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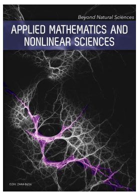

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Numerical Solution of Abel′s Integral Equations using Hermite Wavelet

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Abstract

A numerical method is developed for solving the Abel′s integral equations is presented. The method is based upon Hermite wavelet approximations. Hermite wavelet method is then utilized to reduce the Abel′s integral equations into the solution of algebraic equations. Illustrative examples are included to demonstrate the validity, efficiency and applicability of the proposed technique. Algorithm provides high accuracy and compared with other existing methods.

Keywords: Abel′s integral equations, Hermite wavelets, collocation method
AMS 2010 codes: 65T60, 65R20, 97N40.

1 Introduction

Wavelets theory is a new emerging tool in applied mathematical research area. It is applicable in various fields, such as, signal analysis for waveform representation and segmentations, time-frequency analysis and Harmonic analysis. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [1, 2]. Since 1991 the various types of wavelet methods have been applied for the numerical solution of different kinds of integral equations [3]. Namely, the Haar wavelets method [3], Legendre wavelets method [4], Rationalized haar wavelet [5], Hermite cubic splines [6], Coifman wavelet scaling functions [7], CAS wavelets [8], Bernoulli wavelets [9], wavelet preconditioned techniques [25-28]. Some of the papers are found for solving Abel′s integral equations using the wavelet based methods, such as Legendre wavelets [10] and Chebyshev wavelets [11].

Abel′s integral equations have applications in various fields of science and engineering. Such as microscopy, seismology, semiconductors, scattering theory, heat conduction, metallurgy, fluid flow, chemical reactions, plasma diagnostics, X-ray radiography, physical electronics, nuclear physics [12-14].

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In 1823, Abel, when generalizing the tautochrone problem derived the equation:

$$\int_0^x \frac{y(t)}{\sqrt{(x-t)}} dt = f(x), \tag{1.1}$$

where $f(x)$ is a known function and $y(x)$ is an unknown function to be determined. This equation is a particular case of a linear Volterra integral equation of the first kind. For solving eq. (1) different numerical based methods have been developed over the past few years, such as product integration methods [15], collocation method [16, 17], homotopy analysis transform method [18]. The generalized Abel′s integral equations on a finite segment appeared for the first time in the paper of Zeilon [19]. There are several numerical methods for approximating the solution of singular integral equations is known. Baker [20] studied the numerical treatment of integral equations. A numerical solution of weakly singular volterra integral equations was introduced in [21]. Babolian and Salimi [22] discussed an operational matrix method based on block-pulse functions for singular integral equations. In this paper, we introduced the Hermite wavelets based numerical method for solving Abel′s integral equations.
The article is organized as follows: In Section 2, formulation of Hermite wavelets and function approximation is presented. Section 3 is devoted the method of solution. In section 4, numerical results are demonstrated the accuracy of the proposed method by some of the illustrative examples. Lastly, the conclusion is given in section 5.

2 Properties of Hermite Wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called mother wavelet. When the dilation parameter a and translation parameter b varies continuously, we have the following family of continuous wavelets:

$$\psi_{a,b}(x) = |a|^{-1/2}\psi(\frac{x-b}{a}), \forall a, b \varepsilon R, a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}, b = nb_0 a_0^{-k}, a_0 > 1, b_0 > 0$. We have the following family of discrete wavelets

$$\psi_{k,n}(x) = |a|^{1/2}\psi(a_0^k x - nb_0), \forall a, b \varepsilon R, a \neq 0,$$

where $\psi_{k,n}$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$,then $\psi_{k,n}(x)$ forms an orthonormal basis. Hermite wavelets are defined as [24]

$$\psi_{k,n}(x) = \begin{cases} \frac{2^{k+1/2}}{\sqrt{\pi}} H_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}} \\ 0, & otherwise \end{cases} \tag{2.1}$$

where $m = 0, 1, ..., M-1$. Here is Hermite polynomials of degree m with respect to weight function $W(x) = \sqrt{1-x^2}$ on the real line R and satisfies the following recurrence formula $H_0(x) = 1, H_1(x) = 2x$,

$$H_{m+2}(x) = 2xH_{m+1}(x) - 2(m+1)H_m(x) \tag{2.2}$$

where$m = 0, 1, 2, ...$

Function approximation: Here we approximating the solution $y(x)$ of Abel's integral equations using Hermite wavelet basis as follows:

$$y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x) \tag{2.3}$$

where $\psi_{n,m}(x)$ is given in eq.(2). We approximate $y(x)$ by truncating the series represented in eq.(4) as,

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) = C^T \Psi(x) \tag{2.4}$$

where C and $\Psi(x)$ are $2^{k-1}M \times 1$ matrix,

$$C^T = [c_{1,0}, \dots c_{1,M-1}, c_{2,0}, \dots c_{2,M-1}, \dots c_{2^{k-1},0}, \dots c_{2^{k-1},M-1}] \tag{2.5}$$

$$\Psi(x) = [\psi_{1,0}, \dots \psi_{1,M-1}, \psi_{2,0}, \dots \psi_{2,M-1}, \dots \psi_{2^{k-1},0}, \dots \psi_{2^{k-1},M-1}] \tag{2.6}$$

3 Convergence and Error Analysis

Theorem 3.1. The series solution of Hermite wavelet $y(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x)$ is converges to $y(x)$.

Proof: Let $L^2(R)$ be the infinite dimensional Hilbert space and $\psi_{n,m}$ is defined as eq.(2) forms an orthonormal basis.

Let $y(x) = \sum_{i=0}^{M-1} c_{n,i} \psi_{n,i}(x)$ where $c_{n,i} = \langle y(x), \psi_{n,i}(x) \rangle$ for a fixed n.

Let us denote the sequence of partial sums S_n of $\{c_{n,i}\psi_{n,i}(x)\}$,

Let S_n and S_m be the partial sums with $n \geq m$. Now we have to prove S_n is a Cauchy sequence in Hilbert space $L^2(R)$.

Choose, $S_n = \sum_{i=0}^{n} c_{n,i}\psi_{n,i}(x)$, Now $\langle y(x), S_n \rangle = \left\langle y(x), \sum_{i=0}^{n} c_{n,i}\psi_{n,i}(x) \right\rangle = \sum_{i=m+1}^{n} |c_{n,i}|^2$

We claim that $\|S_n - S_m\|^2 = \sum_{i=m+1}^{n} |c_{n,i}|^2, \ \forall n > m$

Now $\left\| \sum_{i=m+1}^{n} c_{n,i}\psi_{n,i}(x) \right\|^2 = \left\langle \sum_{i=m+1}^{n} c_{n,i}\psi_{n,i}(x), \sum_{i=m+1}^{n} c_{n,i}\psi_{n,i}(x) \right\rangle = \sum_{i=m+1}^{n} |c_{n,i}|^2, \forall\, n > m$

thus, $\left\| \sum_{i=m+1}^{n} c_{n,i}\psi_{n,i}(x) \right\|^2 = \sum_{i=m+1}^{n} |c_{n,i}|^2, \ \forall n > m$

Since, Bessel's inequality, we have $\sum_{i=m+1}^{n} |c_{n,i}|^2 \leq \|y(x)\|^2$ is bounded and convergent.

Hence, $\left\| \sum_{i=m+1}^{n} c_{n,i}\psi_{n,i}(x) \right\|^2 \to 0 \ as \ m,n \to \infty.$

This implies, $\left\| \sum_{i=m+1}^{n} c_{n,i}\psi_{n,i}(x) \right\| \to 0.$ and

Therefore $\{S_p\}$ is a Cauchy sequence and it converges to s (say).

We assert that $y(x) = s$

Now $\langle s - y(x), \psi_{n,i}(x) \rangle = \langle s, \psi_{n,i}(x) \rangle - \langle y(x), \psi_{n,i}(x) \rangle = \langle s, \psi_{n,i}(x) \rangle - \left\langle \lim_{n\to\infty} S_n, \psi_{n,i}(x) \right\rangle = 0,$

This implies,

$$\langle s - y(x), \psi_{n,i}(x) \rangle = 0$$

Hence $y(x) = s$ and $\sum_{i=0}^{n} c_{n,i}\psi_{n,i}(x)$ converges to $y(x)$ as $n \to \infty$ and proved.

Theorem 3.2. Suppose that $y(x) \in C^m[0,1]$ and $C^T\Psi(x)$ is the approximate solution using Hermite wavelet. Then the error bound would be given by,

$$\|E(x)\| \leq \left\| \frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in[0,1]} |y^m(x)| \right\|.$$

Proof: Applying the definition of norm in the inner product space, we have,

$$\|E(x)\|^2 = \int_0^1 \left[y(x) - C^T\Psi(x)\right]^2 dx.$$

Divide interval [0, 1] into 2^{k-1} subintervals $I_n = \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$, $n = 1,2,3,...,2^{k-1}$.

$$\|E(x)\|^2 = \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[y(x) - C^T\Psi(x)\right]^2 dx.$$

$$\|E(x)\|^2 \leq \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[y(x) - P_m(x)\right]^2 dx.$$

where $P_m(x)$ is the interpolating polynomial of degree m which approximates $y(x)$ on I_n. By using the maximum error estimate for the polynomial on I_n, then

$$\|E(x)\|^2 \leq \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[\frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in I_n} |y^m(x)|\right]^2 dx.$$

$$\|E(x)\|^2 \leq \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[\frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in[0,1]} |y^m(x)|\right]^2 dx.$$

$$\|E(x)\|^2 = \int_0^1 \left[\frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in[0,1]} |y^m(x)|\right]^2 dx$$

$$\|E(x)\| \leq \left\| \frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in[0,1]} |y^m(x)| \right\|.$$

which, we have used the well-known maximum error bound for the interpolation.

4 Method of Solution

Consider the Abel's integral equation of the form,

$$\lambda y(x) = f(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}} dt, \quad 0 \leq x\,t \leq 1 \tag{4.1}$$

where $\lambda = 0$ or $\lambda = 1$. We first approximate $y(x)$ as truncated series defined in eq.(4). That is,

$$y(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) = C^T\Psi(x) \tag{4.2}$$

where C and $\Psi(x)$ are defined in eq.(6) and (7). Then substituting eq.(9) in eq.(8), we get

$$\lambda C^T\Psi(x) = f(x) + \int_0^x \frac{C^T\Psi(t)}{\sqrt{x-t}} dt \tag{4.3}$$

Next, assume eq.(10) is precise at following collocation points $x_i = \frac{2i-1}{2^k M}, i = 1,2,...,2^{k-1}M$. Then we obtain

$$\lambda C^T \Psi(x_i) = f(x_i) + \int_0^{x_i} \frac{C^T \Psi(t)}{\sqrt{x_i - t}} dt \tag{4.4}$$

Next, we obtain the system of algebraic equations with $2^{k-1}M$ unknown coefficients. By solving this system of equations, we get Hermite wavelet coefficients and then substituting these coefficients in eq.(9), we get the approximate solution of eq.(8).

5 Numerical Examples

In this section, we present Hermite wavelets method for the numerical solution of Abel′s integral equation to demonstrate the capability of the present method.

$$Error function = \|y_e(x_i) - y_a(x_i)\|_\infty = \sqrt{\sum_{i=1}^{n} (y_e(x_i) - y_a(x_i))^2}$$

where y_e and y_a are the exact and approximate solution respectively.

Example 1. Consider the Abel′s integral equation of first kind [22],

$$\frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3) = \int_0^x \frac{y(t)}{x-t} dt, \ \ 0 \le x \le 1. \tag{5.1}$$

We apply the present method to solve eq.(12) with $k = 1$ and $M = 4$. Then we get truncating approximate solution with unknowns as,

$$y(x) \approx \sum_{m=0}^{3} c_{1,m} \psi_{1,m}(x) = C^T \Psi(x) \tag{5.2}$$

Then applying the procedure discussed in the section 3. We get a system of four algebraic equations with four unknowns and solving this system, we obtain the Hermite wavelet coefficients as, $c_{1,0} = \frac{1257}{1513}, c_{1,1} = -\frac{69}{5000}, c_{1,2} = \frac{277}{10000}, c_{1,3} = \frac{69}{5000}$, and substituting in eq.(13), we obtain: $y(x) = \frac{1257}{1513}\psi_{10}(x) - \frac{69}{5000}\psi_{11}(x) + \frac{277}{10000}\psi_{12}(x) + \frac{69}{5000}\psi_{13}(x)$ On simplifying, we get $y(x) = x^3 - x^2 + 1$, which is exact solution of eq.(12). Numerical results with exact solutions are shown in table 1 and graphically shown in figure 1.

Example 2. Consider the Abel′s integral equation of the first kind [22, 23],

$$x = \int_0^x \frac{y(t)}{\sqrt{x-t}} dt, \ \ 0 \le x \le 1. \tag{5.3}$$

which has the exact solution $y(x) = \frac{2}{\pi}\sqrt{x}$. We solved the eq.(14) using the present method and obtained approximate solution is compared with exact and other existing methods which reflects in table 2 and figure 2. Error analysis is shown in table 3 and figure 3.

Example 3. Consider the Abel′s integral equation of the second kind [22, 23],

$$4y(x) = \frac{4}{\sqrt{x+1}} - arcsin\left(\frac{1-x}{1+x}\right) + \frac{\pi}{2} - \int_0^x \frac{y(t)}{\sqrt{x-t}} dt, \ \ 0 \le x \le 1. \tag{5.4}$$

Table 1 Numerical results of example 1.

x	Exact solution	Present method (k = 1, M = 8)	Abs. Error
0.1	0.9910	0.9910	1.61e-12
0.2	0.9680	0.9680	4.71e-13
0.3	0.9370	0.9370	2.27e-12
0.4	0.9040	0.9040	3.04e-12
0.5	0.8750	0.8750	2.52e-12
0.6	0.8560	0.8560	9.67e-13
0.7	0.8530	0.8530	9.64e-13
0.8	0.8720	0.8720	2.36e-12
0.9	0.9190	0.9190	2.36e-12

Table 2 Numerical results of example 2.

x	Exact solution	Present method (k = 1, M = 10)	Method [22] (k = 1, M = 8)	Method [23] (m=16)
0.1	0.201317	0.200842	0.200128	0.200460
0.2	0.284705	0.284667	0.286092	0.297987
0.3	0.348691	0.348628	0.347394	0.337588
0.4	0.402634	0.402609	0.404161	0.405769
0.5	0.450158	0.450129	0.449568	0.464014
0.6	0.493124	0.493113	0.492704	0.490550
0.7	0.532634	0.532607	0.532315	0.539721
0.8	0.569410	0.569440	0.569156	0.562698
0.9	0.603951	0.603690	0.603742	0.606044

Table 3 Error analysis of example 2.

x	Present method (k = 1, M = 10)	Method [22] (k = 1, M = 8)	Method [23] (m=16)
0.1	4.73e-04	1.18e-03	8.57e-04
0.2	3.77e-05	1.38e-03	1.32e-02
0.3	6.21e-05	1.29e-03	1.11e-02
0.4	2.37e-05	1.52e-03	3.13e-03
0.5	2.85e-05	5.90e-04	1.38e-02
0.6	9.87e-06	4.19e-04	2.57e-03
0.7	2.72e-05	3.19e-04	7.08e-03
0.8	3.05e-05	2.54e-04	6.71e-03
0.9	2.59e-04	2.08e-04	2.09e-03

which has the exact solution $y(x) = \dfrac{1}{\sqrt{x+1}}$. Applying the Hermite wavelet method for solving eq.(15), then obtained approximate solution is compared with the exact solution and method[23] are shown in table 4 and figure 4. Error analysis is shown in table 5. **Example 4.** Consider the Abel′s integral equations of the second kind [22, 23],

$$y(x) = 2\sqrt{x} - \int_0^x \frac{y(t)}{\sqrt{x-t}} dt, \quad 0 \le x \le 1. \tag{5.5}$$

which has the exact solution $y(x) = 1 - exp(\pi t) erfc(\sqrt{\pi t})$. We solved the eq.(16) by the present method, we

Table 4 Numerical results of example 3.

x	Exact solution	Present method ($k = 1, M = 10$)	Method [23] (m=16)
0.1	0.953462589245592	0.953462604453520	0.95646081381695
0.2	0.912870929175277	0.912870928482700	0.90601007037324
0.3	0.877058019307029	0.877058021105882	0.88361513925322
0.4	0.845154254728517	0.845154255308354	0.84340093819493
0.5	0.816496580927726	0.816496581976408	0.80822420481499
0.6	0.790569415042095	0.790569415421744	0.79221049469412
0.7	0.766964988847370	0.766964989996025	0.76284677221990
0.8	0.745355992499930	0.745355991505969	0.74933888037055
0.9	0.725476250110012	0.725476258975992	0.72434536240934

Table 5 Error analysis of example 3.

x	Present method ($k = 1, M = 10$)	Method [23] (m=16)
0.1	1.52e-08	2.99e-03
0.2	6.92e-10	6.86e-03
0.3	1.79e-09	6.55e-03
0.4	5.79e-10	1.75e-03
0.5	1.04e-09	8.27e-03
0.6	3.79e-10	1.64e-03
0.7	1.14e-09	4.11e-03
0.8	9.93e-10	3.98e-03
0.9	8.86e-09	1.13e-03

obtain the approximate solution and is compared with exact and other existing methods as shown in table 6 and figure 5. Error analysis is shown in table 7 and figure 6.

Table 6 Numerical results of example 4.

x	Exact solution	Present method ($k = 1, M = 10$)	Method [22] ($k = 0, M = 16$)	Method [23] (m=16)
0.1	0.414059	0.411229	0.415689	0.402472
0.2	0.508352	0.507572	0.505528	0.519751
0.3	0.564309	0.563685	0.566205	0.554755
0.4	0.603347	0.602926	0.601908	0.605031
0.5	0.632868	0.632521	0.634188	0.640487
0.6	0.656323	0.656059	0.655109	0.654785
0.7	0.675601	0.675358	0.676588	0.678700
0.8	0.691842	0.691690	0.691596	0.688860
0.9	0.705787	0.705398	0.704377	0.706495

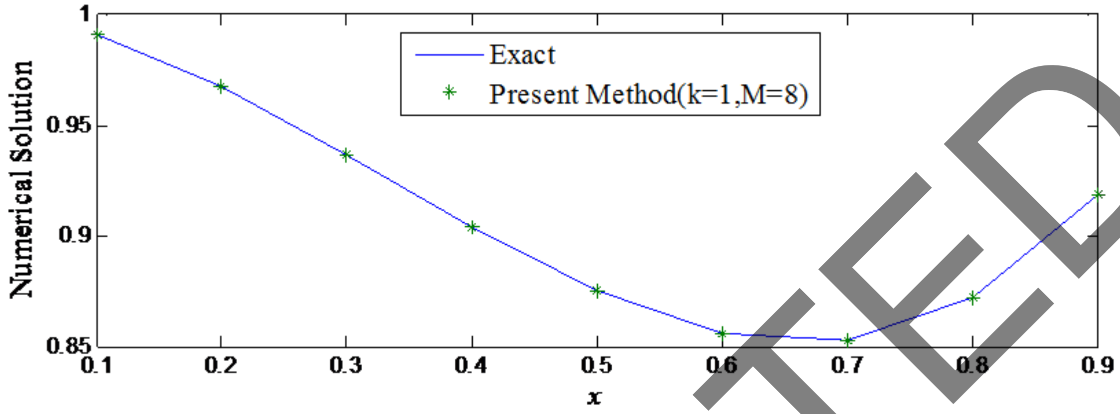

Fig. 1 Comparison of numerical solutions with exact solutions of example 1.

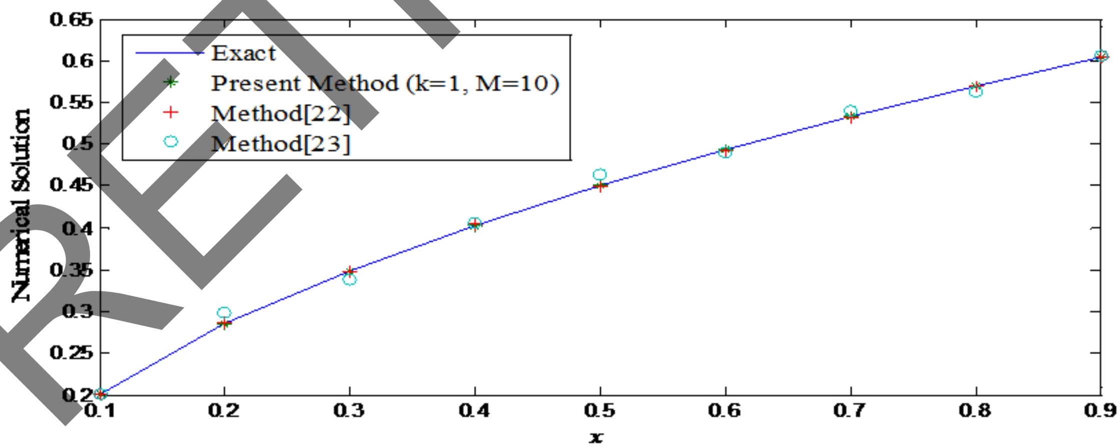

Fig. 2 Comparison of numerical solutions, exact solutions and existing methods of example 2.

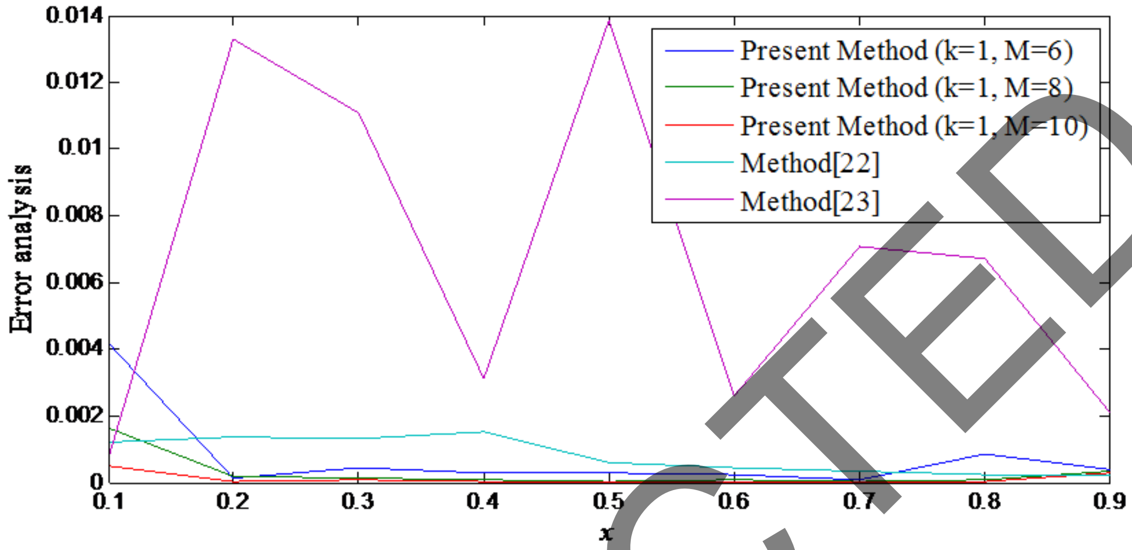

Fig. 3 Comparison of error analysis of example 2.

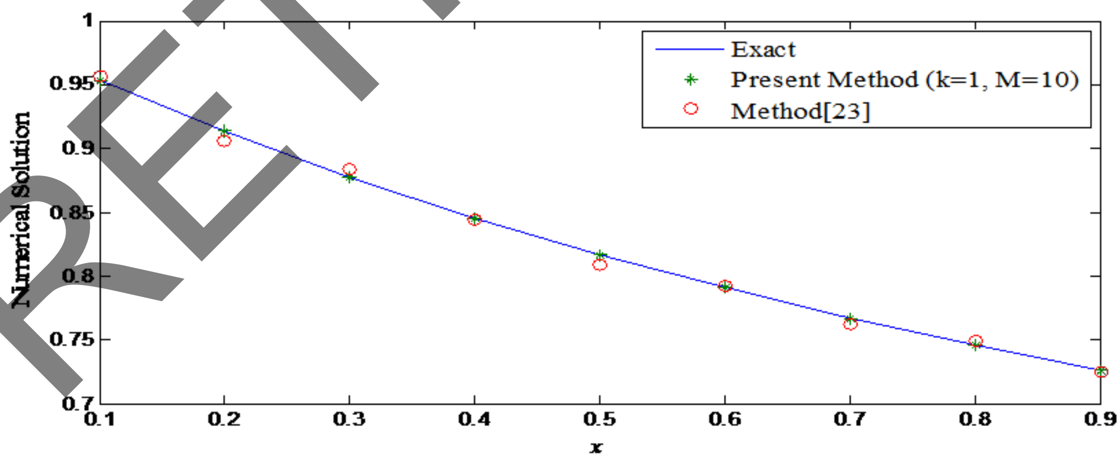

Fig. 4 Comparison of Numerical solutions of example 3.

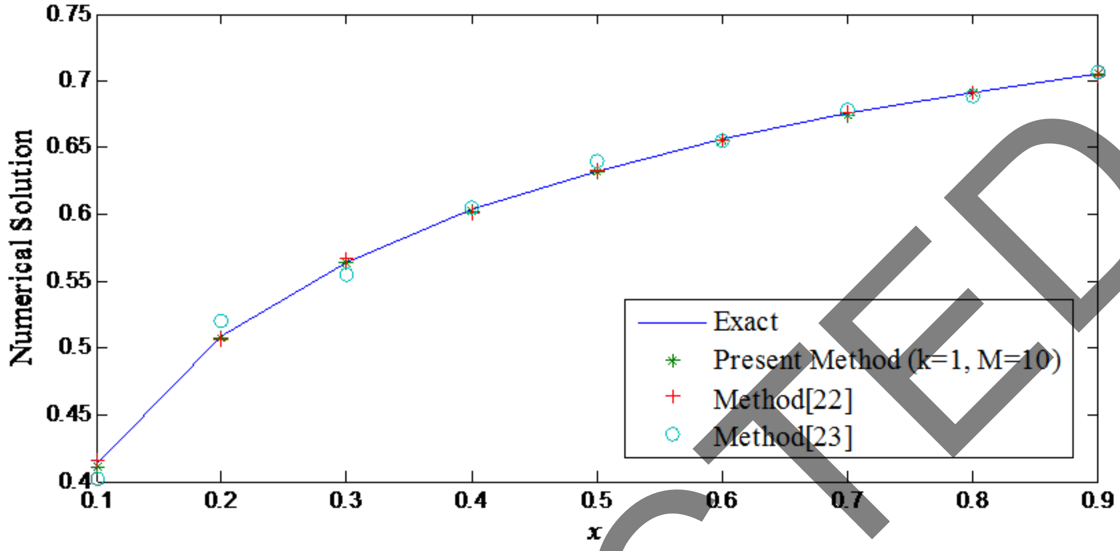

Fig. 5 Comparison of Numerical solutions of example 4.

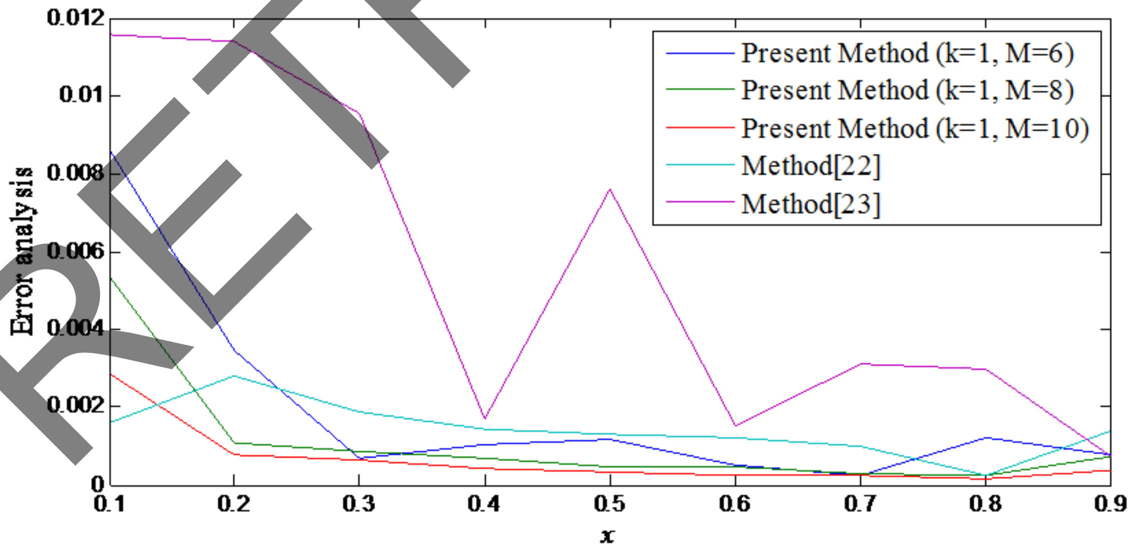

Fig. 6 Comparison of Error analysis of example 4.

Table 7 Error analysis of example 4.

x	Present method ($k = 1, M = 10$)	Method [22] ($k = 0, M = 16$)	Method [23] (m=16)
0.1	2.82e-03	1.62e-03	1.15e-02
0.2	7.79e-04	2.82e-03	1.13e-02
0.3	6.23e-04	1.89e-03	9.55e-03
0.4	4.21e-04	1.43e-03	1.68e-03
0.5	3.46e-04	1.32e-03	7.61e-03
0.6	2.63e-04	1.21e-03	1.53e-03
0.7	2.42e-04	9.86e-04	3.09e-03
0.8	1.50e-04	2.45e-04	2.98e-03
0.9	3.87e-04	1.40e-03	7.08e-04

6 Conclusion

The Hermite wavelet method is applied for the numerical solution of Abel′s integral equations. The present method reduces an integral equation into a set of algebraic equations. Obtained results are higher accuracy with exact ones and existing methods [22, 23], which can be observed in section 5. The numerical results shows that the accuracy improves with increasing the values of M for better accuracy. Convergence theorem reveals that existence of solution.

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