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A New Approach For Weighted Hardy's Operator In VELs

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Abstract

A considerable number of research has been carried out on the generalized Lebesgue spaces $L^{p(x)}$ and boundedness of different integral operators therein. In this study, a new approach for weighted increasing near the origin and decreasing near infinity exponent function that provides a boundedness of the Hardy's operator in variable exponent space is given.

Keywords: Hardy inequality, Variable exponent, Boundedness.

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1 Introduction

Variable exponent studies have been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with a non-standard growth condition, we refer to monographs [7, 11, 12, 13, 15, 18, 21, 23, 29, 30, 31]. The boundedness problems for weighted Hardy's operator in variable exponent Lebesgue spaces $L^{p(\cdot)}$ are well studied, we refer to monographs [4, 9, 14, 16, 17, 19, 20, 22, 24, 26, 27, 28]. In this connection, a necessary and sufficient condition that assumes a log-regularity of exponent function near origin and infinity has been proved in [1, 5, 6, 8, 10, 17, 23]. For a compactness problem of main integral operators in variable exponent Lebesgue space, we refer to [2, 3, 25] and especially for Hardy's operator [4].

In this paper, we establish an integral-type necessary and sufficiency condition on a monotone weighted exponent function $p : (0, \infty) \rightarrow (1, \infty)$ governing the boundedness

$$\left\| \frac{1}{x} v(x) \int_0^x f(t) w(t) dt \right\|_{L^{p(x)}(0, \infty)} \leq C \|f\|_{L^p(0, \infty)} \quad (1)$$

of Hardy's operator $\frac{1}{x} v(x) \int_0^x f(t) w(t) dt$ in $L^{p(x)}(0, \infty)$.

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The following main result has been obtained in this study.

Theorem 1.1. *Let $p : (0, \infty) \rightarrow (1, \infty)$ be a measurable function monotony increasing on some neighborhood of origin $(0, \varepsilon)$ and decreasing on some neighborhood of infinity (N, ∞) such that $p^+ < \infty$ and $p(\infty) > 1$. Then it holds the inequality*

$$\left\| \frac{1}{x} v(x) \int_0^x f(t) w(t) dt \right\|_{L^p(0, \infty)} \leq C_1 \|f\|_{L^p(0, \infty)}$$

for any positive measurable function $f(x)$ on $(0, \infty)$ with a positive constant C_1 depending on constant C_2 below and $p^+, p(\infty)$ if and only if the condition

$$\int_b^\infty x^{-\frac{1}{p'(x)}} \frac{dx}{x} \leq C_2 b^{-\frac{1}{p'(b)}}, \quad b > 0. \quad (2)$$

is satisfied.

We use the following notation. By C, C_i we denote a positive constant depending on $p(\infty), p^+$ and C_2 from the conditions (2). We use also notation $p^+ = \sup\{p(x) : x \in (0, \infty)\}$ and $p^- = \inf\{p(x) : x \in (0, \infty)\}$. Recall the norm in variable exponent Lebesgue space $L^{p(\cdot)}(0, \infty)$ given as $\|f\| = \inf\left\{\lambda > 0 : I_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1\right\}$ makes it as a Banach space, where the modular $I_{p(\cdot)}(f) = \int_0^\infty |f(t)|^{p(t)} dt$. Denote $\|f\|_{p(\cdot)}$ or $\|f\|_{L^{p(\cdot)}(0, \infty)}$ for the $L^{p(\cdot)}(0, \infty)$ variable exponent Lebesgue norm of function f . For a function $p : [0, \infty) \rightarrow [1, \infty)$ denote $p'(x)$ the function satisfying $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ and $p' = \infty$ if $p = 1$.

Denote by χ_E the characteristic function of set $E \subset \mathbb{R}$. The weight functions v and w are assumed to be measurable and having non-negative finite values almost everywhere in $(0, \infty)$.

2 Auxiliary Statements

In this section, we prove some auxiliary assertions in order to prove the main result of this paper.

Lemma 2.1. *Let the condition (2) be satisfied for a monotone increasing near the origin on $(0, \varepsilon)$ and decreasing near infinity on (N, ∞) function $p : (0, \infty) \rightarrow (1, \infty)$ such that $p^+ < \infty$. Then there exists a positive constant C_3 depending on C_2, p^+ such that*

$$[p(2b) - p(b)] \ln \frac{1}{b} \leq C_3 \text{ for } b \in (0, \delta) \quad (3)$$

and

$$[p(b) - p(2b)] \ln b \leq C_4 \text{ for } b \in (N, \infty). \quad (4)$$

Proof. The proof for $b \in (0, 1)$ similar to those in (see [10] the Lemma 4.1 therein). For the completeness of presentation, we consider here the both cases $b \in (0, \varepsilon)$ and $b \in (N, \infty)$.

Let the case $b \in (0, 1)$ be considered. Write the condition (2) over interval $(0, 1)$:

$$\int_b^\infty x^{-\frac{1}{p'(x)}} \frac{dx}{x} \leq C_2 b^{-\frac{1}{p'(b)}}, \quad 0 < b < \frac{1}{4}.$$

By monotony increasing of p , $x^{-\frac{1}{p'(x)}} \geq (4b)^{-\frac{1}{p'(2b)}}$ for $x \in (2b, 4b)$, where $0 < b < \frac{1}{4}$. From this, it follows that

$$C_2 b^{-\frac{1}{p'(b)}} \geq \int_b^{2b} x^{-\frac{1}{p'(x)}} \frac{dx}{x} \geq (4b)^{-\frac{1}{p'(2b)}} \ln 2.$$

Therefore, for $0 < b < \frac{1}{4}$, we have

$$\left(\frac{1}{b}\right)^{\frac{1}{p(b)} - \frac{1}{p(2b)}} \leq \frac{C_2}{\ln 2} 4^{\frac{1}{p'(2b)}} \leq \frac{4C_2}{\ln 2} = C_3.$$

This proves (3).

Let $b > N$ and $p(x)$ decreases near infinity, say for $x > M$. Then $x^{-\frac{1}{p'(x)}} \geq x^{-\frac{1}{p'(2b)}}$ for $x \in (b, 2b)$ and $b > N$. Therefore, from condition (2) over (N, ∞) , it follows that,

$$\int_b^\infty x^{-\frac{1}{p'(x)}} \frac{dx}{x} \leq C_2 b^{-\frac{1}{p'(b)}}, \quad b > N.$$

Then

$$C_2 b^{-\frac{1}{p'(b)}} \geq \int_b^{2b} x^{-\frac{1}{p'(x)}} \frac{dx}{x} \geq \int_b^{2b} x^{-\frac{1}{p'(2b)}} \frac{dx}{x} \geq a^{-\frac{1}{p'(2b)}} \ln 2.$$

and a decreasing of p over $(b, 2b)$ provides

$$x^{-\frac{1}{p'(2b)}} \geq 2^{\frac{1}{p'(2b)}} b^{-\frac{1}{p'(2b)}} \geq b^{-\frac{1}{p'(2b)}} \ln 2.$$

Therefore, for $b > N$, we have

$$b^{\frac{1}{p(2b)} - \frac{1}{p(b)}} \leq \frac{C_2}{\ln 2} = C_5 \quad \text{or} \quad b^{p(b) - p(2b)} \leq C_6.$$

Whence,

$$b^{p(b) - p(2b)} \leq C_6, \quad b > N$$

where $C_6 = C_5^{p(N)^2}$. Therefore (4) satisfied by a constant $C_4 = (p^+)^2 \ln C_5$. This proves Lemma 2.1

Lemma 2.2. Let the condition (2) be satisfied for a monotone increasing near origin on $(0, \varepsilon)$ and decreasing near infinity on (N, ∞) function $p : (0, \infty) \rightarrow (1, \infty)$ such that $p^+ < \infty$. Then there exists an $\delta > 0$ depending on C_2, p^+ such that the function $x^{\delta - \frac{1}{p'(x)}}$ is almost decreasing near origin $(0, \varepsilon)$ and infinity (N, ∞) .

Proof. Set $g(x) = \int_x^\infty t^{-\frac{1}{p'(t)}} \frac{dt}{t}$, $x > 0$. From (2), it follows that

$$g(x) \leq -C_2 x g'(x), \quad 0 < x < \varepsilon.$$

First, integrating this inequality over (t_1, t_2) with $0 < t_1 < t_2 < \varepsilon$, we get

$$t_2^{\frac{1}{C_2}} g(t_2) \leq C_2 t_1^{-\frac{1}{p'(t_1)} + \frac{1}{C_2}}. \quad (5)$$

On the other hand, using the conditions (2) and (3) for $0 < t_2 < \varepsilon$, we get

$$g(t_2) \geq \int_{t_2}^{2t_2} x^{-\frac{1}{p'(x)}} \frac{dx}{x} \geq \left(\frac{1}{2}\right)^{\frac{1}{p'(2t_2)}} \ln 2 \left(\frac{1}{t_2}\right)^{\frac{1}{p'(t_2)}} \geq \frac{\ln 2}{2} t_2^{-\frac{1}{p'(t_2)}}.$$

Inserting this in (5) for all $0 < t_1 < t_2 < \varepsilon$ we get

$$t_2^{-\frac{1}{p'(t_2)} + \frac{1}{c_2}} \leq \frac{2C_2}{\ln 2} t_1^{-\frac{1}{p'(t_1)} + \frac{1}{c_2}}.$$

Now, let $N < t_1 < t_2 < \infty$. We will show that again (5) holds for all $N < t_1 < t_2 < \infty$. From (3), it follows that

$$g(x) \leq -C_2 x g'(x), \quad N < x < \infty.$$

Integrating this inequality over (t_1, t_2) for $N < t_1 < t_2 < \infty$ from (2), it follows that

$$g(x) \leq -C_2 x g'(x), \quad N < x < \infty.$$

Integrating this inequality over (t_1, t_2) , for $N < t_1 < t_2 < \infty$, we get

$$t_2^{\frac{1}{c_2}} g(t_2) \leq C_2 t_1^{-\frac{1}{p'(t_1)} + \frac{1}{c_2}}. \quad (6)$$

On the other hand, using conditions (2) and (4), we get

$$g(t_2) \geq \int_{t_2}^{2t_2} x^{-\frac{1}{p'(x)}} \frac{dx}{x} \geq \ln 2 \left(\frac{1}{2t_2} \right)^{\frac{1}{p'(2t_2)}} \geq 2^{-\frac{1}{p'(N)}} \ln 2 t_2^{-\frac{1}{p'(t_2)}}, \quad t_2 \geq N.$$

Inserting this in (6), we obtain,

$$t_2^{-\frac{1}{p'(t_2)} + \frac{1}{c_2}} \leq \frac{2C_2}{\ln 2} t_1^{-\frac{1}{p'(t_1)} + \frac{1}{c_2}}$$

for all $N < t_1 < t_2 < \infty$. Whence, the function $x^{\delta - \frac{1}{p'(x)}}$ is almost decreasing on (N, ∞) . Therefore, we have proved that

$$t_2^{-\frac{1}{p'(t_2)} + \delta} \leq C_7 t_1^{-\frac{1}{p'(t_1)} + \delta}$$

for all $0 < t_1 < t_2 < \varepsilon$ and $N < t_1 < t_2 < \infty$. The constants $C_7 = \frac{2C_2}{\ln 2}$, $\delta = \frac{1}{c_2}$.

This proves Lemma 2.2

Lemma 2.3. *Let the condition (2) be satisfied for a monotone increasing near origin on $(0, \varepsilon)$ and decreasing near infinity on (N, ∞) function $p : (0, \infty) \rightarrow (1, \infty)$ such that $p^+ < \infty$. Then for $0 < x < \varepsilon$ or for $x > N$ and $t \in (2^{-n-1}x, 2^{-n}x)$, the following inequality holds:*

$$x^{-\frac{1}{p'(x)}} \leq C_7 2^{-n\delta} t^{-\frac{1}{p'(t)}}.$$

Proof. By applying Lemma 2.2 for $2^{-n-1}x < t < 2^{-n}x$, $0 < x < \varepsilon$ or $x > N$, we have

$$x^{\delta - \frac{1}{p'(x)}} \leq C_7 t^{\delta - \frac{1}{p'(t)}},$$

since in both cases $t < x$. Indeed, using that $2^{-n-1}x < t < 2^{-n}x$ and Lemma 2.2, we get

$$x^{-\frac{1}{p'(x)}} \leq C_7 \left(\frac{t}{x} \right)^{\delta} t^{-\frac{1}{p'(t)}} \leq C_7 2^{-n\delta} t^{-\frac{1}{p'(t)}}$$

or

$$x^{-\frac{1}{p'(x)}} \leq C_7 2^{-n\delta} t^{-\frac{1}{p'(t)}} \quad \text{for } 0 < x < \varepsilon \quad \text{or } x > N.$$

This completes the proof of Lemma 2.3

By applying preceding lemmas, we easily get the following assertion.

Lemma 2.4. Let the condition (2) be satisfied for a monotone increasing near origin on $(0, \varepsilon)$ and decreasing near infinity on (N, ∞) function $p : (0, \infty) \rightarrow (1, \infty)$ such that $p^+ < \infty$. Then for $0 < x < \varepsilon$ or $x > N$ and $t \in (2^{-n-1}x, 2^{-n}x)$, the following estimate holds

$$(2^{-n}x)^{\frac{1}{(p_{x,n}^-)'}} \leq C_8 t^{\frac{1}{p'(t)}}, \quad (7)$$

where $p_{x,n}^- = \inf\{p(s) : s \in (2^{-n-1}x, 2^{-n}x)\}$.

Proof. Using the assertions of Lemma 2.1 and condition (2), we get

$$\begin{aligned} (2^{-n}x)^{\frac{1}{(p_{x,n}^-)'}} &\leq (2t)^{\frac{1}{(p_{x,n}^-)'}} \leq 2t^{\frac{1}{(p_{x,n}^-)'}} \\ &\leq 2t^{\frac{1}{p'(t)}} t^{\frac{1}{(p_{x,n}^-)' - \frac{1}{p'(t)}}} \leq 2t^{\frac{1}{p'(t)}} t^{\frac{p_{x,n}^- - p(t)}{p_{x,n} p(t)}} \leq 2t^{\frac{1}{p'(t)}} \left(\frac{1}{t}\right)^{\frac{p(t) - p_{x,n}^-}{p_{x,n} p(t)}} \\ &\leq C_8 t^{\frac{1}{p'(t)}}. \end{aligned}$$

3 Proof of Theorem 1.1

3.1 Sufficiency

Let the condition (2) be satisfied. we will show that inequality (1) holds. Take a measurable positive function f with

$$\|f\|_{p(\cdot)} \leq 1. \quad (8)$$

In order to prove sufficiency, we have to show that $\|\frac{1}{x}v(x) \int_0^x f(t)w(t)dt\|_{p(\cdot)} \leq 1$ or the same is to show that $I_{p(\cdot)}(\frac{1}{x}v(x) \int_0^x f(t)w(t)dt) \leq 1$ (see, e.g. in [10]).

Using Minkowski's inequality for $p(\cdot)$ -norms we have

$$\begin{aligned} &\left\| \frac{1}{x}v(x) \int_0^x f(t)w(t)dt \right\|_{p(\cdot);(0,\infty)} \\ &\leq \left\| \frac{1}{x}v(x) \int_0^x f(t)w(t)dt \right\|_{p(\cdot);(0,N)} + \left\| \frac{1}{x}v(x) \int_0^x f(t)w(t)dt \right\|_{p(\cdot);(N,\infty)} \end{aligned} \quad (9)$$

For the first summand it follows from the results of the paper [10] (for $\alpha = 0$ therein) that it may be estimated as

$$\left\| \frac{1}{x}v(x) \int_0^x f(t)w(t)dt \right\|_{p(\cdot);(0,N)} \leq C_1 \|f\|_{p;(0,N)} \leq 1$$

since an integration interval $(0, N)$ is finite in this part and condition (2) is satisfied.

Now, we pass to an estimation for a second summand in (9), i.e. we get an estimate for the term $i =$
 $\left\| x^{-1} \int_0^x f(t)dt \right\|_{L^{p(\cdot)}(N,\infty)}$. By using Minkowski's inequality for $p(\cdot)$ -norms, it follows that,

$$\begin{aligned} \left\| x^{-1}v(x) \int_0^x f(t)w(t)dt \right\|_{L^{p(\cdot)}(N,\infty)} &= \left\| x^{-1}v(x) \int_0^x f(t)\chi_{\{t>N\}}(t)w(t)dt \right\|_{L^{p(\cdot)}(N,\infty)} \\ &+ \left\| x^{-1}v(x) \int_0^N f(t)w(t)dt \right\|_{L^{p(\cdot)}(N,\infty)}. \end{aligned} \quad (10)$$

Since $p^- > 1$ and $\|f\|_{p(\cdot)} \leq 1$ it follows that

$$\int_0^N f(t) dt \leq \|f\|_{p(\cdot)} \|\chi_{\{0 < t < N\}}\|_{p'(\cdot)} \leq \|\chi_{\{0 < t < N\}}\|_{p'(\cdot)} \leq N^{\frac{1}{(p^+)'}}.$$

Whence

$$\begin{aligned} & \left\| x^{-1} v(x) \int_0^N f(t) w(t) dt \right\|_{L^{p(\cdot)}(N, \infty)} \\ & \leq \|\chi_{\{0 < t < N\}}\|_{p'(\cdot)} \|x^{-1}\|_{L^{p(\cdot)}(N, \infty)} \leq C_9(N, p^+, p(\infty)) \end{aligned} \quad (11)$$

since $p(\infty) > 1$,

$$I_{p(\cdot)}(t^{-1} \chi_{(N, \infty)}(t)) = \int_N^\infty \frac{dt}{t^{p(t)}} \leq \int_N^\infty \frac{dt}{t^{p(\infty)}} = \frac{N^{1-p(\infty)}}{p(\infty) - 1}$$

and therefore,

$$\|x^{-1}\|_{L^{p(\cdot)}(n, \infty)} \leq N^{\frac{1-p(\infty)}{p(\infty)}} (p(\infty) - 1)^{-\frac{1}{p(\infty)}}.$$

Thus, in order to get an estimation for i it suffices to estimate

$$\left\| x^{-1} v(t) \int_0^x f(u) \chi_{\{t > N\}}(t) w(t) dt \right\|_{L^{p(\cdot)}(N, \infty)}.$$

We have

$$\begin{aligned} & \left\| x^{-1} v(x) \int_N^x f(t) w(t) dt \right\|_{L^{p(\cdot)}(N, \infty)} \leq \left\| x^{-1} \sum_{n=0}^\infty v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t > N\}}(t) w(t) dt \right\|_{L^{p(\cdot)}(N, \infty)} \\ & \leq \sum_{n=0}^\infty \left\| x^{-1} v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t > N\}}(t) w(t) dt \right\|_{L^{p(\cdot)}(N, \infty)}. \end{aligned} \quad (12)$$

We shall estimate every summand on the right hand side. By using Lemma 2.3 for $N < t < x < \infty$, it follows that $x^{-\frac{1}{p'(x)}} \leq C_7 2^{-n\delta} t^{-\frac{1}{p'(t)}}$. Therefore

$$\begin{aligned} & x^{-\frac{1}{p'(x)}} x^{-\frac{1}{p'(x)}} \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t > N\}}(t) dt \\ & \leq C_7 2^{-n\delta} x^{-\frac{1}{p'(x)}} t^{-\frac{1}{p'(x)}} \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t > N\}}(t) dt. \end{aligned}$$

By using Hölder's inequality for $p(x)$ -norms and assumption (8), we get

$$\begin{aligned} & \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t > N\}}(t) dt \\ & \leq C_0 \|f(t) \chi_{\{t > N\}}(t)\|_{L^{p(\cdot)}(2^{-n-1}x, 2^{-n}x)} \times \|\chi_{\{t > N\}}(\cdot)\|_{L^{p'(\cdot)}(2^{-n-1}x, 2^{-n}x)} \end{aligned}$$

$$\leq \left\| \chi_{\{t>N\}}(\cdot) \right\|_{L^{p'(\cdot)}(2^{-n-1}x, 2^{-n}x)}.$$

Lemma 3.1. *There exists a positive constant $C_9 > 1$ depending on $C_2, p^+, p(\infty)$, such that*

$$\left\| \chi_{\{t>N(\cdot)\}} \chi_{(2^{-n-1}x, 2^{-n}x)}(\cdot) \right\|_{L^{p'(\cdot)}} \leq C_9 y^{\frac{1}{p'(y)}}, \quad \forall y \in (2^{-n-1}x, 2^{-n}x) \quad (13)$$

for $x > N$.

Proof. We prove estimation (13) from the opposite. Let inequality (13) be violated. We show that a contradiction occurs. Let

$$\left\| \chi_{\{t>N\}}(\cdot) \chi_{(2^{-n-1}x, 2^{-n}x)}(\cdot) \right\|_{L^{p'(\cdot)}} > C_9 y^{\frac{1}{p'(y)}}. \quad (14)$$

From the definition of $p'(\cdot)$ -norms, it follows that for any sufficiently small $\delta > 0$ the inequality holds

$$\int_{2^{-n-1}x}^{2^{-n}x} \left(\frac{1}{\left\| \chi_{\{t>N\}}(\cdot) \chi_{(2^{-n-1}x, 2^{-n}x)}(\cdot) \right\|_{L^{p'(\cdot)}} - \delta} \right)^{p'(t)} dt > 1. \quad (15)$$

Indeed, if (15) does not hold, we get a contradiction, that is,

$$\left\| \chi_{\{t>N\}}(\cdot) \chi_{(2^{-n-1}x, 2^{-n}x)}(\cdot) \right\|_{L^{p'(\cdot)}}$$

is a list number satisfying

$$\int_{2^{-n-1}x}^{2^{-n}x} \left(\frac{1}{\lambda} \right)^{p'(t)} dt \leq 1.$$

We set $\delta = \frac{1}{2} \left\| \chi_{\{t>N\}}(\cdot) \chi_{(2^{-n-1}x, 2^{-n}x)}(\cdot) \right\|_{L^{p'(\cdot)}}$ in (15), to get

$$\int_{2^{-n-1}x}^{2^{-n}x} \left(\frac{2}{\left\| \chi_{\{t>N\}}(\cdot) \chi_{(2^{-n-1}x, 2^{-n}x)}(\cdot) \right\|_{L^{p'(\cdot)}}} \right)^{p'(t)} dt > 1.$$

By applying here (14), it follows that

$$\int_{2^{-n-1}x}^{2^{-n}x} \left(\frac{2}{C_9 y^{\frac{1}{p'(y)}}} \right)^{p'(t)} dt > 1, \quad \forall y \in (2^{-n-1}x, 2^{-n}x), \quad x > N.$$

For any points t, y lying in $(2^{-n-1}x, 2^{-n}x)$ it follows from Lemma 2.1 that $y^{\frac{1}{p'(y)}}$ is comparable with $t^{\frac{1}{p'(t)}}$. This means, there exist two positive constants C_{10}, C_{11} depending on C_2, p^+ such that

$$C_{10} t^{\frac{1}{p'(t)}} \leq y^{\frac{1}{p'(y)}} \leq C_{11} t^{\frac{1}{p'(t)}}.$$

Therefore,

$$\int_{2^{-n-1}x}^{2^{-n}x} \left(\frac{2}{C_{10} C_9 t^{\frac{1}{p'(t)}}} \right)^{p'(t)} dt > 1, \quad x > N.$$

Whence

$$\frac{2(p^-)' \ln 2}{(C_{10}C_9)^{(p^+)'}} \geq 1. \quad (16)$$

By choosing constant C_9 sufficiently large, we get a contradiction with (16). This proves estimate (13). Thus we have proved that

$$\int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t>N\}}(t) dt \leq C_9 y^{\frac{1}{p'(y)}}, \quad \forall y \in (2^{-n-1}x, 2^{-n}x), \quad x > N. \quad (17)$$

This proves Lemma 3.1.

Now, we shall derive an estimate for

$$\left\| \frac{1}{x} v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t>N\}}(t) w(t) dt \right\|_{L^p(N, \infty)}.$$

In order to carry out it, we get an estimation for the proper modular

$$\begin{aligned} & I_{p(\cdot);(N, \infty)} \left(\frac{1}{x} v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t>N\}}(t) w(t) dt \right) \\ &= I_{p(\cdot);(N, \infty)} \left(x^{-\frac{1}{p(x)}} x^{-\frac{1}{p'(x)}} v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t>M\}}(t) w(t) dt \right) \end{aligned} \quad (18)$$

It follows from Lemma 2.3 that

$$x^{-\frac{1}{p'(x)}} \leq C_7 2^{-n\epsilon} t^{-\frac{1}{p'(t)}}$$

for $\forall y, t \in (2^{-n-1}x, 2^{-n}x)$ and $x > N$. By using the last estimate the expression (18) is exceeded

$$\begin{aligned} & \int_N^\infty \left(C_7 2^{-n\delta} y^{-\frac{1}{p'(y)}} v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t>N\}}(t) w(t) dt \right)^{p(x)} \frac{dx}{x} \\ &= \int_N^\infty C_7^{p(x)} 2^{-np(x)\delta} y^{-\frac{p(x)}{p'(y)}} \left(\frac{v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t>N\}}(t) w(t) dt}{C_7 t^{\frac{1}{p'(t)}}} \right)^{p(x)} \left(C_{11} t^{\frac{1}{p'(t)}} \right)^{p(x)} \frac{dx}{x}. \end{aligned}$$

By applying here Lemma 2.1, we get that, the term $y^{-\frac{p(x)}{p'(y)}}$ is comparable with $t^{-\frac{p(x)}{p'(t)}}$ in the integral terms, i.e.

$$\left(y^{-\frac{1}{p'(y)}} \cdot t^{\frac{1}{p'(t)}} \right)^{p^+} \leq C_{12} \quad \text{for all } t, y \text{ lying in } (2^{-n-1}x, 2^{-n}x).$$

By a use of (13) a parentheses term in the preceding integral is less than 1. By decreasing the power $p(x)$ on the power of parentheses to $p_{x,n}^-$, we will increase the fraction. Then the last expression is less then

$$\begin{aligned} & C_{12} 2^{-n\delta p^-} \int_M^\infty \left(\frac{v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t>M\}}(t) w(t) dt}{C_7 t^{\frac{1}{p'(t)}}} \right)^{p_{x,n}^-} \frac{dx}{x} \\ & \leq C_{12} 2^{-n\delta p^-} \int_N^\infty \left(\frac{v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t>N\}}(t) w(t) dt}{C_9 t^{\frac{1}{p'(t)}}} \right)^{p_{x,n}^-} \frac{dx}{x}. \end{aligned} \quad (19)$$

Using Holder's inequality, it follows that

$$\begin{aligned} & v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t>N\}}(t) w(t) dt \\ & \leq \left(v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t)^{p_{x,n}^-} \chi_{\{t>N\}}(t) w(t) dt \right)^{\frac{1}{p_{x,n}^-}} (2^{-n-1}x)^{\frac{1}{(p_{x,n}^-)'}} \end{aligned}$$

with $x \geq N2^{n+1}$. By inserting this in the interior integral (19), that is exceeded by

$$C_{12} 2^{-n\delta p^-} \int_N^\infty \left(v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t)^{p_{x,n}^-} \chi_{\{t>N\}}(t) w(t) dt \right) (2^{-n-1}x)^{\frac{p_{x,n}^-}{(p_{x,n}^-)'}} t^{-\frac{p_{x,n}^-}{p'(t)}} \frac{dx}{x}.$$

Since $2^{-n-1}x \leq t$ and by using Lemma 2.4 and Lemma 2.1, it follows that, the last expression is exceeded by

$$\begin{aligned} & \int_N^\infty \left(\frac{1}{x} v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t>N\}}(t) w(t) dt \right)^{p(x)} dx \\ & \leq C_{13} 2^{-n\delta p^-} \int_N^\infty \left(v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t)^{p_{x,n}^-} \chi_{\{t>N\}}(t) w(t) dt \right) \frac{dx}{x}. \end{aligned} \quad (20)$$

Now, we estimate the interior integral through $I_{p(\cdot)}(f)$. The interior integral here is exceeded

$$\begin{aligned} & \int_{B_{x,n} \cap \left\{ t: f(t) > \frac{1}{1+t^2} \right\}} \left(\frac{f(t) \chi_{\{t>N\}}(t)}{\frac{1}{1+t^2}} \right)^{p_{x,n}^-} \left(\frac{1}{1+t^2} \right)^{p_{x,n}^-} dt \\ & + \int_{B_{x,n} \cap \left\{ t: f(t) \leq \frac{1}{1+t^2} \right\}} (f(t))^{p_{x,n}^-} dt \end{aligned}$$

$$\begin{aligned}
&\leq \int_{B_{x,n}} \left(\frac{f(t) \chi_{\{t>N\}}(t)}{\frac{1}{1+t^2}} \right)^{p(t)} \left(\frac{1}{1+t^2} \right)^{p_{x,n}^-} dt + \int_{B_{x,n}} \left(\frac{\chi_{\{t>N\}}(t)}{1+t^2} \right)^{p_{x,n}^-} dt \\
&\leq \int_{B_{x,n}} (f(t))^{p(t)} (1+t^2)^{p(t)-p_{x,n}^-} \chi_{\{t>N\}}(t) dt + \int_{B_{x,n}} \left(\frac{\chi_{\{t>N\}}(t)}{1+t^2} \right)^{p_{x,n}^-} dt \\
&\leq C_{14} \int_{B_{x,n}} f(t)^{p(t)} \chi_{\{t>N\}}(t) dt + \int_{B_{x,n}} \left(\frac{\chi_{\{t>N\}}(t)}{1+t^2} \right) dt; \quad B_{x,n} = (2^{-n-1}x, 2^{-n}x)
\end{aligned}$$

since the assumption (2) is made using (4) from Lemma 2.1 and $p_{x,n}^- \geq 1$, it follows that

$$(1+t^2)^{p(t)-p_{x,n}^-} \leq C_{15}.$$

Evidently, $\left(\frac{1}{1+t^2} \right)^{p_{x,n}^-} \leq \frac{1}{1+t^2}$. Therefore, we get an estimate

$$\int_{B_{x,n}} (f(t))^{p_{x,n}^-} \chi_{\{t>N\}}(t) dt \leq C_{15} \int_{B_{x,n}} f(t)^{p(t)} \chi_{\{t>N\}}(t) dt + \int_{B_{x,n}} \left(\frac{\chi_{\{t>N\}}(t)}{1+t^2} \right) dt. \quad (21)$$

Inserting this inequality in (20) and applying Fubini's theorem, it follows that

$$\begin{aligned}
&I_{p(\cdot), (N, \infty)} \left(\frac{1}{x} v(x) \int_{B_{n,x}} f(t) \chi_{\{t>N\}}(t) w(t) dt \right) \\
&\leq C_{15} 2^{-n\delta p^-} \int_N^\infty \left(v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t)^{p(t)} \chi_{\{t>N\}}(t) w(t) dt + v(x) \int_{B_{x,n}} \left(\frac{\chi_{\{t>N\}}(t)}{1+t^2} \right) w(t) dt \right) \frac{dx}{x} \\
&= C_{15} 2^{-n\delta p^-} v(x) \int_{2^{-n}N}^\infty \left(\int_{2^{nt}}^{2^{n+1}t} \frac{dx}{x} \right) f(t)^{p(t)} \chi_{\{t>N\}}(t) w(t) dt + v(x) \int_{2^{-n}M}^\infty \left(\int_{2^{nt}}^{2^{n+1}t} \frac{dx}{x} \right) \frac{\chi_{\{t>N\}}(t) w(t) dt}{1+t^2} \\
&\leq C_{15} 2^{-n\delta p^-} \ln 2 \left(\int_{2^{-n}N}^\infty f(t)^{p(t)} \chi_{\{t>N\}}(t) dt + \ln 2 \int_{2^{-n}M}^\infty \frac{\chi_{\{t>N\}}(t) dt}{1+t^2} \right) \\
&= C_{15} 2^{-n\delta p^-} \ln 2 \left(\int_N^\infty f(t)^{p(t)} dt + \int_N^\infty \frac{dt}{1+t^2} \right) \leq C_{15} 2^{-n\delta p^-} \ln 2 \left(1 + \frac{\pi}{2} \right) = C_{16} 2^{-n\delta p^-}.
\end{aligned}$$

Whence,

$$I_{p(\cdot), (N, \infty)} \left(\frac{1}{x} v(x) \int_{B_{n,x}} f(t) \chi_{\{t>N\}}(t) w(t) dt \right) \leq C_{16} 2^{-n\delta p^-}. \quad (22)$$

From this it follows that

$$\left\| \frac{1}{x} v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t)^{p(t)} \chi_{\{t>N\}}(t) w(t) dt \right\|_{L^p(N,\infty)} \leq C_{17} 2^{-\frac{n\delta p^-}{p^+}}. \quad (23)$$

Inserting (23) in (12), we get

$$\left\| x^{-1} v(x) \int_N^x f(x) dx \right\|_{L^{p(\cdot)}(N,\infty)} \leq \sum_{n=1}^{\infty} C_{17} 2^{-\frac{n\delta p^-}{p^+}} = C_{18}.$$

Further, substitute this estimate and (11) into (10), we complete the proof of the sufficiency part of Theorem 1.1.

3.2 Necessity.

Let us note, for an increasing near origin exponent functions $p(\cdot)$ it was proved in [19] that a necessity condition for inequality (1) to hold in the class of measurable positive functions f with support in finite interval $(0, N)$ is the same condition (2) over the points $b \in (0, N)$ (observe not over all axes $(0, \infty)$). To finish the necessity in Theorem 1.1 it remains to get this condition over the points $b > N$, where we shall essentially use the decreasing of exponent near infinity.

Below, we shall prove the necessity of condition (2) over points $x > N$ for decreasing near infinity exponent functions. We insert a function

$$f_0(x) = x^{-\frac{1}{p(x)}} \chi_{(b, 2b)}(x), \quad x > N$$

into inequality (1.2) with a parameter $b > N$ be fixed. It is clear that, $I_{p(\cdot)}(f_0) = \ln 2$. It follows from the inequality (1) that

$$I_{p(\cdot)} \left(\frac{1}{x} \int_b^x f_0(x) dx \right) \leq C_{19}.$$

From this it follows

$$\int_{2b}^{4b} \left(x^{-\frac{1}{p'(x)}} \int_b^{2b} t^{-\frac{1}{p(t)}} dt \right)^{p(x)} \frac{dx}{x} \leq C_{19}. \quad (24)$$

By monotony of p in (N, ∞) it follows the functions $x^{-\frac{1}{p'(x)}}$ and $t^{-\frac{1}{p(t)}}$ are decreasing therein. This yields

$$x^{-\frac{1}{p'(x)}} \geq (4b)^{-\frac{1}{p'(4b)}}, \quad 2b < x < 4b$$

and

$$t^{-\frac{1}{p(t)}} \geq (2b)^{-\frac{1}{p'(2b)}}, \quad b < t < 2b.$$

By taking into account this inequalities, it follows from (24) that

$$\int_{2b}^{4b} \left((4b)^{-\frac{1}{p'(4b)}} (2b)^{\frac{1}{p'(2b)}} \right)^{p(x)} \frac{dx}{x} \leq C_{19}.$$

This inequality yields

$$\left(\frac{1}{2} (2b)^{\frac{1}{p'(2b)} - \frac{1}{p'(4b)}} \right)^{p^-} \ln 2 \leq C_{19}$$

if the parenthesis term is greater 1. If not, we have

$$\frac{1}{2} (2b)^{\frac{1}{p'(2b)} - \frac{1}{p'(4b)}} \leq 1,$$

i.e. it follows that

$$(2b)^{\frac{1}{p'(2b)} - \frac{1}{p'(4b)}} \leq C_{20},$$

or

$$(2b)^{\frac{p(2b)-p(4b)}{p(2b)p(4b)}} \leq C_{20}.$$

Replacing $2b$ by b , we get the inequality

$$b^{\frac{p(b)-p(2b)}{p(b)p(2b)}} \leq C_{20}, \quad b > 2N.$$

Whence,

$$b^{p(b)-p(2b)} \leq C_{21}, \quad C_{21} = C_{20}^{(p^+)^2}, \quad b > 2N,$$

or

$$[p(b) - p(2b)] \ln b \leq C_{22}, \quad C_{22} = (p^+)^2 \ln C_{21}. \quad (25)$$

Now, having property (25) and using inequality (1), we shall derive condition (2) for $b > N$, too. In connection, we need some assertions below, e.g. following lemma is similar to those in [10].

Lemma 3.2. *For any $\frac{1}{2}x < y < 2x$ with $x > N$ the estimates*

$$C_{23}x^{-\frac{1}{p'(x)}} \leq y^{-\frac{1}{p'(y)}} \leq C_{24}x^{-\frac{1}{p'(x)}}$$

are satisfied with positive constants C_{23}, C_{24} depending on C_{22} .

Proof. By using estimate (25) and decreasing of $\frac{1}{p'(x)}$, it follows that

$$\begin{aligned} y^{-\frac{1}{p'(y)}} &\leq \left(\frac{x}{2}\right)^{-\frac{1}{p'(y)}} \leq x^{\frac{1}{p'(x)} - \frac{1}{p'(2x)}} \cdot x^{-\frac{1}{p'(x)}} 2^{\frac{1}{p'(N)}} \\ &\leq 2x^{p(x)-p(2x)} \cdot x^{-\frac{1}{p'(x)}} \leq 2C_{21}x^{-\frac{1}{p'(x)}}. \end{aligned}$$

By the same way,

$$\begin{aligned} x^{-\frac{1}{p'(x)}} &\leq \left(\frac{y}{2}\right)^{-\frac{1}{p'(x)}} \leq y^{\frac{1}{p'(y)} - \frac{1}{p'(2y)}} \cdot y^{-\frac{1}{p'(y)}} 2^{\frac{1}{p'(N)}} \\ &\leq 2y^{p(y)-p(2y)} \cdot y^{-\frac{1}{p'(y)}} \leq 2C_{21}y^{-\frac{1}{p'(y)}}. \end{aligned}$$

This proves Lemma 3.2.

Lemma 3.3. *The function $x^{-\frac{1}{p'(x)}}$ almost decreases on (N, ∞) .*

Proof. Take any $N < t_1 \leq t_2 < \infty$, we show that there exists a constant $C_{25} > 1$ depending on the constant C_2 of inequality (1) and p^+ such that

$$t_2^{-\frac{1}{p'(t_2)}} \leq C_{25}t_1^{-\frac{1}{p'(t_1)}}.$$

We fix any $t_1 = b > 2N$ and choose $n \in \mathbb{N}$ such that $2^{n-1}a < t_2 \leq 2^n b$. Then

$$C_2 \geq \int_{2b}^{\infty} \left(x^{-\frac{1}{p'(x)}} \int_b^{2b} t^{-\frac{1}{p'(t)}} dt \right)^{p(x)} \frac{dx}{x}$$

$$\geq \sum_{k=1}^{\infty} \int_{2^{k-1}b}^{2^k b} \left(x^{-\frac{1}{p(x)}} \int_b^{2b} t^{-\frac{1}{p(t)}} dt \right)^{p(x)} \frac{dx}{x},$$

by using Lemma 3.2,

$$\geq \sum_{k=1}^{\infty} \int_{2^{k-1}b}^{2^k b} \left[C_{24}^{-1} b^{\frac{1}{p'(b)}} x^{-\frac{1}{p'(x)}} \right]^{p(x)} C_{24}^{p^-} \frac{dx}{x},$$

by using Lemma 3.2, we obtain

$$\begin{aligned} &\geq \sum_{k=1}^{\infty} \int_{2^{k-1}b}^{2^k b} \left[C_{24}^{-1} b^{\frac{1}{p'(b)}} (2^k b)^{-\frac{1}{p'(2^k b)}} \right]^{p(x)} C_{24}^{p^-} \frac{dx}{x} \\ &\geq \sum_{k \in N'} \left[C_{24}^{-1} b^{\frac{1}{p'(b)}} (2^k b)^{-\frac{1}{p'(2^k b)}} \right]^{p^+} C_{24}^{p^-} \ln 2 \\ &\quad + \sum_{k \in N''} \left[C_{24}^{-1} b^{\frac{1}{p'(b)}} (2^k b)^{-\frac{1}{p'(2^k b)}} \right]^{p^-} C_{24}^{p^-} \ln 2, \end{aligned}$$

where $\sum_{k \in N'}$ is a summation over $k \in N$ such that the inequality

$$b^{\frac{1}{p'(b)}} (2^k b)^{-\frac{1}{p'(2^k b)}} \leq 1$$

holds, and $\sum_{k \in N''}$ is a summation of the opposite case. Therefore, for any $k \in N$ one gets

$$b^{\frac{1}{p'(b)}} (2^k b)^{-\frac{1}{p'(2^k b)}} \leq C_{25}.$$

This yields

$$t_2^{-\frac{1}{p'(t_2)}} \cdot t_1^{\frac{1}{p'(t_1)}} \leq C_{26}$$

i.e. function $x^{-\frac{1}{p'(x)}}$ is almost decreasing. Since $x^{-\frac{1}{p'(x)}}$ is almost decreasing on $[N, \infty)$, we get

$$C_2 \geq \int_b^{\infty} C_{26}^{p^-} \left[\frac{1}{C_{26}} b^{\frac{1}{p'(b)}} x^{-\frac{1}{p'(x)}} \right]^{p^+} dx$$

or

$$\int_b^{\infty} x^{-\frac{p^+}{p'(x)}} \frac{dx}{x} \leq C_{27} b^{\frac{p^+}{p'(b)}}, \quad b > N. \quad (26)$$

Now, let $N < t_1 < t_2 < \infty$. We show that condition (26) entails (2). We take any $N < t_1 < t_2 < \infty$ and set $K(x) = \int_x^{\infty} t^{-\frac{p^+}{p'(t)}} \frac{dt}{t}$, $x > N$. From (26) it follows that

$$K(x) \leq -C_{27} x K'(x), \quad N < x < \infty.$$

Integrating this inequality over (t_1, t_2) , for $N < t_1 < t_2 < \infty$ and using (26), we get

$$t_2^{\frac{1}{C_{27}}} K(t_2) \leq C_{27} t_1^{-\frac{p^+}{p'(t_1)} + \frac{1}{C_{27}}} \quad (27)$$

On the other hand, by using monotone decreasing of $x^{-\frac{p^+}{p'(x)}}$, the conditions (25) and (26), it follows that

$$k(t_2) \geq \int_{t_2}^{2t_2} x^{-\frac{p^+}{p'(x)}} \frac{dx}{x} \geq \ln 2 \left(\frac{1}{2t_2} \right)^{\frac{p^+}{p'(2t_2)}} \geq 2^{-\frac{p^+}{p'(N)}} \ln 2 t_2^{-\frac{p^+}{p'(t_2)}}, \quad t_2 \geq N.$$

By inserting this in (27) one gets,

$$t_2^{-\frac{p^+}{p'(t_2)} + \frac{1}{C_{27}}} \leq C_{28} t_1^{-\frac{p^+}{p'(t_1)} + \frac{1}{C_{27}}}$$

for all $N < t_1 < t_2 < \infty$. Whence, the function $x^{\delta - \frac{p^+}{p'(x)}}$ is almost decreasing on (N, ∞) . Therefore, we have proved that

$$t_2^{-\frac{1}{p'(t_2)} + \delta_1} \leq C_{28} t_1^{-\frac{1}{p'(t_1)} + \delta_1}$$

for all $N < t_1 < t_2 < \infty$; a constant $\delta_1 = \frac{\delta}{p^+}$.

Therefore $x^{-\frac{1}{p'(x)} + \delta_1}$ is almost decreasing with $\delta_1 = \frac{\delta}{p^+}$, $\delta = \frac{1}{C_{27}}$. From this it follows

$$\begin{aligned} \int_b^\infty x^{-\frac{1}{p'(x)}} \frac{dx}{x} &\leq C_{28} \int_a^\infty b^{\delta - \frac{1}{p'(x)}} \frac{dx}{x^{1+\delta}} \\ &\leq C_{28} b^{\delta - \frac{1}{p'(b)}} \int_b^\infty \frac{dx}{x^{1+\delta}} = \frac{C_{28}}{\delta} b^{-\frac{1}{p'(a)}}, \quad b > N \end{aligned}$$

This proves the necessity of condition (2) near infinity.

Conclusion

A new method of weighted increasing near origin and decreasing near infinity exponent function that provides a boundedness of the Hardy's operator in variable exponent Lebesgue space was obtained. The method we use here leads us to the most general sense. We don't have to work in a particular interval. This method can be applied to different operators. And this method brings important facilities in the study of operator theory.

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