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Application of modified wavelet and homotopy perturbation methods to nonlinear oscillation problems

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Abstract

In this paper, an accurate and efficient Chebyshev wavelet-based technique is successfully employed to solve the nonlinear oscillation problems. Numerical examples are also provided to illustrate the efficiency and performance of these methods. Homotopy perturbation methods may be viewed as an extension and generalization of the existing methods for solving nonlinear equations. In addition, the use of Chebyshev wavelet is found to be simple, flexible, accurate, efficient and less computational cost. Our analytical results are compared with simulation results and found to be satisfactory.

Keywords: Chebyshev wavelet, nonlinear oscillation, homotopy perturbation, operational matrix, numerical solutions
AMS 2010 codes: 34E10, 34E13, 35K20, 35K60

1 Introduction

Most of the oscillation problems in engineering sciences are nonlinear, and it is difficult to solve such equations analytically. Recently, nonlinear oscillator models have been widely considered in physical and chemical sciences. Due to the limitation of existing exact solutions, many approximate analytical and numerical approaches have been investigated. Many real-life problems that arise in several branches of pure and applied science can be expressed using the nonlinear differential equations. Therefore, these nonlinear equations must be solved using analytical/numerical methods. Many researchers have been working on various analytical methods for solving nonlinear oscillation systems in the last decades. Nowadays, the computational experience is significant, and several numerical methods have been suggested and analyzed under certain conditions. These numerical methods have been developed using different techniques such as Taylor series, homotopy perturbation method, quadrature formula, variation iteration method and decomposition method [1–7]. Noor et al. [8] have applied a sixth-order predictor–corrector iterative method for solving the nonlinear equations. Chun [9] proposed a class of fifth-order and sixth-order iterative methods. In this paper, we apply the wavelet transform

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method and homotopy perturbation method for solving the nonlinear oscillation equations. The homotopy perturbation method was developed by He [10, 11] and has been applied to a wide class of nonlinear and linear problems arising in various branches.

Recently, the homotopy perturbation method (HPM) [12–16] with expanding parameter is applied to solve some of the nonlinear equations. Ren and Hu [17], Yu et al. [18] and Wu and He [19] developed the nonlinear oscillation problem using HPM. Abbasbandy et al. [20] developed the homotopy perturbation method and the decomposition method [21] for solving nonlinear equations. Ganji et al. [22] applied the oscillation systems with nonlinearity terms such as the motion of a rigid rod rocking. Khudayarov and Turaev [23] developed the mathematical model of the problem of nonlinear oscillations of a viscoelastic pipeline conveying fluid. Nasab et al. [24] solved the nonlinear singular boundary value problems.

To our knowledge, no exact analytical expressions for oscillation problems are reported. However, it is difficult for us to obtain the exact solution for these problems. The purpose of this paper was to derive the approximate analytical expressions for some oscillation problems in engineering sciences.

2 Mathematical formulation of the problems

The general form of the differential equation describing the oscillations of single-degree-of-freedom systems can be written as [25]:

$$\ddot{x}+F(x)=0 \tag{1}$$

with initial conditions:

$$x(0)=l \text{ and } \dot{x}(0)=0 \tag{2}$$

where $F(x)$ represents the linear and nonlinear terms. A typical nonlinear conservative system that has been the subject of many investigations is Duffing-type oscillator. This nonlinear oscillator represents the dynamic behaviour of many engineering problems. Exact analytical solutions to the oscillatory problem in the form of Eq. (1) are generally impossible, and therefore, some numerical solution is to be reported. The main goal is to apply the discretization and linearization concepts to develop the HPM and CWM which can provide periodic solutions for oscillatory problems.

3 Some properties of shifted second kind Chebyshev polynomials [26]

In this section, we discuss some relevant properties of the function

$$U_n(x)=\frac{\sin(n+1)\theta}{\sin\theta},\ x=\cos\theta \tag{3}$$

These polynomials are orthogonal on $[-1,1]$

$$ie., \int_{-1}^{1}\sqrt{1-x^2}U_m(x)U_n(x)dx=\begin{cases}0,\ for\ m\neq n,\\ \frac{\pi}{2},\ for\ m=n.\end{cases} \tag{4}$$

The following properties of second kind Chebyshev polynomials [26] are of fundamental importance in the sequel. They are eigenfunctions of the following singular Sturm–Liouville equation:

$$(1-x^2)D^2\varphi_k(x)-3xD\varphi_k(x)+k(k+2)\varphi_k(x)=0, \tag{5}$$

where $D \equiv \frac{d}{dx}$, and orthogonal polynomial may be generated using the recurrence relation

$$U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x),\ k = 1,2,3,..., \tag{6}$$

starting from $U_0(x) = 1$ and $U_1(x) = 2x$, or from Rodriguez formula

$$U_n(x) = \frac{(-2)^n(n+1)!}{(2n+1)!\sqrt{1-x^2}} D^n \left[(1-x^2)^{n+\frac{1}{2}}\right] \tag{7}$$

3.1 Shifted second kind Chebyshev polynomials

The shifted second kind Chebyshev polynomials are defined on the interval [0,1] by $U_n^*(x) = U_n(2x-1)$. All the results of second kind Chebyshev polynomials can be easily transformed to give the corresponding results for their shifted forms. The orthogonality relation with respect to the weight function $\sqrt{x-x^2}$ is given by

$$\int_0^1 \sqrt{x-x^2}U_n^*(x)U_m^*(x)dx = \begin{cases} 0, \ for\ m \neq n, \\ \frac{\pi}{8}, \ for\ m = n. \end{cases} \tag{8}$$

The first derivative $U_n^*(x)$ is given in the following corollary.
Corollary 1. The first derivative of the shifted second kind Chebyshev polynomial is given by

$$DU_n^*(x) = 4 \sum_{k=0,(k+n)odd}^{n-1} (k+1)U_k^*(x) \tag{9}$$

3.2 Shifted second kind Chebyshev operational matrix of derivatives (S2KCOM)

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelet:

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}} \psi\left(\frac{t-b}{a}\right),\ a,b \in \Re,\ a \neq 0 \tag{10}$$

Shifted second kind Chebyshev wavelets $\psi_{nm}(t) = \psi(k,n,m,t)$ have four arguments: k, n can assume any positive integer, m is the order of second kind Chebyshev polynomials and $t$is the normalized time. They are defined on the interval [0,1] by

$$\psi_{nm}(t) = \begin{cases} \frac{2^{\frac{(k+3)}{2}}}{\sqrt{\pi}} U_m^*\left(2^k t - n\right) \ for\ t \in \left[\frac{n}{2^k}, \frac{n+1}{2^k}\right] \\ 0, \ for\ otherwise \end{cases} \quad m = 0,1,.....M,\ n = 0,1,.....,2^k-1 \tag{11}$$

3.3 Function Approximation

A function $f(t)$ defined on the interval [0,1] may be expanded in terms of second kind Chebyshev wavelets as

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t), \tag{12}$$

where

$$c_{nm} = \langle f(t), \psi_{nm}(t)\rangle_{\omega} = \int_0^1 \omega(t)f(t)\psi_{nm}(t)dt, \tag{13}$$

and $\omega(t)=\sqrt{t-t^2}$. If the infinite series is truncated, then $f(t)$ can be approximated as

$$f(t)\approx\sum_{n=0}^{2^k-1}\sum_{m=0}^{M}c_{nm}\psi_{nm}(t)=C^T\psi(t), \tag{14}$$

where C and $\psi(t)$ are $2^k(M+1)\times 1$ matrices defined by

$$C=\left[c_{0,0},c_{0,1},\ldots c_{0,M},\ldots.c_{2^k-1,M},\ldots.c_{2^k-1,1},\ldots\ldots\ldots.c_{2^k-1,M}\right]^T \tag{15}$$

$$\psi(t)=\left[\psi_{0,0}(t),\psi_{0,1}(t),\ldots\ldots\ldots\psi_{0,M}(t),\ldots..\psi_{2^k-1,M}(t),\ldots\ldots\ldots\ldots..\psi_{2^k-1,1}(t),\ldots..\psi_{2^k-1,M}(t)\right]^T \tag{16}$$

3.4 Solution pertaining to the nonlinear differential equations using shifted second kind Chebyshev wavelet method (S2KCWM) and homotopy perturbation method

In this section, we give some numerical results obtained using the algorithms presented in the previous sections. We consider the following examples.

Example 1.

We consider the nonlinear differential equation [27]

$$\ddot{x}+\frac{x}{1+\mu x^2}=0 \tag{17}$$

with initial conditions:

$$x(0)=l \text{ and } \dot{x}(0)=0 \tag{18}$$

Wavelet method:

Eq. (17) is solved using the wavelet scheme,

$$C^TD^2\psi(t)+\frac{C^T\psi(t)}{1+\mu(C^T\psi(t))^2}=0 \tag{19}$$

Moreover, the initial conditions are

$$C^T\psi(t)=l,\ C^TD\psi(t)=0,\ \text{at } t=0 \tag{20}$$

Using Eq. (19) and Eq. (20), the following system of algebraic equations can be obtained

$$64c_2+\frac{2c_0+(8t-4)\,c_1+\left(32t^2-32t+6\right)\,c_2}{1+\mu\left(2c_0+(8t-4)\,c_1+(32t^2-32t+6)\,c_2\right)^2}=0 \tag{21}$$

$$2c_0-4c_1+6c_2=l \tag{22}$$

$$8c_1-32c_2=0 \tag{23}$$

The suggested method of solution $x(t)$ is approximated as follows (Appendix B):

$$x(t)=C^T\psi(t) \tag{24}$$

Homotopy perturbation method:

Solving the nonlinear Eq. (17) using a new approach of homotopy perturbation method (Appendix A)

$$x(t)=\cos t\left(l+\frac{l^3\mu}{64}-\frac{50l^3\mu^2}{18432}\right)+\cos 3t\left(\frac{13l^4\mu^2}{6144}-\frac{l^3\mu}{64}\right)+\cos 5t\left(\frac{11l^4\mu^2}{18432}\right) \tag{25}$$

The velocity becomes

$$\dot{x}(t)=\sin t\left(-l-\frac{l^3\mu}{64}+\frac{50l^3\mu^2}{18432}\right)+3\sin 3t\left(-\frac{13l^4\mu^2}{6144}+\frac{l^3\mu}{64}\right)-5\sin 5t\left(\frac{11l^4\mu^2}{18432}\right) \tag{26}$$

Example 2.

We consider equation [28]

$$\ddot{x}+\frac{x}{1+\alpha x+\beta x^2}=0 \tag{27}$$

with the initial conditions

$$x(0)=l \text{ and } \dot{x}(0)=0 \tag{28}$$

Wavelet method:

The wavelet scheme of Eq. (27) is

$$C^T D^2\psi(t)+\frac{C^T\psi(t)}{1+\alpha\,(C^T\psi(t))+\beta(C^T\psi(t))^2}=0 \tag{29}$$

The initial conditions are

$$C^T\psi(t)=l,\ C^T D\psi(t)=0, \text{ at } t=0 \tag{30}$$

Using Eq. (29) and Eq. (30), the following system of algebraic equations can be obtained

$$64c_2+\frac{2c_0+(8t-4)\,c_1+\left(32t^2-32t+6\right)\,c_2}{1+\alpha\,(2c_0+(8t-4)\,c_1+(32t^2-32t+6)\,c_2)+\beta\,(2c_0+(8t-4)\,c_1+(32t^2-32t+6)\,c_2)^2}=0 \tag{31}$$

$$2c_0-4c_1+6c_2=l \tag{32}$$

$$8c_1-32c_2=0 \tag{33}$$

The solution of $x(t)$ is approximated as follows (Appendix B):

$$x(t)=C^T\psi(t) \tag{34}$$

Homotopy perturbation method:

Solving the nonlinear Eq. (27) using a new approach of homotopy perturbation method, we get

$$x(t)=l\cos t+\sin^2\left(\frac{t}{2}\right)\left[\left(\frac{2\alpha l^2}{3}+\frac{\beta l^3}{4}\right)\cos t+\frac{\beta l^3}{8}\cos 2t+\frac{4\alpha l^2}{3}+\frac{\beta l^3}{8}\right] \tag{35}$$

The velocity becomes

$$\dot{x}(t)=-l\sin t+2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)\left[-\left(\frac{2\alpha l^2}{3}+\frac{\beta l^3}{4}\right)\sin t-\frac{\beta l^3}{4}\sin 2t\right] \tag{36}$$

Example 3.

We consider the differential equation [29]

$$\ddot{u}+u+\varsigma u^3=0 \tag{37}$$

with initial conditions:

$$u(0)=l \text{ and } \dot{u}(0)=0 \tag{38}$$

Wavelet method:

Eq. (37) is solved using the wavelet scheme,

$$C^T D^2\psi(t)+\left(C^T\psi(t)\right)+\varsigma\left(C^T\psi(t)\right)^3=0 \tag{39}$$

Moreover, the initial conditions are

$$C^T\psi(t)=l,\ C^T D\psi(t)=0, \text{ at } t=0 \tag{40}$$

Using Eq. (39) and Eq. (40), the following system of algebraic equations can be obtained

$$64c_2+\left(2c_0+(8t-4)c_1+\left(32t^2-32t+6\right)c_2\right)+\varsigma\left(2c_0+(8t-4)c_1+\left(32t^2-32t+6\right)c_2\right)^3=0 \tag{41}$$

$$2c_0-4c_1+6c_2=l \tag{42}$$

$$8c_1-32c_2=0 \tag{43}$$

The solution of $u(t)$ is approximated as follows (Appendix B):

$$u(t)=C^T\psi(t) \tag{44}$$

Homotopy perturbation method:

Solving the nonlinear Eq. (37) using a new approach of homotopy perturbation method, we get

$$u(t)=l\cos t-\frac{\varsigma l^3}{32}(\cos t-\cos 3t) \tag{45}$$

The velocity becomes

$$\dot{u}(t)=-l\sin t+\frac{\varsigma l^3}{32}(\sin t-3\sin 3t) \tag{46}$$

4 Numerical simulation and discussion

To illustrate the applicability, accuracy and effectiveness of the proposed method, we have compared the approximate analytical solution of the nonlinear differential equations with numerical data. The function ode 45 (Runge–Kutta method) in MATLAB software, which is a function of solving the initial value problems, is used. In Figure 1(a–c), we have plotted the numerical solution and the approximate solution derived by our proposed method using HPM and CWM. The figure shows the behaviour of the solution for various values of the parameter. From the figure, it is observed that the variation in the approximate solution is small, when $\mu \leq 1$. From Figure 1(a–c), it is also observed that the amplitude depends upon the initial conditions. Figure 2 represents the displacement and velocity versus time t. From the figure, it is noted that the amplitude of displacement and velocity are equal.

Figure 3(a–c) denotes the displacement versus time for various values of the parameters α and β. The numerical solution is compared with our analytical results in Figure 3(a–c) and found to be satisfactory. Displacement and velocity are shown in Figure 4. Here also, the amplitude of displacement and velocity are equal. Figure 5 shows the displacement versus time for the oscillation problem (Eq. (45)). Displacement and velocity versus time are shown in Figure 6. All the result also confirmed for the problem Eq. (37).

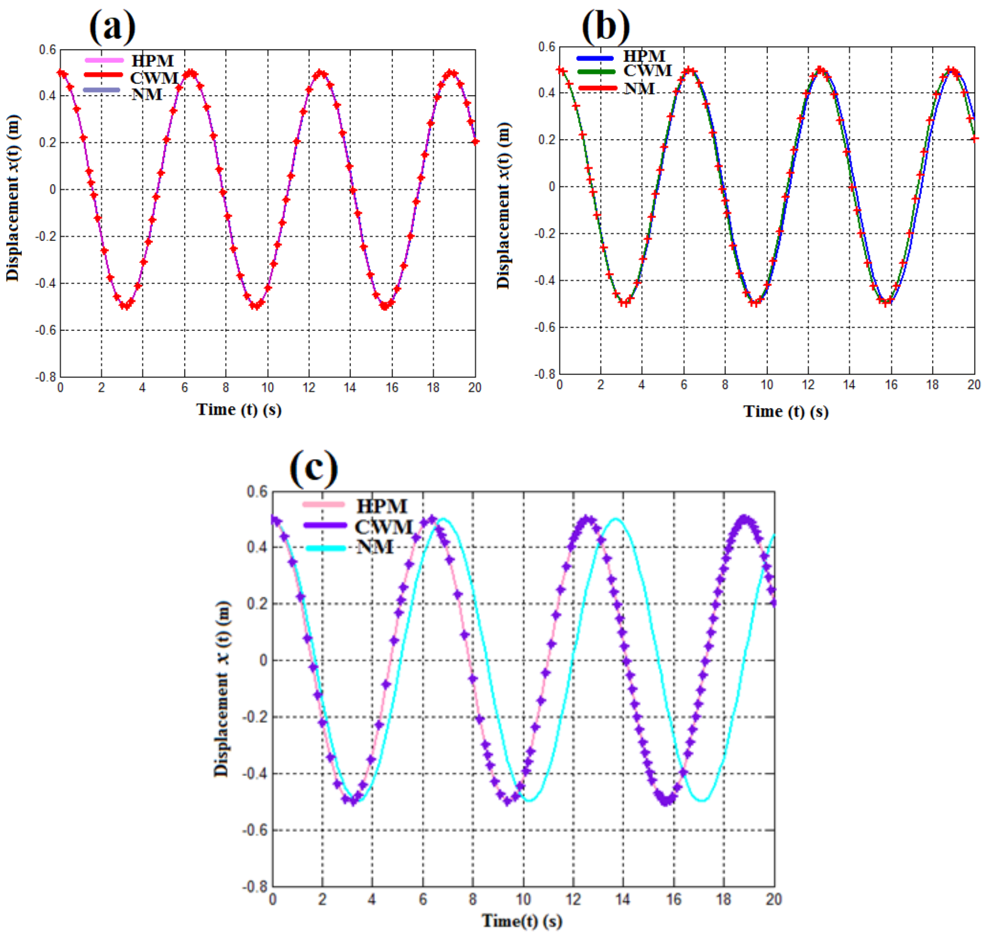

Fig. 1 (a–c) Comparison of CWM (Eq.(24), HPM (Eq.(25) and numerical method (MATLAB result) for various parameter values. Fig.1(a) $l = 0.5$ and $\mu = 0.01$ Fig.1(b) $l = 0.5$ and $\mu = 0.1$ Fig.1(c) $l = 0.5$ and $\mu = 1$.

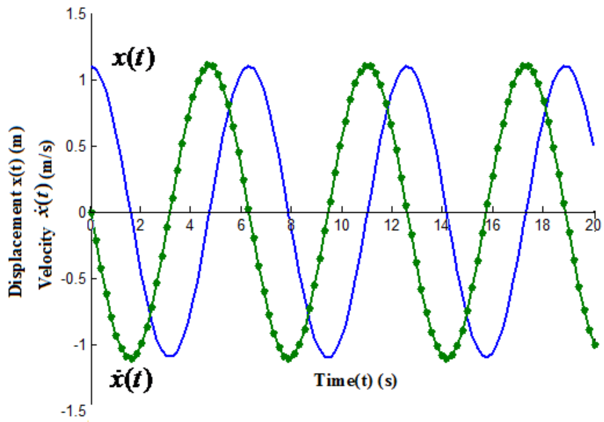

Fig. 2 Plot of displacement and velocity for oscillator Eq. (26) with weak nonlinearity and small amplitude oscillations $l = 1.1$ and $\mu = 0.1$.

5 Conclusion

In this paper, a wavelet technique has been employed for the approximate solution successfully to solve the well-known nonlinear oscillator differential equations such as Duffing equation with different parameters. There

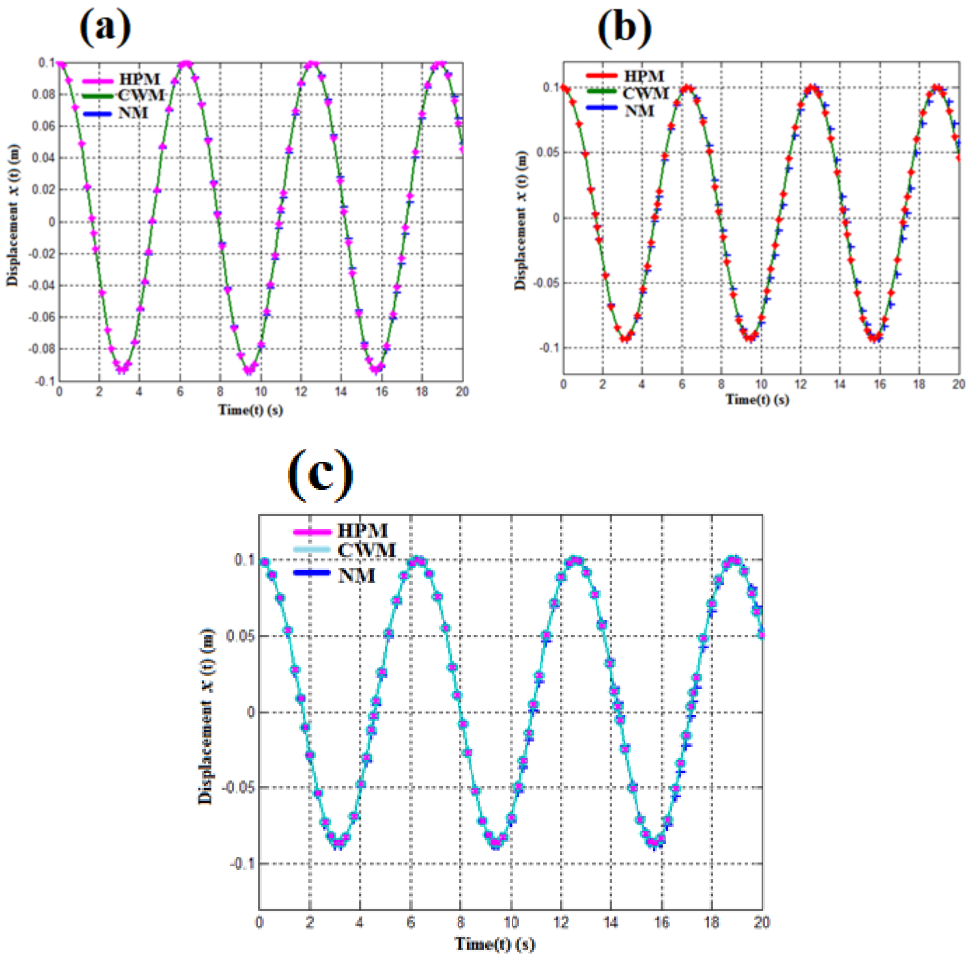

Fig. 3 Comparison of CWM (Eq. (34), HPM (Eq. (35) and numerical method (MATLAB result) for various parameter values. Fig. 3 (a) $l = 0.1,\ \alpha = 1$and $\beta = 0.5$ Fig.3 (b) $l = 0.1,\ \alpha = 1$ and $\beta = 2$ Fig. 3 (c) $l = 0.1,\ \alpha = 2$ and $\beta = 0.5$.

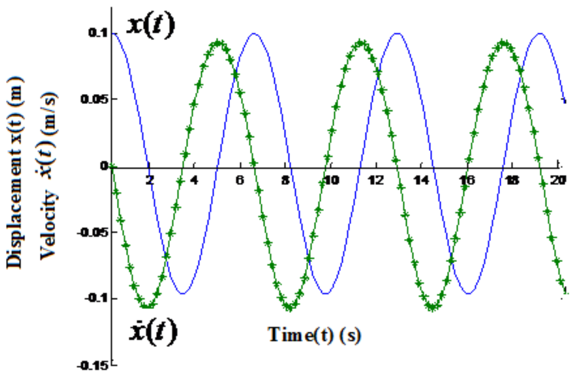

Fig. 4 Plot of displacement and velocity for oscillator Eq. (36) with weak nonlinearity and small amplitude oscillations $l = 0.1,\ \alpha = 1,\ \beta = 2$.

is no need for iterations for achieving sufficient accuracy in numerical results. The results are also obtained via CWM, HPM and numerical solution. Moreover, the proposed method is used to compare CWM, HPM, and NM iteration with the nonlinear part. The effects of constant parameters on responses of the system for approximate

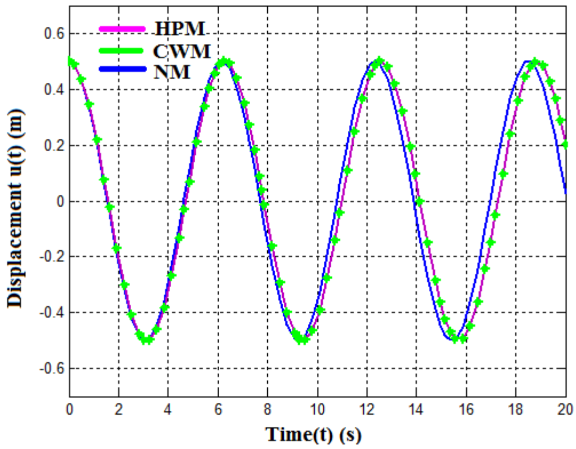

Fig. 5 Comparison of CWM (Eq. (44), HPM (Eq. (45) and numerical method (MATLAB result) for fixed parameter values $l = 0.5,\ \varsigma = 0.2$.

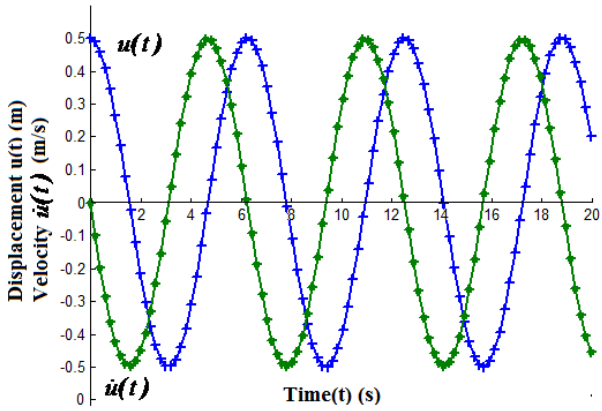

Fig. 6 Plot of displacement and velocity for oscillator Eq. (46) with weak nonlinearity and small amplitude oscillations $l = 0.5,\ \varsigma = 0.2$.

solution are also shown in figures.

Acknowledgment

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APPENDIX

Appendix A

Approximate analytical solution of oscillations [27] using HPM

Eq. (17) can be rewritten as follows:

$$\ddot{x}(t) + x(t) + \mu\,(x(t))^2\ddot{x}(t) = 0 \tag{A1}$$

This equation is in the form

$$\ddot{x}(t)+x(t)=-\mu f\left(x,\dot{x},\overset{(i)}{x}\right) \tag{A2}$$

where

$$f\left(x,\dot{x},\overset{(i)}{x}\right)=[x(t)]^{2}\ddot{x}(t) \tag{A3}$$

The solution of Eq. (A.2) is

$$x(t)=x_0+\sum_{k\geq 1}\frac{\mu^{k}x_0^{(k)}(t)}{k!} \tag{A4}$$

Using HPM, we get the following Eq. (A.1)

$$\ddot{x}_0^{(0)}(t)+x_0^{(0)}(t)=0 \tag{A5}$$

$$\ddot{x}_0^{(1)}(t)+x_0^{(1)}(t)=f\left(x_0,\dot{x}_0,\overset{(i)}{x_0}\right)-\Lambda_0^{(1)}x_0(t) \tag{A6}$$

$$\ddot{x}_0^{(2)}(t)+x_0^{(2)}(t)=2f^{(1)}\left(x_0,\dot{x}_0,\overset{(i)}{x_0}\right)-2\Lambda_0^{(1)}\_\Lambda_0^{(2)}x_0(t) \tag{A7}$$

The solution of Eq. (A.5) with initial conditions and is $x_0^{(0)}(0)=l\dot{x}_0^{(0)}(0)=0$,

$$x_0^{(0)}(t)=l\cos t \tag{A8}$$

From Eq. (A.5), Eq. (A.8) and Eq. (A.6), we yield,

$$\ddot{x}_0^{(1)}(t)+x_0^{(1)}(t)=\mu l^3\cos^3 t-\Lambda_0^{(1)}l\cos t=l\cos t\left(\frac{3l^2}{4}-\Lambda_0^{(1)}\right)+\frac{l^3\mu}{4}\cos 3t \tag{A9}$$

Neglect the presence of a secular term in Eq. (A.9),

$$\Lambda_0^{(1)}-\frac{3l^2}{4}=0 \tag{A10}$$

From Eq. (A.9) using initial conditions $x_0^{(1)}(0)=0$ and $\dot{x}_0^{(1)}(0)=0$,

$$x_0^{(1)}(t)=\frac{l^3\mu}{32}(\cos t-\cos 3t) \tag{A11}$$

Using Eqns. (A.8), (A.10), and (A.11) in Eq. (A.7), we yields,

$$\ddot{x}_0^{(2)}(t)+x_0^{(2)}(t)=\left(\frac{3l^4}{128}-\Lambda_0^{(2)}\right)l\cos t-\frac{13l^4\mu^2}{128}\cos 3t-\frac{11l^4\mu^2}{128}\cos 5t \tag{A12}$$

Neglect the presence of a secular term in Eq. (A.12),

$$\Lambda_0^{(2)}-\frac{3l^4}{128}=0 \tag{A13}$$

From Eq. (A.12) using initial conditions: $x_0^{(2)}(0)=0$ and $\dot{x}_0^{(2)}(0)=0$, we get

$$x_0^{(2)}(t)=\frac{13l^4\mu^2}{1024}(\cos 3t-\cos t)+\frac{11l^4\mu^2}{3072}(\cos 5t-\cos t) \tag{A14}$$

Substituting $x_0^{(0)}(t), x_0^{(1)}(t), x_0^{(2)}(t)$ in Eq. (A.4), we get the solution of Eq. (25) in the text.

Approximate analytical solution of oscillations [28] using HPM

Eq. (27) can be rewritten as follows:

$$\ddot{x}(t)+x(t)+\left(\alpha x(t)+\beta (x(t))^2\right)\ddot{x}(t)=0 \tag{A15}$$

This equation is in the form

$$\ddot{x}(t)+x(t)=-f\left(x,\dot{x},\overset{(i)}{x}\right) \tag{A16}$$

where $f\left(x,\dot{x},\overset{(i)}{x}\right)=\left(\alpha x(t)+\beta (x(t))^{2}\right)\ddot{x}(t)$

The above equation can be rewritten as follows:

$$\ddot{x}_0^{(0)}(t)+x_0^{(0)}(t)=0 \tag{A17}$$

$$\ddot{x}_0^{(1)}(t)+x_0^{(1)}(t)=-f\left(x,\dot{x},\overset{(i)}{x}\right)-\Lambda_0^{(1)}x_0(t) \tag{A18}$$

The solution of Eq. (A.17) with initial conditions and $x_0^{(0)}(0)=l$ is $\dot{x}_0^{(0)}(0)=0$,

$$x_0^{(0)}(t)=l\cos t \tag{A19}$$

From Eq. (A.17), Eq. (A.19) and Eq. (A.18), we yield,

$$\ddot{x}_0^{(1)}(t)+x_0^{(1)}(t)=\alpha l^2\cos^2 t+\beta l^3\cos^3 t-\Lambda_0^{(1)}l\cos t \tag{A20}$$

$$=\left(\frac{3\beta l^3}{4}-\Lambda_0^{(1)}\right)l\cos t+\frac{\alpha l^2}{2}(1+\cos 2t)+\frac{\beta l^3}{4}\cos 3t \tag{A21}$$

Avoid the presence of a secular term in Eq. (A.21), i.e.,

$$\Lambda_0^{(1)}-\frac{3\beta l^3}{4}=0 \tag{A22}$$

The solution of Eq. (A.22) using initial conditions $x_0^{(1)}=0$ and $\dot{x}_0^{(1)}=0$, can be obtained as follows:

$$x_0^{(1)}(t)=\sin^2\left(\frac{t}{2}\right)\left[\left(\frac{2\alpha l^2}{3}+\frac{\beta l^3}{4}\right)\cos t+\frac{\beta l^3}{8}\cos 2t+\frac{4\alpha l^2}{3}+\frac{\beta l^3}{8}\right] \tag{A23}$$

Substituting $x_0^{(0)}(t), x_0^{(1)}(t)$ in Eq. (A.4), we get a solution of Eq. (35) in the text.

Approximate analytical solution of oscillations [29] using HPM

Eq. (37) can be rewritten as follows:

$$\ddot{u}(t)+u(t)=-\varsigma u^3 \tag{A24}$$

This equation is in the form

$$\ddot{u}(t)+u(t)=-\varsigma f\left(u,\dot{u},\overset{(i)}{u}\right) \tag{A25}$$

where $f\left(u,\dot{u},\overset{(i)}{u}\right)=[u(t)]^{3}$

The above equation can be rewritten as follows:

$$\ddot{u}_0^{(0)}(t)+u_0^{(0)}(t)=0 \tag{A26}$$

$$\ddot{u}_0^{(1)}(t)+u_0^{(1)}(t)=-\varsigma f\left(u,\dot{u},\overset{(i)}{u}\right)-\Lambda_0^{(1)}\,l\cos t \tag{A27}$$

We obtain the solution of Eq. (A.26) with initial conditions $u_0^{(0)}(0)=l$ and $\dot{u}_0^{(0)}(0)=0$, and we get

$$u_0^{(0)}(t)=l\cos t \tag{A28}$$

From Eq. (A.26), Eq. (A.28) and Eq. (A.27), we yield,

$$\ddot{u}_0^{(1)}(t)+u_0^{(1)}(t)=-\varsigma\, l^3\cos^3 t-\Lambda_0^{(1)}\,l\cos t \tag{A29}$$

$$=-\left(\frac{3\varsigma\, l^3}{4}-\Lambda_0^{(1)}\right)l\cos t-\frac{\varsigma\, l^3}{4}\cos 3t \tag{A30}$$

Avoid the presence of a secular term in Eq. (A.30), i.e.,

$$\Lambda_0^{(1)}-\frac{3\varsigma\, l^3}{4}=0 \tag{A31}$$

We obtain the solution of Eq. (A.30) using initial conditions $u_0^{(1)}(0)=l$ and $\dot{u}_0^{(1)}(0)=0$, and we get

$$u_0^{(1)}(t)=\frac{-\varsigma\, l^3}{32}\left(\cos(t)-\cos(3t)\right) \tag{A32}$$

Substituting $u_0^{(0)}(t)$, $u_0^{(1)}(t)$ in Eq. (A.4), we get a solution of Eq. (45) in the text.

Appendix B

Shifted second Chebyshev kind wavelets operational matrix of derivatives:

We solve the nonlinear equation using the algorithm described in shifted second kind Chebyshev wavelet method for the case corresponding to $m=2$, $k=0$. To obtain the approximate solution of $x(t)$ and $u(t)$. Using Eq. (17), Eq. (27), and Eq. (37), the two operational matrices D and D^2 can be obtained

$$D=\begin{bmatrix}0&0&0\\4&0&0\\0&8&0\end{bmatrix},D^2=\begin{bmatrix}0&0&0\\0&0&0\\32&0&0\end{bmatrix} \tag{B1}$$

$$\psi(t)=\sqrt{\frac{2}{\pi}}\begin{bmatrix}2\\8t-4\\32t^2-32t+6\end{bmatrix},C^T=\sqrt{\frac{\pi}{2}}\begin{bmatrix}c_0\;c_1\;c_2\end{bmatrix} \tag{B2}$$

The second kind Chebyshev wavelet expansion of a function, to define the residual, $\Re(t)$ of this equation can be written as follows:

$$\Re(t) = C^T D^2 \psi(t) + F_1^T \psi(t)(\psi(t))^T D^T C + F_2^T \psi(t)(\psi(t))^T C - G_1^{TT} \psi(t) \tag{B3}$$

The approximate solution is

$$x(t) = C^T \psi(t) \tag{B4}$$

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