^{n-1} = \partial_t^2 a^n + \partial_t^2 a^{n-1} = \Delta t\left(dta^n + dta^{n-1}\right)$, it can rewrite or estimate the two ends of the error equation.

$$\begin{aligned}&\left(h\left(U^n\right)\partial_t^2 a^n, V\right) = \left(h\left(U^n\right)dta^n, dta^n\right) - \left(h\left(U^n\right)dta^{n-1}, dta^{n-1}\right) = \left[\left(h\left(U^n\right)dta^{n-1}, dta^{n-1}\right)\right.\\&\left.-\left(h\left(U^{n-1}\right)dta^{n-1}, dta^{n-1}\right)\right] - \left(h\left(U^n\right) - h\left(U^{n-1}\right)dta^{n-1}, dta^{n-1}\right)\end{aligned} \tag{18}$$

It is known

$$|A^n + D_1^n| \le C\Delta t(||a^n||_1^2 + ||Z^n||^2) + |||dta^n||^2 + ||dta^{n-1}||^2 + ||\partial_t^2 Z^n||^2 \tag{19}$$

The following estimates are highlighted:

$$B = \left(\left[a\left(U^n\right) - a\left(u^n\right)\right]5\tilde{u}^n.5\left(a^{n+1} - a^{n-1}\right)\right) \tag{20}$$

Be aware:

$$\sum_{n=1}^{M-1} B_1^{(1)} = \left(\left[a\left(U^M\right) - a\left(u^M\right)\right]5\tilde{u}^M, 5a^M\right) - \left(\left[a\left(U^1\right) - a\left(u^1\right)\right]5\tilde{u}^1, 5a^1\right) \tag{21}$$

And $U^M - u^M = a^1 + Z^1 + \Delta t\sum\limits_{n=1}^{M-1}\left(dta^n + dtZ^n\right)$. We can get:

$$\left|\sum_{n=1}^{M-1} B_1^{(1)}\right| \le C\Delta t\sum_{n=1}^{M-1}\left(\|dta^n\|^2 + \|dtZ^n\|^2\right) + X\left\|a^M\right\|_1^2 + C\left(\left\|Z^1\right\|^2 + \left\|a^1\right\|_1^2\right) \tag{22}$$

Similar estimates can be obtained as follows:

$$\left|\sum_{n=1}^{M-1} B_1^{(1)}\right| \le C\Delta t\sum_{n=1}^{M-1}\left(\|dta^n\|^2 + \|dtZ^n\|^2 + \|a^n\|_1^2 + \left\|a^{n+1}\right\|_1^2 + \|Z^n\|^2 + \left\|Z^{n+1}\right\|^2\right) + X\left\|a^M\right\|_1^2 + C\left(\left\|a^1\right\|^2 + \left\|Z^1\right\|^2\right) \tag{23}$$

Using inverse estimation and distribution integral, it can get:

$$
\begin{aligned}
&\left|\left(C^n, 5\left(a^{n+1}-a^{n+1}\right)\right)\right| \le C\Delta t\left[(\Delta t)^4 h^{-2}\right]\left\|\partial_t^2 Z^n\right\|_1^2+(\Delta t)^4+\left\|dta^n\right\|^2+\left\|dta^{n-1}\right\| \\
&\left|\sum_{n=1}^{M}\left(b_i\left(u^n\right) Z_x^n, a^{n+1}-a^{n-1}\right)\right| \le \left|\sum_{n=1}^{M}\left(b_i\left(u^n\right) Z_x^n, a^{n+1}-a^{n-1}\right)\right|+2\left|\left(b_i\left(u^{M-1}\right) Z^{M-1}, a_x^{M-\frac{1}{2}}\right)\right. \\
&\left.-\left(b_i\left(u^1\right) Z^1, a_x^{\frac{1}{2}}\right)-\Delta t \sum_{n=2}^{M-1}\left(b_i\left(u^n\right) \frac{Z^n-Z^{n-1}}{\Delta t}\right)\right|+Z^{n-1} \frac{bt\left(u^n\right)-bt\left(u^{n-1}\right)}{\Delta t}, a_x^{n-\frac{1}{2}}\Bigg| \\
&\le C\Delta t \sum_{n=1}^{M}\left(\left\|Z^n\right\|^2+\left\|dta^n\right\|^2+\left\|dta^{n-1}\right\|^2+X\left(\left\|a^M\right\|_1^2+\left\|a^{M-1}\right\|_1^2\right)\right) \\
&+C\left(\left\|Z^{M-1}\right\|^2+\left\|Z^1\right\|^2+\left\|a^{\frac{1}{2}}\right\|_1^2+C\Delta t \sum_{n=2}^{M-1}\left(\left\|dtZ^{n-1}\right\|^2\right)+\left\|Z^{n-1}\right\|^2+\left\|a^{\frac{1}{2}}\right\|_1^2\right)
\end{aligned} \tag{24}
$$

It is noticed that $m \ge 1$, $(\Delta t)^2 = O\left(h^{m+1}\right)$, the lemma shows that:

$$
\left\|Z^{M-1}\right\|^2+\left\|Z^M\right\|^2+C\Delta t \sum_{n=1}^{M-1}\left(\left\|Z^n\right\|^2+\left\|Z^{n+1}\right\|^2+\left\|dtZ^n\right\|^2+\left\|Z^{n-1}\right\|^2+(\Delta t)^4 h^{-2}\left\|\partial_t^2 Z^n\right\|_1^2\right) \le Ch^{2m+2} \tag{25}
$$

For error equation, the sum can be solved from n=$1,2,\ldots,M-1$, and it is noted that $\left\|a^0\right\|_1 = 0$, $\left\|a^1\right\|_1 + \left\|dta^0\right\| \le C(\Delta t)^2$. Using inductive hypothesis and the above estimates, we can get:

$$
\begin{aligned}
&C_2\left\|dta^{M-1}\right\|^2+\left(a\left(U^{M-1}\right) 5a^{M-1}, 5a^M\right)+\lambda\left|a^M-a^{M-1}\right|_1^2 \le C\left(h^{2m+2}+(\Delta t)^4\right) \\
&+2X\left(\left\|a^M\right\|^2+\left\|a^{M-1}\right\|^2\right)+C\Delta t \sum_{n=1}^{M-1}\left(\left\|dta^n\right\|^2+\left\|a^n\right\|_1^2+\left\|a^{n+1}\right\|_1^2\right)
\end{aligned} \tag{26}
$$

If $\lambda > \frac{dC_1}{4}$, $V = \min\left\{\lambda - \frac{dC_1}{4}, \frac{C_0}{4}\right\} > 0$, so:

$$
\lambda\left|a^M-a^{M-1}\right|_1^2+\left(a\left(U^{M-1}\right) 5a^M\right) \ge V\left(\left|a^M-a^{M-1}\right|_1^2+\left|a^M+a^{M-1}\right|_1^2\right) \ge C\left(\left\|a^M\right\|_1^2+\left\|a^{M-1}\right\|_1^2\right) \tag{27}
$$

If $\Delta t, X$ is appropriately small, the Gronwall inequality can be applied to equation (27):

$$
\left\|dta^{M-1}\right\|^2+\left\|a^M\right\|_1^2+\left\|a^{M-1}\right\|_1^2 \le C\left((\Delta t)^4+h^{2m+2}\right) \tag{28}
$$

It can be seen immediately that $h, \Delta t$ is sufficiently small and $\max\limits_{0 \le n \le M-1}\left\|dta^n\right\|_{L^\infty} \le Q$, so the inductive hypothesis holds for m = N - 1 [18].

Theorem: If a_{ij}, b_i, f, h and u satisfy the above conditions, $m+1 > \frac{d}{2}$, $m \ge 1$, $\lambda \ge \frac{dC_1}{4}$, then when h, Δt are sufficiently small:

$$
\max_{0 \le n \le M-1}\left\{\left\|dt(U-u)^{n-1}\right\|+\left\|(U-u)^{n-\frac{1}{2}}\right\|+h\left\|(U-u)^{n-\frac{1}{2}}\right\|_1\right\} \le C\left(h^{m+1}+(\Delta t)^2\right) \tag{29}
$$

If the super-convergence analysis is completed by $V = U^{n+1} - U^{n-1}$ in the question (1), then the question (1) is stable.

2.2 Fully Discrete Convergence Analysis of Fourth-Order Non-linear Hyperbolic Equations

2.2.1 Question Description

The following fourth-order non-linear hyperbolic equations are considered:

$$
\begin{cases}
u_{tt}+\gamma\Delta^2 u-\Delta u_t+f(u)=0, (X,t) \in \Omega \times (0,T] \\
u(X,t)=\Delta u(X,t)=0, (X,t) \in \partial\Omega \times (0,T] \\
u(X,0)=u_0(X), u_t(X,0)=u_1(X), X \in \Omega
\end{cases} \tag{30}
$$

where $\Omega \in R^2$is a bounded convex polygon region with Lipschitz continuous boundary, $\partial\Omega$ is the boundary of Ω, $T \in (0,+\infty)$, γ is a positive fixed value, $X = (x,y)$, $f(u)$ is a global Lipschitz continuous about u, that is, there exists a constant C greater than 0. Let

$$|f(u_1) - f(u_2)| \leq C|u_1 - u_2| \tag{31}$$

In this paper, $W^{m,p}$ is used to denote the usual Sobolev spaces, whose norms and seminorms are denoted as $\|\cdot\|_{m,p}$ and $|\cdot|_{m,p}$, respectively. Especially when $p = 2$, $W^{m,p}$ is denoted as $H^m(\Omega)$, and the corresponding norms and seminorms are denoted as $\|\cdot\|_m$ and $|\cdot|_m$.

$$\|\varphi\|_{L^\infty(0,Y;H^K(\Omega))} \triangleleft \|\varphi\|_{H^K(\Omega)}, \|\varphi\|_{L^2(0,Y;H^K(\Omega))} \triangleleft \left(\int_0^t \|\varphi\|^2{}_{H^K(\Omega)} ds\right)^{\frac{1}{2}} \tag{32}$$

2.2.2 Fully Discrete Convergence Analysis of Fourth-Order Non-linear Hyperbolic Equations based on Finite Element Analysis

Let $\vec{p} = -\nabla u$, $v = \nabla\cdot\vec{p}$, then the problem equation (30) is equivalent to the following problem:

$$\begin{cases} u_{tt} + \gamma\Delta v + v + v_t + f(u) = 0, (X,t) \in \Omega\times(0,T] \\ v - \nabla\cdot\vec{p} = 0, (X,t) \in \Omega\times(0,T] \\ \vec{p} + \nabla u = 0, (X,t) \in \Omega\times(0,T] \\ u(X,t) = v(X,t) = 0, (X,t) \in \partial\Omega\times(0,T] \\ u(X,0) = u_0, u_t(X,0) = u_1, X \in \Omega \end{cases} \tag{33}$$

The variational question of equation (30) is to find $\{u,v,\vec{p}\}[0,T] \to H_0^1(\Omega)\times H_0^1(\Omega)\times\left(L^2(\Omega)\right)^2$ so that:

$$\begin{cases} (u_{tt},\phi) = \gamma(\nabla v,\nabla\phi) + (v,\phi) + (f(u),\phi) = 0, \forall\phi \in H_0^1(\Omega) \\ (v,\chi) + (\vec{p},\nabla\chi) = 0, \forall\phi \in H_0^1(\Omega) \\ (\vec{p},\vec{w}) + (\nabla u,\vec{w}) = 0, \forall\vec{w} \in \left(L^2(\Omega)\right)^2 \\ u(X,0) = u_0, u_t(X,0) = u_1, X \in \Omega \end{cases} \tag{34}$$

Considering the semi-discrete scheme of equation (34), $\{u_h, v_h, \vec{p}_h\} : [0,T] \to M_h\times M_h\times\vec{W}_h$ is obtained, so that:

$$\begin{cases} (u_{htt},\phi_h) + \gamma(\nabla v_h,\nabla\phi_h) + (v_h,\phi_h) + (v_{ht},\phi_h) + (f(u_h),\phi_h) = 0, \forall\phi_h \in M_h \\ (v_h,\chi_h) + (\vec{p}_h,\nabla\chi_h) = 0, \forall\chi_h \in M_h \\ (\vec{p}_h,\vec{w}_h) + (\nabla u_h,\vec{w}_h) = 0, \forall\vec{w}_h \in \vec{W}_h \\ u_h(0) = R_h u_0, u_{ht}(0) = R_h u_1, X \in \Omega \\ u_h(0) = \widehat{R}_h(-\nabla u_0), X \in \Omega \end{cases} \tag{35}$$

It is easy to verify that equation (35) has unique solutions. The super-approximation and super-convergence results of mixed element solutions are given in the semi-discrete scheme [19].

Theorem 1: Supposing that $\{u,v,\vec{p}\}$ and $\{u_h,v_h,\vec{p}_h\}$ are the solutions of equation (30) and equation (35), respectively. When $u,v \in H^3(\Omega)$, u_{tt}, v_t, $v_{tt} \in H^2(\Omega)$, $\vec{p} \in \left(H^2(\Omega)\right)^2$, there are the following super-approximation properties:

$$\|I_h u - u_h\|_1 \leq Ch^2\left[\|u\|_3 + \left(\int_0^t R_2 ds\right)^{\frac{1}{2}}\right] \tag{36}$$

$$\|I_h v - v_h\|_1 \leq Ch^2\left[\|v\|_3 + \left(\|v\|_2^2 + \int_0^t R_1 ds\right)^{\frac{1}{2}}\right] \tag{37}$$

$$\|\Pi h\vec{p}-\vec{p}_h\|_0 \le Ch^2\left[\|\vec{p}\|_2+\|v\|_3+\left(\int_0^t R_2 ds\right)^{\frac{1}{2}}\right] \tag{38}$$

where $R_1 = \|u\|_2^2+\|u_{tt}\|_2^2+\|v\|_2^2+\|v_{tt}\|_2^2, R_2 = \|v_t\|_2^2+R_1$.

Let:

$$u-u_h = (u-R_h u)+(R_h u-u_h) = \eta+\xi \tag{39}$$

$$v-v_h = \left(v-\widehat{R}_h v\right)+\left(\widehat{R}_h v-v_h\right) = \tau+\tau \tag{40}$$

$$\vec{p}-\vec{p}_h = \left(\vec{p}-\widehat{R}_h\vec{p}\right)+\left(\widehat{R}_h\vec{p}-\vec{p}_h\right)=\vec{p}+\vec{\theta} \tag{41}$$

From equations (30) and (35), the following error equation can be obtained:

$$\begin{cases} (\xi_{tt},\phi_h)+\gamma(\nabla\tau,\nabla\phi_h)+(\tau,\phi_h)+(\tau_t,\phi_h) \\ =(\eta_{tt},\phi_h)-(\tau,\phi_h)-(\tau_t,\phi_h)-(f(u)-f(u_h),\phi_h) \\ (\tau,\chi_h)+\left(\vec{\theta},\nabla\chi_h\right) = -(\tau,\chi_h) \\ \left(\vec{\theta},\vec{w}_h\right)+(\nabla\xi,\vec{w}_h)=0 \end{cases} \tag{42}$$

In equation (42), $\phi_h = \tau_t$ in formula 1, and for t in the second and third formulas, derivatives are obtained. And then $\chi_h = \xi_{tt}$ and $\vec{w}_h = \nabla\xi_{tt}$, respectively, there are:

$$\begin{cases} (\xi_{tt},\tau_t)+\gamma(\nabla\tau,\nabla\tau_t)+(\tau,\tau_t)+(\tau_t,\tau_t) \\ =-(\eta_{tt},\tau_t)-(\tau,\tau_t)-(\tau_t,\tau_t)-(f(u)-f(u_h),\tau_t) \\ (\tau_t,\xi_{tt})+\left(\vec{\theta}_t,\nabla\xi_{tt}\right) = -(\tau,\xi_{tt}) \\ \left(\vec{\theta}_t,\nabla\xi_{tt}\right)+(\nabla\xi_t,\nabla\xi_{tt})=0 \end{cases} \tag{43}$$

According to equation (43):

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\|\nabla\xi_t\|_0^2+\tfrac{\gamma}{2}\tfrac{d}{dt}\|\nabla\tau\|_0^2+\tfrac{1}{2}\tfrac{d}{dt}\|\tau\|_0^2+\|\tau\|_0^2 \\ &=-(\eta_{tt},\tau_t)-(r,\tau_t)-(r_t,\tau_t)-(f(u)-f(u_h),\tau_t)+(\tau_t,\xi_{tt}) = \sum_{i=1}^{5} A_i \end{aligned} \tag{44}$$

Let estimate $A_i\,(i=1,2,...,5)$ in turn.

Using Schwarz inequality, Young inequality and interpolation theory, the following conclusions are obtained:

$$\sum_{i=1}^{2} A_i \le C(\|\eta_{tt}\|_0+\|r\|_0)\|\tau_t\|_0 \le Ch^4\left(\|u_{tt}\|_2^2+\|v_{tt}\|_2^2\right)+\frac{1}{2}\|\tau_t\|_2^2 \tag{45}$$

Since f satisfies the Lipschitz condition, there are:

$$A_4 \le C\|u-u_h\|_0\|\tau_t\|_0 \le C\left(\|\xi\|_0^2+h^4\|u\|_2^2+\frac{1}{2}\|\tau_t\|_0^2\right) \tag{46}$$

Using derivative transfer techniques, there are:

$$A_3 = (r_{tt},\tau)-\frac{d}{dt}(r_t,\tau) \le C\left(h^4\|v_{tt}\|_2^2+\|\tau\|_0^2\right)-\frac{d}{dt}(r_t,\tau) \tag{47}$$

$$A_5 = (r_{tt},\xi_t)+\frac{d}{dt}(r_t,\xi_t) \le C\left(h^4\|v_{tt}\|_2^2+\|\xi_t\|_0^2\right)-\frac{d}{dt}(r_t,\xi_t) \tag{48}$$

By introducing the above estimates of A_i into equation (44), we can obtain that:

$$\begin{aligned}&\tfrac{\min\{1,\gamma\}}{2}\tfrac{d}{dt}\left(\|\xi_t\|_1^2+\|\tau\|_1^2\right)\le Ch^4\left(\|u\|_2^2+\|u_{tt}\|_2^2+\|v\|_2^2+\|u_{tt}\|_2^2\right)\\&+C\left(\|\xi\|_0^2+\|\xi_t\|_0^2+\|\tau\|_0^2\right)-\tfrac{d}{dt}\left[(r_t,\tau)-(r_t,\xi_t)\right]\end{aligned} \tag{49}$$

The two ends of the equation are multiplied by $\frac{2}{\min\{1,\gamma\}}$, and then integrated from 0 to t. It is noted that $\tau(0)=\xi_t(0)=\xi(0)=0$, $\|\xi\|_0^2\le C\int_0^t\|\xi_t\|_0^2ds$, and then according to inequality $\int_0^t\int_0^s\varphi^2dsd\tau\le C\int_0^t\varphi^2ds$, there are:

$$\begin{aligned}&\|\xi_t\|_1^2+\|\tau\|_1^2\le Ch^4\int_0^t\left(\|u\|_2^2+\|u_{tt}\|_2^2+\|v\|_2^2+\|u_{tt}\|_2^2\right)ds\\&+C\int_0^t\left(\|\tau\|_1^2+\|\xi_t\|_1^2+\|\xi\|_0^2\right)ds-\tfrac{1}{\min\{1,\gamma\}}\left[(r_t,\tau)-(\xi_t)\right]\\&\le Ch^4\left[\|v_t\|_2^2+\int_0^t\left(\|u\|_2^2+\|u_{tt}\|_2^2+\|v\|_2^2+\|u_{tt}\|_2^2\right)ds\right]\\&+C\int_0^t\left(\|\tau\|_1^2+\|\xi_t\|_1^2\right)ds+\tfrac{1}{2}\left(\|\tau\|_1^2+\|\xi_t\|_1^2\right)\end{aligned} \tag{50}$$

The Gronwall inequality is used to obtain:

$$\|\xi_t\|_1+\|\tau\|_1\le Ch^2\left(\|v_t\|_2^2+\int_0^t R_1ds\right)^{\frac{1}{2}} \tag{51}$$

On the other hand, if it is noticed that $\xi(0)=\nabla\xi(0)=0$, it can get from $\|\xi\|_1^2\le C\int_0^t\|\xi_t\|_1^2ds$:

$$\|\xi\|_1^2\le C\int_0^t\|\xi_t\|_1^2ds\le Ch^4\left[\int_0^t\left(\|v_t\|_2^2+\int_0^s R_1d\tau\right)\le Ch^4\int_0^t R_2ds\right] \tag{52}$$

Namely:

$$\|\xi\|_1\le Ch^2\left(\int_0^t R_2ds\right)^{\frac{1}{2}} \tag{53}$$

In the third form of equation (42), $\vec{w}_h=\vec{\theta}$ is obtained from Schwarz inequality.

$$\left\|\vec{\theta}\right\|_0^2=\left(\vec{\theta},\vec{\theta}\right)=-\left(\nabla\xi,\vec{\theta}\right)\le C\left\|\nabla\xi\;\|0\|\;\vec{\theta}\right\|\Big\|_0 \tag{54}$$

According to equation (54), there are:

$$\left\|\vec{\theta}\right\|_0<C\|\xi\|_1\le Ch^2\left(\int_0^t R_2ds\right)^{\frac{1}{2}} \tag{55}$$

By using the trigonometric inequality, it can obtain:

$$\|I_hu-u_h\|_1\le C\left(\|I_hu-R_hu\|_1+\|R_hu-u_h\|_1\right)\le Ch^2\left[\|u\|_3+\left(\int_0^t R_2ds\right)^{\frac{1}{2}}\right] \tag{56}$$

The theorem is proved.

In order to obtain global super-convergence, the adjacent four elements are merged into one large element processing operators I_{2h}^2 and Π_{2h}^2:

$$\begin{cases}I_{2h}^2I_hu=I_{2h}^2u,\forall u\in H^2(\Omega)\\\left\|I_{2h}^2u-u\right\|_1\le Ch^2\|u\|_3,\forall u\in H^3(\Omega)\\\left\|I_{2h}^2u_h\right\|_1\le C\|u_h\|_1,\forall u_h\in M_h\end{cases} \tag{57}$$

$$\begin{cases} \Pi_{2h}^2 \Pi_h \vec{p} = \Pi_{2h}^2 \vec{p}, \forall \vec{p} \in (H^2(\Omega))^2 \\ \left\|\Pi_{2h}^2 \vec{p} - \vec{p}_0\right\|_0 \le Ch^2 \|\vec{p}\|_2, \forall \vec{p} \in (H^2(\Omega))^2 \\ \left\|\Pi_{2h}^2 \vec{p}_h\right\|_0 \le C\|\vec{p}_h\|_1, \forall \vec{p}_h \in \vec{W}_h \end{cases} \tag{58}$$

Combining theorem 1 and the properties of operators above, the global super-convergence results can be obtained as follows.
Theorem 2: Under the condition of Theorem 1, there are:

$$\left\|I_{2h}^2 u_h - u\right\|_1 \le Ch^2 \left[\|u\|_3 + \left(\int_0^t R_2 ds\right)^{\frac{1}{2}}\right] \tag{59}$$

$$\left\|I_{2h}^2 v_h - v\right\|_1 \le Ch^2 \left[\|v\|_3 + \left(\|v_t\|_2^2 + \int_0^t R_1 ds\right)^{\frac{1}{2}}\right] \tag{60}$$

$$\left\|\Pi_{2h}^2 \vec{p}_h - \vec{p}\right\|_0 \le Ch^2 \left[\|\vec{p}\|_2 + \left(\|v\|_3 + \int_0^t R_2 ds\right)^{\frac{1}{2}}\right] \tag{61}$$

It is proved that: According to equations (36) and (57), using trigonometric inequalities, the following results are obtained:

$$\begin{aligned} &\left\|I_{2h}^2 u_h - u\right\|_1 = \left\|I_{2h}^2 u_h - I_{2h}^2 I_h u + I_{2h}^2 I_h u - u\right\|_1 \\ &\le C\|u_h - I_h u\|_1 + C\left\|I_{2h}^2 u - u\right\|_1 \\ &\le C\|u_h - I_h u\|_1 + Ch^2\|u\|_3 \\ &\le Ch^2 \left[\|u\|_3 + \left(\int_0^t R_2 ds\right)^{\frac{1}{2}}\right] \end{aligned} \tag{62}$$

According to equation (57) and equation (37), using trigonometric inequalities, similar proofs can be obtained as follows:

$$\left\|I_{2h}^2 v_h - v\right\|_1 = Ch^2 \left[\|v\|_3 + \left(\|v_t\|_2^2 + \int_0^t R_1 ds\right)^{\frac{1}{2}}\right] \tag{63}$$

According to equations (58) and (38), the triangular inequality is used to obtain:

$$\begin{aligned} &\left\|\Pi_{2h}^2 \vec{p}_h - \vec{p}\right\|_0 = \left\|\Pi_{2h}^2 \vec{p}_h - \Pi_{2h}^2 \Pi_h \vec{p} + \Pi_{2h}^2 \Pi_h \vec{p} - \vec{p}\right\|_0 \\ &\le C\|\vec{p}_h - \Pi_h \vec{p}\|_0 + C\left\|\Pi_{2h}^2 \vec{p} - \vec{p}\right\|_0 \\ &\le C\|\vec{p}_h - \Pi_h \vec{p}\|_0 + Ch^2\|\vec{p}\|_2 \\ &\le Ch^2 \left[\|\vec{p}\|_2 + \|v\|_3 + \left(\int_0^t R_2 ds\right)^{\frac{1}{2}}\right] \end{aligned} \tag{64}$$

The theorem can be proved.
Note 1: If interpolation is used directly and the high precision results of bilinear elements Q_{11} and $Q_{10} \times Q_{10}$ are used, and the reciprocal transfer technique of time t is used, when $u, u_t, u_{tt}, v, v_t \in H^3(\Omega)$, $v_{tt} \in H^2(\Omega)$v and $\vec{p} \in \left(H^2(\Omega)\right)^2$, the following super-approximation results are obtained:

$$\|I_h u - u_h\|_1 \le Ch^2 \left[\int_0^t \left(\|u\|_2^2 + \|u_{tt}\|_3^2 + \|v_t\|_3^2 + \|v_{tt}\|_2^2 + \|u_t\|_3^2 + \|v\|_3^2\right) ds\right]^{\frac{1}{2}} \tag{65}$$

$$\|I_h u - u_h\|_1 \le Ch^2 \left[\|u_t\|_2^2 + \|v_t\|_3^2 + \|v\|_3^2 + \int_0^t \left(\|u\|_2^2 + \|u_{tt}\|_3^2 + \|v\|_3^2 + \|v_t\|_3^2 + \|v_{tt}\|_2^2\right) ds\right]^{\frac{1}{2}} \tag{66}$$

$$\|\Pi_h \vec{p} - \vec{p}_h\|_0 \le Ch^2 \left[\|\vec{p}\|_2 + \|u\|_3 + \left[\int_0^t \left(\|u\|_2^2 + \|u_{tt}\|_3^2 + \|v_t\|_3^2 + \|v_{tt}\|_3^2 + \|v_t\|_3^2 + \|v\|_3^2\right) ds\right]^{\frac{1}{2}}\right] \tag{67}$$

Compared with theorem 1, we can see that the method of combining interpolation with projection is used to reduce the smoothness of u_t, u_{tt}, v_t.
Note 2: Many well-known incompatible elements can be verified, such as EQ_1^{rot} elements in rectangular meshes, Q_1^{rot} elements in square meshes or constrained rotating Q_1 elements (equivalent to P_1 incompatible elements in rectangular meshes), because their compatibility errors can only be estimated as follows:

$$\sum_{K\in\Gamma_h}\int_{\partial K}\frac{\partial v}{\partial n}\phi_h ds = O\left(h^2\right)\|v\|_3\|\phi_h\|_h = O\left(h\right)\|v\|_3\|\phi_h\|_0, \phi_h \in M_h \tag{68}$$

where $\|\cdot\|_h = \left(\sum_K |\cdot|_{1,K}^2\right)^{\frac{1}{2}}$ is a module on ,M_h so the result of Theorem 1 cannot be obtained up to now. However, under the condition of theorem 1, if the condition $v_t \in H^3\left(\Omega\right)$, $\vec{p}_t \in \left(H^2\left(\Omega\right)\right)^2$ is added, the super-approximation results with $O\left(h^2\right)$ orders in semi-discrete scheme can also be obtained by using the derivative transfer technique. The total degree of freedom of the mixed element scheme given here is only 4NP (where NP is the number of all nodes in the partition of Ω) [20].

3 Conclusions

Hyperbolic PDEs are PDEs describing vibration or wave phenomena. One of its typical examples is the wave equation and the wave equation when $n = 1$. It can be used to describe the small transverse vibration of string, which is called string vibration equation. This is the first PDE to be systematically studied. In the process of neural propagation, neural transmission signals and the rate of change in time and space are mathematically represented as a class of initial boundary value problems for non-linear quasi-hyperbolic equations. The non-linear hyperbolic equation is a new type of non-linear evolution equation with profound physical background. In this paper, the full discrete convergence analysis method of non-linear hyperbolic equation based on finite element analysis is used to analyse the full discrete convergence of second-order and fourth-order non-linear hyperbolic equation and obtain the super-convergence results. There is a certain value in the study of non-linear hyperbolic equation.

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