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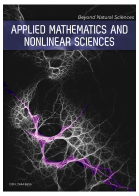

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Anticipated backward doubly stochastic differential equations with non-Liphschitz coefficients

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Abstract

In this work, we deal with a new type of differential equations called anticipated backward doubly stochastic differential equations. We establish existence and uniqueness of solution in the case of non-Lipschitz coefficients.

Keywords: Anticipated backward doubly stochastic differential equation, non-lipschitz coefficients, Itô's representation formula and Gronwall lemma.
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1 Introduction

Backward stochastic differential equations (BSDEs in short) were first introduced by Pardoux and Peng [4]. They proved an existence and uniqueness result under Lipschitz condition. Since then many efforts have been made in relaxing the Lipschitz assumption of the generator of the BSDEs (see among others Mao [3] and Wang and Huang [7]). Few years later, the same authors considered in [5] a new type of BSDEs, that is a class of backward doubly stochastic differential equations (BDSDEs in short) with two different directions of stochastic integrals. These equations are extensively used in the study of stochastic partial differential equations (SPDEs). Their link with SPDEs in the case of Lipchitzian drift was established in [5]. The key point of solvency of such equations is the martingale representation theorem. In this spirit, Bally and Matoussi [1] gave the probabilistic representation of the solution in Sobolev space of semilinear SPDEs in terms of BDSDEs.

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On the other hand, Peng and Yang [6] introduced the following type of anticipated backward stochastic differential equations (ABSDEs in short)

$$\begin{cases} -dY_t = f\left(t, Y_t, Z_t, Y_{t+\delta(t)}, Z_{t+\zeta(t)}\right) dt - Z_t dW_t, & 0 \le t \le T, \\ Y_t = \xi_t, \quad Z_t = \eta_t, & T \le t \le T+K, \end{cases}$$

where δ and ζ are given nonnegative deterministic functions. In these equations, the generator includes not only the values of solutions of the present but also the future. In [6], the authors obtained the existence and uniqueness of the solution of ABSDE under Lipschitz assumption, gave the comparison theorem for one dimensional ABSDEs and finally they solved a stochastic control problem by showing the duality between linear stochastic differential delay equations and ABSDEs.

This paper is devoted to the following anticipated BDSDE

$$\begin{cases} Y_t = \xi_T + \int_t^T f(r, Y_r, Z_r, Y_{r+\delta(r)}, Z_{r+\zeta(r)}) dr + \int_t^T g(r, Y_r, Z_r) dB_r - \int_t^T Z_r dW_r, & 0 \le t \le T, \\ Y_t = \xi_t, \quad Z_t = \eta_t, \qquad T \le t \le T+K \end{cases} \tag{1.1}$$

where K is a positive constant, $\xi_.$, $\eta_.$ are given stochastic processes and $\delta, \zeta : [0,T] \to \mathbf{R}_+$ are continuous functions satisfying:

(A1) : $t + \delta(t) \le T + K, \quad t + \zeta(t) \le T + K.$

(A2) : There exists $M \ge 0$ such that for $0 \le t \le T$ and non negative integrable function h,

$$\int_t^T h(r + \phi(r)) dr \le M \int_t^{T+K} h(r) dr, \quad \phi \in \{\delta, \zeta\}.$$

The paper is organized as follows. In section 2, we study first solvability of our equation in the case of Lipschtzian coefficients. Using this result, in section 3 we prove existence and uniqueness of solution with coefficients satisfying rather weaker conditions.

2 Preliminaries

Let Ω be a non-empty set, $\mathscr{F}$ a $\sigma-$algebra of sets of Ω and $\mathbf{P}$ a probability measure defined on $\mathscr{F}$. The triplet $(\Omega, \mathscr{F}, \mathbf{P})$ defines a probability space, which is assumed to be complete. For a fix real $0 < T \le \infty$, we assume given two mutually independent processes:

- an $\ell-$dimensional Brownian motion $(B_t)_{0 \le t \le T}$,
- a $d-$dimensional Brownian motion $(W_t)_{0 \le t \le T}$.

We consider the family $(\mathscr{F}_t)_{0 \le t \le T}$ given by

$$\mathscr{F}_t = \mathscr{F}_t^W \vee \mathscr{F}_{t,T}^B, \quad 0 \le t \le T, \quad \mathscr{G}_s = \mathscr{F}_{0,s}^W \vee \mathscr{F}_{s,T+K}^B, \quad 0 \le s \le T+K,$$

where for any process $\{\varphi_t\}_{t \ge 0}$, $\mathscr{F}_{s,t}^{\varphi} = \sigma\{\varphi_r - \varphi_s, \ s \le r \le t\} \vee \mathscr{N}$, $\mathscr{F}_t^{\varphi} = \mathscr{F}_{0,t}^{\varphi}$. Here $\mathscr{N}$ denotes the class of $\mathbf{P}-$null sets of $\mathscr{F}$. Note that $(\mathscr{F}_t)_{0 \le t \le T}$ does not constitute a classical filtration.

For $k \in \mathbf{N}^*$ we consider the following sets (where $\mathbf{E}$ denotes the mathematical expectation with respect to the probability measure $\mathbf{P}$):

- $L^2(\mathscr{G}_T, \mathbf{R}^k)$ the space of $\mathscr{G}_T$-measurable random variable such that $\mathbf{E}\left[|\xi_T|^2\right] < +\infty$.

- $\mathscr{S}^2_{[0,T]}(\mathscr{G},\mathbf{R}^k)$ the space of $\mathscr{G}_t-$adapted càdlàg processes
$$\Psi:[0,T]\times\Omega\longrightarrow\mathbf{R}^k,\ ||\Psi||^2_{\mathscr{S}^2(\mathbf{R}^k)}=\mathbf{E}\left(\sup_{0\leq t\leq T}|\Psi_t|^2\right)<\infty.$$
- $\mathscr{M}^2_{[0,T]}(\mathscr{G},\mathbf{R}^{k\times d})$ the space of $\mathscr{G}_t-$progressively measurable processes
$$\Psi:[0,T]\times\Omega\longrightarrow\mathbf{R}^{k\times d},\ ||\Psi||^2_{\mathscr{M}^2(\mathbf{R}^{k\times d})}=\mathbf{E}\int_0^T|\Psi_t|^2\,dt<\infty.$$
- $\mathscr{C}^2_{\mathscr{G}}(0,T)=\mathscr{M}^2_{[0,T]}(\mathscr{G},\mathbf{R}^k)\times\mathscr{M}^2_{[0,T]}(\mathscr{G},\mathbf{R}^{k\times d})$ endowed with the norm
$$||(Y,Z)||^2_{\mathscr{C}^2_{\mathscr{G}}(0,T)}=||Y||^2_{\mathscr{M}^2(\mathbf{R}^k)}+||Z||^2_{\mathscr{M}^2(\mathbf{R}^{k\times d})}.$$
- $\mathscr{B}^2_{\mathscr{G}}(0,T)=\mathscr{S}^2_{[0,T]}(\mathscr{G},\mathbf{R}^k)\times\mathscr{M}^2_{[0,T]}(\mathscr{G},\mathbf{R}^{k\times d})$ endowed with the norm
$$||(Y,Z)||^2_{\mathscr{B}^2_{\mathscr{G}}(0,T)}=||Y||^2_{\mathscr{S}^2(\mathbf{R}^k)}+||Z||^2_{\mathscr{M}^2(\mathbf{R}^{k\times d})}.$$
- $\mathbf{S}$ be the set of all nondecreasing, continuous and concave function $\rho(\cdot):\mathbf{R}_+\to\mathbf{R}_+$ satisfying $\rho(0)=0$, $\rho(s)>0$ for $s>0$ and $\int_{0+}\frac{du}{\rho(u)}=+\infty$.

Remark 2.1. *For any $\rho\in\mathbf{S}$, we can find a pair of positive constants a and b such that $\rho(v)\leq a+bv$ for all $v\geq 0$.*

We denote $\mathscr{A}=\Omega\times[0,T]\times\mathbf{R}^k\times\mathbf{R}^{k\times d}$, $\quad f(r,0)=h(r,0,0,0,0)$, for all $x,y\in\mathbf{R}^k$ $|x|$ the Euclidean norm of x and denote by $\langle x,y\rangle$ the Euclidean inner product.

Definition 2.2. *A pair of processes (Y,Z) is called a solution to ABDSDE (1.1), if $(Y,Z)\in\mathscr{B}^2_{\mathscr{G}}(0,T+K)$ and it satisfies eq.(1.1).*

First we investigate the case of lipschitz coefficients.

3 The case of Lipschitz coefficients

In this subsection, we will mainly study the existence and uniqueness of the solution to ABDSDE (1.1) with Lipschitz coefficients. For this purpose, we first make the following assumptions.

3.0.1 Assumptions

In the following, we assume that there exists $\rho\in\mathbf{S}$ such that f and g satisfy assumptions **(H1)**.

(H1.1): There exists a constant $c>0$ such that
$$\begin{aligned}|f(t,y,z,\theta(r),\varphi(r))-f(t,y',z',\theta'(r),\varphi'(r))|^2&\leq c(|y-y'|^2+|z-z'|^2)\\&\quad+\mathbf{E}^{\mathscr{F}_t}[|\theta(r)-\theta'(r)|^2+|\varphi(r)-\varphi'(r)|^2],\end{aligned}$$
for all $(r,r')\in[t,T+K]$, $(t,y,z,\theta(r),\varphi(r))$, $(t,y',z',\theta'(r),\varphi'(r))\in\mathscr{A}\times\mathscr{C}^2_{\mathscr{G}}(t,T+K)$.

(H1.2): There exists a constant $0<\alpha_1<1$ such that for any (t,y,z), $(t,y',z')\in[0,T]\times\mathbf{R}^k\times\mathbf{R}^{k\times d}$
$$|g(t,y,z)-g(t,y',z')|^2\leq c|y-y'|^2+\alpha_1|z-z'|^2.$$

(H1.3): For any $(t,y,z)\in[0,T]\times\mathbf{R}^k\times\mathbf{R}^{k\times d}$,
$$\mathbf{E}\left[\int_0^T|f(s,0)|^2ds\right]+\mathbf{E}\left[\int_0^T|g(s,y,z)|^2ds\right]<\infty.$$

3.1 Existence and uniqueness of solution

Lemma 3.1. *Suppose that $(Y_t,Z_t)_{0\leq t\leq T}\in\mathscr{C}^2_{\mathscr{G}}(0,T+K)$ is the unique solution to the ABDSDE (1.1). Then $Y\in\mathscr{S}^2_{[0,T]}(\mathscr{G},\mathbf{R}^k)$.*

Proof. Itô's formula applied to eq.(1.1) yields, for $0\leq t\leq T$

$$
\begin{aligned}
|Y_t|^2+\int_t^T|Z_r|^2dr &= |\xi_T|^2+2\int_t^T\langle Y_r,f(r,Y_r,Z_r,Y_{r+\delta(r)},Z_{r+\zeta(r)})\rangle dr+2\int_t^T\langle Y_r,g(r,Y_r,Z_r)dB_r\rangle\\
&\quad -2\int_t^T\langle Y_r,Z_rdW_r\rangle+\int_t^T|g(r,Y_r,Z_r)|^2dr. \qquad (3.1)
\end{aligned}
$$

Using the fact that $2ab\leq\varepsilon a^2+b^2/\varepsilon$ for $\varepsilon>0$ and assumptionn **(H1.1)**, we deduce that

$$
\begin{aligned}
2\mathbf{E}\int_t^T\langle Y_r,f(r,Y_r,Z_r,Y_{r+\delta(r)},Z_{r+\zeta(r)})\rangle dr &\leq \frac{1}{\varepsilon}\mathbf{E}\int_t^T|Y_r|^2dr+\varepsilon c\mathbf{E}\int_t^T(|Y_r|^2+|Z_r|^2)dr\\
&\quad+\varepsilon\mathbf{E}\int_t^T\mathbf{E}^{\mathscr{F}_t}[|Y_{r+\delta(r)}|^2+|Z_{r+\zeta(r)}|^2]dr+2\mathbf{E}\int_t^T|Y_r||f(r,0)|dr.
\end{aligned}
$$

Applying **(A2)**, we obtain finally

$$
\begin{aligned}
2\mathbf{E}\int_t^T\langle Y_r,f(r,Y_r,Z_r,Y_{r+\delta(r)},Z_{r+\zeta(r)})\rangle dr &\leq \left(\frac{1}{\varepsilon}+\varepsilon(c+M)\right)\mathbf{E}\int_t^T|Y_r|^2dr\\
&\quad+\varepsilon(c+M)\mathbf{E}\int_t^T|Z_r|^2dr+2\mathbf{E}\int_t^T|Y_r||f(r,0)|dr+\varepsilon M\mathbf{E}\int_T^{T+K}(|\xi_r|^2+|\eta_r|^2)dr.
\end{aligned}
$$

In addition, for any $0\leq t\leq T$, we have

$$
\begin{aligned}
\mathbf{E}\int_t^T|g(r,Y_r,Z_r)|^2dr &\leq \mathbf{E}\int_t^T|g(r,Y_r,Z_r)-g(r,0,0)|^2dr+\mathbf{E}\int_t^T|g(r,0,0)|^2dr\\
&\leq c\mathbf{E}\int_t^T|Y_r|^2dr+\alpha_1\mathbf{E}\int_t^T|Z_r|^2dr+\mathbf{E}\int_t^T g(r,0,0)|^2dr.
\end{aligned}
$$

Putting pieces together, we deduce from (3.1) that

$$
\begin{aligned}
\mathbf{E}\int_t^T|Z_r|^2dr &\leq \mathbf{E}[|\xi_T|^2]+M\varepsilon\mathbf{E}\int_T^{T+K}(|\xi_r|^2+|\eta_r|^2)dr+\left(\frac{1}{\varepsilon}+\varepsilon(c+M)+c\right)\mathbf{E}\int_t^T|Y_r|^2dr\\
&\quad+2\mathbf{E}\int_t^T|Y_r||f(r,0)|dr+\mathbf{E}\int_t^T|g(r,0,0)|^2dr+(\alpha_1+\varepsilon(c+M))\mathbf{E}\int_t^T|Z_r|^2dr.
\end{aligned}
$$

If we choose $\varepsilon=\varepsilon_0$ satisfying $1/C_0=(1-[\alpha_1+\varepsilon_0(c+M)])^{-1}>0$, we deduce that

$$
\mathbf{E}\int_t^T|Z_r|^2dr\leq\frac{1}{C_0}\mathbf{E}\left[X_t+2\int_t^T|Y_r||f(r,0)|dr+\int_t^T|g(r,0,0)|^2dr\right] \qquad (3.2)
$$

where putting $C_1=\left(\frac{1}{\varepsilon_0}+\varepsilon_0(c+M)+c\right)$,

$$
X_t=\left[|\xi_T|^2+M\varepsilon_0\int_T^{T+K}(|\xi_r|^2+|\eta_r|^2)dr+C_1\int_t^T|Y_r|^2dr\right].
$$

By the same computations as before, we have

$$\mathbf{E}[|\xi_T|^2]+2\mathbf{E}\int_t^T\langle Y_r,f(r,Y_r,Z_r,Y_{r+\delta(r)},Z_{r+\zeta(r)})\rangle dr+\mathbf{E}\int_t^T|g(r,Y_r,Z_r)|^2dr$$
$$\leq \frac{1}{C_0}\mathbf{E}\left[X_t+2\int_t^T|Y_r||f(r,0)|dr+\int_t^T|g(r,0,0)|^2dr\right]. \tag{3.3}$$

Moreover using again eq.(3.1), we have

$$\begin{aligned}\mathbf{E}\left(\sup_{t\leq r\leq T}|Y_r|^2\right)\leq\ &\mathbf{E}[|\xi_T|^2]+2\mathbf{E}\sup_{t\leq s\leq T}\left(\int_s^T\langle Y_r,f(r,Y_r,Z_r,Y_{r+\delta(r)},Z_{r+\zeta(r)})\rangle dr\right)\\&+2\mathbf{E}\sup_{t\leq s\leq T}\left|\int_s^T\langle Y_r,g(r,Y_r,Z_r)dB_r\rangle\right|+2\mathbf{E}\sup_{t\leq s\leq T}\left|\int_s^T\langle Y_r,Z_rdW_r\rangle\right|\\&+\int_t^T|g(r,Y_r,Z_r)|^2dr.\end{aligned} \tag{3.4}$$

By Burkhölder-Davis-Gundy inequality, there exists a constant $C>0$ which may vary from line to line such that

$$\mathbf{E}\sup_{t\leq s\leq T}\left|\int_s^T\langle Y_r,g(r,Y_r,Z_r)dB_r\rangle\right|\leq\frac{1}{8}\mathbf{E}\left(\sup_{t\leq r\leq T}|Y_r|^2\right)+C\int_t^T|g(r,Y_r,Z_r)|^2dr$$
$$2\mathbf{E}\sup_{t\leq s\leq T}\left|\int_s^T\langle Y_r,Z_rdW_r\rangle\right|\leq\frac{1}{8}\mathbf{E}\left(\sup_{t\leq r\leq T}|Y_r|^2\right)+C\int_t^T|Z_r|^2dr.$$

Using the above inequalities, we deduce from (3.4) that

$$\begin{aligned}\frac{3}{4}\mathbf{E}\left(\sup_{t\leq r\leq T}|Y_r|^2\right)\leq\ &\mathbf{E}[|\xi_T|^2]+2\mathbf{E}\sup_{t\leq s\leq T}\left(\int_s^T\langle Y_r,f(r,Y_r,Z_r,Y_{r+\delta(r)},Z_{r+\zeta(r)})\rangle dr\right)\\&+C\int_t^T|g(r,Y_r,Z_r)|^2dr+C\mathbf{E}\int_t^T|Z_r|^2dr\end{aligned}$$

Applying (3.2) and (3.3), we deduce that

$$\frac{3}{4}\mathbf{E}\left(\sup_{t\leq r\leq T}|Y_r|^2\right)\leq\frac{2C}{C_0}\mathbf{E}\left[X_t+2\int_t^T|Y_r||f(r,0)|dr+\int_t^T|g(r,0,0)|^2dr\right]. \tag{3.5}$$

Moreover, we have

$$\frac{4C}{C_0}\mathbf{E}\int_t^T|Y_r||f(r,0)|dr\leq\frac{1}{4}\mathbf{E}\left(\sup_{t\leq r\leq T}|Y_r|^2\right)+4\left(\frac{2C}{C_0}\right)^2\mathbf{E}\left(\int_t^T|f(r,0)|dr\right)^2.$$

Hence gathering (3.2) and (3.5) we obtain

$$\frac{1}{2}\mathbf{E}\left(\sup_{t\leq r\leq T}|Y_r|^2\right)+\mathbf{E}\int_t^T|Z_r|^2dr\leq C_2\mathbf{E}\left[X_t+\left(\int_t^T|f(r,0)|dr\right)^2+\int_t^T|g(r,0,0)|^2dr\right], \tag{3.6}$$

where C_2 is a positive constant (which may change from line to line).
Then, applying the fubini's theorem to (3.6), this leads to

$$\begin{aligned}\mathbf{E}\left(\sup_{t\leq r\leq T}|Y_r|^2\right)\leq\ &C_2\mathbf{E}\left[|\xi_T|^2+\int_T^{T+K}(|\xi_r|^2+|\eta_r|^2)dr+\left(\int_t^T|f(r,0)|dr\right)^2+\int_t^T|g(r,0,0)|^2dr\right]\\&+C_3\int_t^T\mathbf{E}[\sup_{r\leq s\leq T}|Y_s|^2]dr,\end{aligned}$$

where $C_3 = 2C_1C_2$. Hence Gronwall's inequality yields

$$\mathbf{E}\left(\sup_{t\leq r\leq T}|Y_r|^2\right) \leq +\infty.$$

This implies that $Y \in \mathscr{S}^2_{[0,T]}(\mathscr{G},\mathbf{R}^k)$. This completes the proof. □

To solve our equations, we examine first the cases where the coefficients do not depend on the variables. Namely, we consider the stochastic equation

$$Y_t = \xi_T + \int_t^T f(r)dr + \int_t^T g(r)dB_r - \int_t^T Z_r dW_r, \quad 0 \leq t \leq T. \tag{3.7}$$

where $f \in \mathscr{M}^2_{[0,T]}(\mathscr{G},\mathbf{R}^k)$, $\quad g \in \mathscr{M}^2_{[0,T]}(\mathscr{G},\mathbf{R}^{k\times\ell})$ and $\xi_T \in L^2(\mathscr{G}_T,\mathbf{R}^k)$.

Let us recall the following result which will be useful in the sequel (the proof is omitted since it is an adaptation of Theorem 3.1 in Xu [9]).

Proposition 3.2. *Given $\xi_T \in L^2(\mathscr{G}_T,\mathbf{R}^k)$, eq.(3.7) has a unique solution $(Y_t,Z_t)_{0\leq t\leq T} \in \mathscr{C}^2_{\mathscr{G}}(0,T)$.*

We are now in position to give our main results of this section.

Theorem 3.3. *Assume that the assumptions* **(A1)**, **(A2)** *and* **(H1)** *are true and let $\xi_T \in L^2(\mathscr{G}_T,\mathbf{R}^k)$. Then for any $(\xi,\eta) \in \mathscr{S}^2_{[T,T+K]}(\mathscr{G},\mathbf{R}^k) \times \mathscr{M}^2_{[T,T+K]}(\mathscr{G},\mathbf{R}^{k\times d})$ the ABDSDE (1.1) has a unique solution $(Y_t,Z_t)_{0\leq t\leq T} \in \mathscr{B}^2_{\mathscr{G}}(0,T+K)$.*

Proof. **(i) Existence**. Let us consider the mapping

$$\begin{aligned}\Psi : \mathscr{C}^2_{\mathscr{G}}(0,T+K) &\to \mathscr{C}^2_{\mathscr{G}}(0,T+K),\\ (y,z) &\to (Y,Z)\end{aligned}$$

where the pair $(Y_t,Z_t)_{0\leq t\leq T+K} \in \mathscr{C}^2_{\mathscr{G}}(0,T+K)$ is s.t. $(Y_t,Z_t)_{T\leq t\leq T+K} = (\xi_t,\eta_t)$ and it satisfies the equation

$$\begin{cases}\forall 0\leq t\leq T,\\ Y_t = \xi_T + \displaystyle\int_t^T f(r,y_r,z_r,y_{r+\delta(r)},z_{r+\zeta(r)})dr + \int_t^T g(r,y_r,z_r)dB_r - \int_t^T Z_r dW_r,\\ \forall t\in[T,T+K], \quad Y_t = \xi_t, \quad Z_t = \eta_t.\end{cases} \tag{3.8}$$

Thanks to Proposition 3.2, the mapping Ψ is well defined. Let (Y,Z) and $(\widetilde{Y},\widetilde{Z})$ be two solutions of eq.(3.8), i.e :

$$(Y,Z) = \Psi(y,z) \quad \textit{and} \quad (\widetilde{Y},\widetilde{Z}) = \Psi(\widetilde{y},\widetilde{z}).$$

Fix $\beta \in \mathbf{R}$. The pair $(\overline{Y},\overline{Z})$ solves the ABDSDE

$$\begin{cases}\overline{Y}_t = \displaystyle\int_t^T \Delta f(r)dr + \int_t^T \Delta g(r)dB_r - \int_t^T \overline{Z}_r dW_r, \quad \forall 0\leq t\leq T,\\ \forall t\in[T,T+K], \quad \overline{Y}_t = 0, \quad \overline{Z}_t = 0,\end{cases} \tag{3.9}$$

where for $\rho \in \{Y,Z\}, \overline{\rho} = \rho - \widetilde{\rho}$, $\Delta g(r) = g(r,y_r,z_r) - g(r,\widetilde{y}_r,\widetilde{z}_r)$ and

$$\Delta f(r) = f(r,y_r,z_r,y_{r+\delta(r)},z_{r+\zeta(r)}) - f(r,\widetilde{y}_r,\widetilde{z}_r,\widetilde{y}_{r+\delta(r)},\widetilde{z}_{r+\zeta(r)}).$$

Applying Ito's formula, we obtain

$$\mathbf{E}\left[e^{\beta t}|\overline{Y}_t|^2+\beta\int_t^T e^{\beta r}|\overline{Y}_r|^2dr+\int_t^T e^{\beta r}|\overline{Z}_r|^2dr\right]=2\mathbf{E}\int_t^T e^{\beta r}\langle\overline{Y}_r,\Delta f(r)\rangle dr+\mathbf{E}\int_t^T e^{\beta r}|\Delta g(r)|^2dr. \tag{3.10}$$

Using the inequality $2ab\leq\varepsilon a^2+b^2/\varepsilon$ (where $\varepsilon>0$ will be chosen later) and assumption **(H1.1)**, we obtain

$$\mathbf{E}\int_t^T e^{\beta r}|\Delta g(r)|^2dr\leq c\mathbf{E}\int_t^{T+K}e^{\beta r}|\overline{y}_r|^2dr+\alpha_1\mathbf{E}\int_t^{T+K}e^{\beta r}|\overline{z}_r|^2dr.$$

Similarly, we have

$$2e^{\beta r}\langle\overline{Y}_r,\Delta f(r)\rangle\leq e^{\beta r}\varepsilon|\overline{Y}_r|^2+\frac{c}{\varepsilon}e^{\beta r}(|\overline{y}_r|^2+|\overline{z}_r|^2)+\frac{1}{\varepsilon}e^{\beta r}\mathbf{E}^{\mathscr{F}_t}\left[|\overline{y}_{r+\delta(r)}|^2+|\overline{z}_{r+\zeta(r)}|^2\right].$$

Which implies by virtue of condition **(A2)** that

$$2\mathbf{E}\int_t^T e^{\beta r}\langle\overline{Y}_r,\Delta f(r)\rangle dr\leq\varepsilon\mathbf{E}\int_t^{T+K}e^{\beta r}|\overline{Y}_r|^2dr+\frac{1}{\varepsilon}(c+M)\mathbf{E}\int_t^{T+K}e^{\beta r}(|\overline{y}_r|^2+|\overline{z}_r|^2)dr.$$

Therefore, we can write (where $\gamma=\frac{\frac{1}{\varepsilon}(c+M)+c}{\frac{1}{\varepsilon}(c+M)+\alpha_1}$)

$$\begin{aligned}\mathbf{E}\left(e^{\beta t}|\overline{Y}_t|^2\right)&+(\beta-\varepsilon)\mathbf{E}\int_t^{T+K}e^{\beta r}|\overline{Y}_r|^2dr+\mathbf{E}\int_t^{T+K}e^{\beta r}|\overline{Z}_r|^2dr\\&\leq\left(\frac{1}{\varepsilon}(c+M)+c\right)\mathbf{E}\int_t^{T+K}e^{\beta r}|\overline{y}_r|^2dr+\left(\frac{1}{\varepsilon}(c+M)+\alpha_1\right)\mathbf{E}\int_t^{T+K}e^{\beta r}|\overline{z}_r|^2dr\\&=\left(\frac{1}{\varepsilon}(c+M)+\alpha_1\right)\mathbf{E}\int_t^{T+K}e^{\beta r}\left[\gamma|\overline{y}_r|^2+|\overline{z}_r|^2\right]dr.\end{aligned}$$

Hence if we choose $\varepsilon=\varepsilon_0$ satisfying $\overline{c}=\left(\frac{1}{\varepsilon_0}(c+M)+\alpha_1\right)<1$, choose $\beta=\varepsilon_0+\gamma$, then we deduce

$$\mathbf{E}\int_t^{T+K}e^{\beta r}\left[\gamma|\overline{Y}_r|^2+|\overline{Z}_r|^2\right]dr\leq\overline{c}\mathbf{E}\int_t^{T+K}e^{\beta r}\left[\gamma|\overline{y}_r|^2+|\overline{z}_r|^2\right]dr.$$

Thus, the mapping Ψ is a strict contraction on $\mathscr{C}^2_{\mathscr{G}}(0,T+K)$ and it has a unique fixed point

$$(Y,Z)\in\mathscr{C}^2_{\mathscr{G}}(0,T+K).$$

It remains to prove that the above solution is in $\mathscr{B}^2_{\mathscr{G}}(0,T+K)$. Indeed, by Lemma 3.1, we have $Y\in\mathscr{S}^2_{[0,T]}(\mathscr{G},\mathbf{R}^k)$. Thus, we obtain $(Y_t,Z_t)_{0\leq t\leq T}\in\mathscr{B}^2_{\mathscr{G}}(0,T+K)$.

(ii) Uniqueness. Let (Y,Z) and $(\widetilde{Y},\widetilde{Z})$ two solutions of eq.(1.1). Itô's formula applied to eq.(3.9) yields, for $0\leq t\leq T$

$$\mathbf{E}[|\overline{Y}_t|^2]+\mathbf{E}\int_t^T|\overline{Z}_r|^2dr\leq2\mathbf{E}\int_t^T\langle\overline{Y}_r,\Delta f(r)\rangle dr+\mathbf{E}\int_t^T|\Delta g(r)|^2dr. \tag{3.11}$$

Using assumption **(H1)**, we have :

$$2\mathbf{E}\int_t^T\langle\overline{Y}_r,\Delta f(r)\rangle dr\leq(\frac{1}{\varepsilon}(c+M)+\varepsilon)\mathbf{E}\int_t^T|\overline{Y}_r|^2dr+\frac{1}{\varepsilon}(c+M)\mathbf{E}\int_t^T|\overline{Z}_r|^2dr,$$

$$\mathbf{E}\int_t^T|\Delta g(r)|^2dr\leq c\mathbf{E}\int_t^T|\overline{Y}_r|^2dr+\alpha_1\mathbf{E}\int_t^T|\overline{Z}_r|^2dr.$$

Hence if we choose $\xi = \xi_0$ satisfying $\overline{\alpha} = \frac{1}{\varepsilon_0}(c+M)+\alpha_1 < 1$ and denote $\overline{c} = \frac{c(\varepsilon_0+1)+M}{\varepsilon_0} + \varepsilon_0$, then using the above inequalities, from (3.11), we obtain

$$\mathbf{E}\left(|\overline{Y}_t|^2\right) + (1-\overline{\alpha})\mathbf{E}\int_t^T |\overline{Z}_r|^2 dr \leq \overline{c}\mathbf{E}\int_t^T |\overline{Y}_r|^2 dr.$$

Then we can use Gronwall's inequality to deduce $\overline{Y} = 0$ and $\overline{Z} = 0$. This completes the proof. □

4 The case of non-Lipschitz coefficients

In this subsection, we will mainly study the existence and uniqueness of the solution to ABDSDE (1.1) with non-Lipschitz coefficients. For this purpose, we first make the following assumptions.

4.0.1 Assumptions

In the following, we assume that there exists $\rho \in \mathbf{S}$ such that f and g satisfy assumptions **(H2)**.

(H2.1): There exists a constant $c > 0$ such that

$$\begin{aligned}|f(t,y,z,\theta(r),\varphi(r)) - f(t,y',z',\theta'(r),\varphi'(r))|^2 &\leq c(\rho(|y-y'|^2) + |z-z'|^2) \\ &\quad + \mathbf{E}^{\mathscr{F}_t}[\rho(|\theta(r)-\theta'(r)|^2) + |\varphi(r)-\varphi'(r)|^2],\end{aligned}$$

for all $(r,r') \in [t,T+K]$, $(t,y,z,\theta(r),\varphi(r))$, $(t,y',z',\theta'(r),\varphi'(r)) \in \mathscr{A} \times \mathscr{C}^2_{\mathscr{G}}(t,T+K)$.

(H2.2): There exists a constant $0 < \alpha_1 < 1$ such that for any (t,y,z), $(t,y',z') \in [0,T] \times \mathbf{R}^k \times \mathbf{R}^{k\times d}$

$$|g(t,y,z) - g(t,y',z')|^2 \leq \rho(|y-y'|^2) + \alpha_1|z-z'|^2.$$

(H2.3): **(H1.3)** holds.

4.1 Existence and uniqueness of solution

We consider now the sequence $(\Theta^n)_{n\in\mathbf{N}} = (Y^n, Z^n)_{n\in\mathbf{N}}$ given by

$$\begin{cases} Y_t^0 = 0,\ Z_t^0 = 0, \quad 0 \leq t \leq T+K, \\ Y_t^n = \xi_T + \displaystyle\int_t^T f(r,Y_r^{n-1},Z_r^n,Y_{r+\delta(r)}^{n-1},Z_{r+\zeta(r)}^n)dr + \int_t^T g(r,Y_r^{n-1},Z_r^n)dB_r - \int_t^T Z_r^n dW_r, \qquad 0 \leq t \leq T, \\ Y_t^n = \xi_t, \quad Z_t^n = \eta_t, \qquad T \leq t \leq T+K. \end{cases} \tag{4.1}$$

Thanks to Theorem 3.3, this sequence is well defined since the generators $f(r,Y_r^{n-1},\cdot,Y_{r+\delta(r)}^{n-1},\cdot)$ and $g(r,Y_r^{n-1},\cdot)$ are Γ-Lipschitz. Let us state the following previous result

Lemma 4.1. *Assume that the assumptions* **(A1)**, **(A2)** *and* **(H2)** *are true and let* $\xi_T \in L^2(\mathscr{G}_T,\mathbf{R}^k)$. *Then for any* $(\xi,\eta) \in \mathscr{S}^2_{[T,T+K]}(\mathscr{G},\mathbf{R}^k) \times \mathscr{M}^2_{[T,T+K]}(\mathscr{G},\mathbf{R}^{k\times d})$ *there exists a positive constant* C' *such that*

$$\sup_{n\geq 0}\mathbf{E}|Y_t^n|^2 \leq C'(1+\mathbf{E}[X]), \qquad 0 \leq t \leq T+K \tag{4.2}$$

where

$$X = |\xi_T|^2 + \int_T^{T+K}(|\xi_r|^2 + |\eta_r|^2)dr + \int_0^T |f(r,0)|^2 dr + \int_0^T |g(r,0,0)|^2 dr.$$

Proof. For $\beta > 0$, apply Itô's formula to $e^{\beta t}|Y_t^n|^2$,

$$\mathbf{E}[e^{\beta t}|Y_t^n|^2] + \beta\mathbf{E}\int_t^T e^{\beta r}|Y_r^n|^2dr + \mathbf{E}\int_t^T e^{\beta r}|Z_r^n|^2dr = \mathbf{E}[e^{\beta T}|\xi_T|^2]$$
$$+2\mathbf{E}\int_t^T e^{\beta r}\langle Y_r^n, f(r,Y_r^{n-1},Z_r^n,Y_{r+\delta(r)}^{n-1},Z_{r+\zeta(r)}^n)\rangle dr + \mathbf{E}\int_t^T e^{\beta r}|g(r,Y_r^{n-1},Z_r^n)|^2dr.$$

Using the inequality $2ab \le \varepsilon a^2 + b^2/\varepsilon$ (where $\varepsilon > 0$ will be chosen later), we deduce from assumptions **(H2.1)** and **(H2.2)**

$$\mathbf{E}\int_t^T e^{\beta r}|g(r,Y_r^{n-1},Z_r^n)|^2dr \le \mathbf{E}\int_t^T e^{\beta r}|g(r,0,0)|^2dr + \mathbf{E}\int_t^T e^{\beta r}\left[\rho(|Y_r^{n-1}|^2) + \alpha_1|Z_r^n|^2\right]dr$$

and

$$2\mathbf{E}\int_t^T e^{\beta r}\langle Y_r^n, f(r,Y_r^{n-1},Z_r^n,Y_{r+\delta(r)}^{n-1},Z_{r+\zeta(r)}^n)\rangle dr \le \frac{\varepsilon}{2}\mathbf{E}\int_t^T e^{\beta r}|Y_r^n|^2dr + \frac{2}{\varepsilon}\mathbf{E}\int_t^T e^{\beta r}|f(r,0)|^2dr$$
$$+\frac{2}{\varepsilon}c\mathbf{E}\int_t^T e^{\beta r}\left(\rho(|Y_r^{n-1}|^2) + |Z_r^n|^2\right)dr + \frac{2}{\varepsilon}\mathbf{E}\int_t^T e^{\beta r}\mathbf{E}^{\mathscr{F}_t}\left[\rho(|Y_{r+\delta(r)}^{n-1}|^2) + |Z_{r+\zeta(r)}^n|^2\right]dr. \tag{4.3}$$

Applying condition **(A2)**, the last term on the right-hand side of (4.3) is less than

$$\frac{2}{\varepsilon}M\mathbf{E}\int_t^T e^{\beta r}\left[\rho(|Y_r^{n-1}|^2) + |Z_r^n|^2\right]dr + \frac{2}{\varepsilon}M\mathbf{E}\int_T^{T+K} e^{\beta r}\left[\rho(|\xi_r|^2) + |\eta_r|^2\right]dr.$$

Putting pieces together, we obtain finally

$$\mathbf{E}[e^{\beta t}|Y_t^n|^2] + \beta\mathbf{E}\int_t^T e^{\beta r}|Y_r^n|^2dr + \mathbf{E}\int_t^T e^{\beta r}|Z_r^n|^2dr \le \mathbf{E}[e^{\beta T}|\xi_T|^2] + \frac{\varepsilon}{2}\mathbf{E}\int_t^T e^{\beta r}|Y_r^n|^2dr$$
$$+\left(\frac{2}{\varepsilon}(c+M)+1\right)\mathbf{E}\int_t^T e^{\beta r}\rho(|Y_r^{n-1}|^2)dr + \left(\frac{2}{\varepsilon}(c+M)+\alpha_1\right)\mathbf{E}\int_t^T e^{\beta r}|Z_r^n|^2dr$$
$$+\frac{2}{\varepsilon}M\mathbf{E}\int_T^{T+K} e^{\beta r}\left[\rho(|\xi_r|^2) + |\eta_r|^2\right]dr + \frac{2}{\varepsilon}\mathbf{E}\int_t^T e^{\beta r}|f(r,0)|^2dr + \mathbf{E}\int_t^T e^{\beta r}|g(r,0,0)|^2dr.$$

This implies thanks to Remark 2.1 that

$$\mathbf{E}[e^{\beta t}|Y_t^n|^2] + \beta\mathbf{E}\int_t^T e^{\beta r}|Y_r^n|^2dr + \mathbf{E}\int_t^T e^{\beta r}|Z_r^n|^2dr \le \mathbf{E}[e^{\beta T}|\xi_T|^2] + \frac{\varepsilon}{2}\mathbf{E}\int_t^T e^{\beta r}|Y_r^n|^2dr$$
$$+\left(\frac{2}{\varepsilon}(c+M)+1\right)b\mathbf{E}\int_t^T e^{\beta r}|Y_r^{n-1}|^2dr + \left(\frac{2}{\varepsilon}(c+M)+\alpha_1\right)\mathbf{E}\int_t^T e^{\beta r}|Z_r^n|^2dr$$
$$+\frac{2}{\varepsilon}M\mathbf{E}\int_T^{T+K} e^{\beta r}\left[b|\xi_r|^2 + |\eta_r|^2\right]dr + \frac{2}{\varepsilon}\mathbf{E}\int_t^T e^{\beta r}|f(r,0)|^2dr + \mathbf{E}\int_t^T e^{\beta r}|g(r,0,0)|^2dr + C_\varepsilon$$

where $C_\varepsilon = \frac{a}{\beta}\left[\frac{2}{\varepsilon}(c+2M)+1\right]e^{\beta(T+K)}$. Choose $\varepsilon = \varepsilon_0$ such that $\beta = \dfrac{1}{2}(\varepsilon_0+1)$, $C_0 = C_{\varepsilon_0}$ and $\dfrac{2}{\varepsilon_0}(M+c)+\alpha_1 = 1/2$. Therefore, we obtain

$$\mathbf{E}[e^{\beta t}|Y_t^n|^2] + \frac{1}{2}\mathbf{E}\int_t^T e^{\beta r}|Y_r^n|^2dr + \frac{1}{2}\mathbf{E}\int_t^T e^{\beta r}|Z_r^n|^2dr \le \left(\frac{2}{\varepsilon_0}(c+M)+1\right)b\mathbf{E}\int_t^T e^{\beta r}|Y_r^{n-1}|^2dr$$
$$+\mathbf{E}[e^{\beta T}|\xi_T|^2] + \frac{2}{\varepsilon_0}M\mathbf{E}\int_T^{T+K} e^{\beta r}\left[b|\xi_r|^2 + |\eta_r|^2\right]dr + \frac{2}{\varepsilon_0}\mathbf{E}\int_0^T e^{\beta r}|f(r,0)|^2dr$$
$$+\mathbf{E}\int_0^T e^{\beta r}|g(r,0,0)|^2dr + C_0, \quad 0 \le t \le T.$$

This leads to

$$\mathbf{E}[|Y_t^n|^2]+\frac{1}{2}\mathbf{E}\int_t^T|Y_r^n|^2dr+\frac{1}{2}\mathbf{E}\int_t^T|Z_r^n|^2dr\leq C'\left(\mathbf{E}[X]+1\right)+C'\mathbf{E}\int_t^T|Y_r^{n-1}|^2dr,\quad 0\leq t\leq T$$

where C' is a positive constant(which may vary from line to line).
In particular, putting $q_n(t)=\sup_{n\in\mathbb{N}}\mathbf{E}[|Y_t^n|^2]$, we have

$$q_n(t)\leq C'\left(\mathbf{E}[X]+1\right)+C'\int_t^T q_n(r)dr,\quad 0\leq t\leq T.$$

Gronwall's inequality yields

$$\sup_{n\in\mathbb{N}}\mathbf{E}[|Y_t^n|^2]\leq C'(1+\mathbf{E}[X]),\quad 0\leq t\leq T.$$

This immediately gives (4.2). □

Now we establish the main result of this section.

Theorem 4.2. *Assume that the assumptions* **(A1)**, **(A2)** *and* **(H2)** *are true and let* $\xi_T\in L^2(\mathscr{G}_T,\mathbf{R}^k)$. *Then for any* $(\xi,\eta)\in\mathscr{S}^2_{[T,T+K]}(\mathscr{G},\mathbf{R}^k)\times\mathscr{M}^2_{[T,T+K]}(\mathscr{G},\mathbf{R}^{k\times d})$ *the ABDSDE* (1.1) *has a unique solution* $(Y,Z)\in\mathscr{B}^2_{\mathscr{G}}(0,T+K)$.

Proof. **(i) Existence**. We consider the sequence defined in eq.(4.1). For a process $\rho\in\{Y,Z\}$, and $n\in\mathbf{N},m\in\mathbf{N}$, $\overline{\rho}_t^{n,m}=\rho_t^n-\rho_t^m$, $\Delta g^{(n,m)}(r)=f(r,Y_r^{n-1},Z_r^n)-g(r,Y_r^{m-1},Z_r^m)$ and
$\Delta f^{(n,m)}(r)=f(r,Y_r^{n-1},Z_r^n,Y_{r+\delta(r)}^{n-1},Z_{r+\zeta(r)}^n)-f(r,Y_r^{m-1},Z_r^m,Y_{r+\delta(r)}^{m-1},Z_{r+\zeta(r)}^m)$.

Note that the pair $(\overline{Y}^{n,m},\overline{Z}^{n,m})$ solves the following equation

$$\begin{cases}\overline{Y}_t^{n,m}=\displaystyle\int_t^T\Delta f^{(n,m)}(r)dr+\int_t^T\Delta g^{(n,m)}(r)dB_r-\int_t^T\overline{Z}_r^{n,m}dW_r, & 0\leq t\leq T,\\ \overline{Y}_t^{n,m}=0,\quad \overline{Z}_t^{n,m}=0, \quad T\leq t\leq T+K.\end{cases}$$

By the same computations as in the proof of Lemma 4.1, we have

$$\mathbf{E}|\overline{Y}_t^{n,m}|^2+\frac{1}{2}\mathbf{E}\int_t^T|\overline{Y}_r^{n,m}|^2dr+\frac{1}{2}\mathbf{E}\int_t^T|\overline{Z}_r^{n,m}|^2dr\leq C'\mathbf{E}\int_t^T\rho(|\overline{Y}_r^{n-1,m-1}|^2)dr,\quad 0\leq t\leq T. \tag{4.4}$$

Applying Fatou's lemma and the fact that $\rho\in\mathbf{S}$, we deduce that

$$q(t)\leq C'\int_t^{T+K}\rho(q(r))dr,\quad 0\leq t\leq T+K$$

where $q(t)=\lim_{n,m\to\infty}\sup\mathbf{E}|\overline{Y}_t^{n,m}|^2$, $\quad 0\leq t\leq T+K$. Therefore, we can use Bihari's inequality to get $q(t)=0$, i.e. $\lim_{n,m\to\infty}\sup\mathbf{E}|\overline{Y}_t^{n,m}|^2=0$ for all $0\leq t\leq T+K$.
So, from inequality (4.4), we obtain

$$\lim_{n,m\to\infty}\mathbf{E}\left(|Y_t^n-Y_t^m|^2+\int_t^{T+K}|Z_r^n-Z_r^m|^2dr\right)=0,\quad 0\leq t\leq T+K.$$

Then, there exists $(Y,Z)\in\mathscr{B}^2_{\mathscr{G}}(0,T+K)$ such that

$$\lim_{n\to\infty}\mathbf{E}\left(|Y_t^n-Y_t|^2+\int_t^{T+K}|Z_r^n-Z_r|^2dr\right)=0,\quad 0\leq t\leq T+K.$$

Finally, taking limit in eq.(4.1) as $n \to +\infty$, we conclude that (Y,Z) solves

$$\begin{cases} 0 \le t \le T, \\ Y_t = \xi_T + \int_t^T f(r, Y_r, Z_r, Y_{r+\delta(r)}, Z_{r+\zeta(r)})dr + \int_t^T g(r, Y_r, Z_r)dB_r - \int_t^T Z_r dW_r, \\ Y_t = \xi_t, \quad Z_t = \eta_t, \qquad T \le t \le T+K. \end{cases}$$

This shows that $(Y,Z) \in \mathscr{B}^2_{\mathscr{G}}(0,T+K)$ solves ABDSDE (1.1). The proof of existence is complete.

(ii) Uniqueness. Let $(Y^i, Z^i) \in \mathscr{B}^2_{\mathscr{G}}(0,T+K)$, $i = 1,2$ be two solutions of ABDSDE (1.1). Define $\overline{Y}_t = Y_t^1 - Y_t^2$, $\overline{Z}_t = Z_t^1 - Z_t^2$, $\Delta g(r) = g(r, Y_r^1, Z_r^1) - g(r, Y_r^2, Z_r^2)$ and $\Delta f(r) = f(r, Y_r^1, Z_r^1, Y^1_{r+\delta(r)}, Z^1_{r+\zeta(r)}) - f(r, Y_r^2, Z_r^2, Y^2_{r+\delta(r)}, Z^2_{r+\zeta(r)})$.
We obtain the following equation

$$\begin{cases} \overline{Y}_t = \int_t^T \Delta f(r)dr + \int_t^T \Delta g(r)dB_r - \int_t^T \overline{Z}_r dW_r, \qquad 0 \le t \le T, \\ \overline{Y}_t = 0, \quad \overline{Z}_t = 0, \qquad T \le t \le T+K. \end{cases} \tag{4.5}$$

By the same computations as in Lemma 4.1 , we obtain

$$\mathbf{E}[|\overline{Y}_t|^2] + \frac{1}{2}\mathbf{E}\int_t^T |\overline{Y}_r|^2 dr + \frac{1}{2}\mathbf{E}\int_t^T |\overline{Z}_r|^2 dr \le C'\mathbf{E}\int_t^T \rho(|\overline{Y}_r|^2)dr, \quad 0 \le t \le T.$$

This leads to

$$\mathbf{E}[|\overline{Y}_t|^2] \le C'\mathbf{E}\int_t^{T+K} \rho(|\overline{Y}_r|^2)dr, \quad 0 \le t \le T+K.$$

Using Fubini's theorem and Jensen's inequality, we deduce that

$$\mathbf{E}|\overline{Y}_t|^2 \le C'\int_t^{T+K} \rho(\mathbf{E}|\overline{Y}_r|^2)dr, \quad 0 \le t \le T+K.$$

Then we can use Bihari's inequality to obtain $\mathbf{E}|\overline{Y}_t|^2 = 0$, $0 \le t \le T+K$. This implies $\overline{Z}_t = 0$. □

References

[1] V. Bally, A. Matoussi: *Weak Solutions for SPDEs and Backward doubly stochastic differential equations. Journal of Theoretical Probability*, **14**, 125-164, (2001).
[2] B. Han, Y. Shi, B. Zhu: *Backward doubly stochastic differential equations with non-lipschitz coefficients*, Acta Mathematica Scientia A, **28**, 977-984, (2008).
[3] X. Mao: *Adapted solution of backward stochastic differential equations with non-Lipschitz coefficients*, Stochastic Processes and Their Applications **58**, 281-292, (1995).
[4] E. Pardoux, S. Peng: *Adapted solution of a backward stochastic differential equation*, Systems Control Lett, **114**, 55-61, (1990).
[5] É. Pardoux, S. Peng: *Backward doubly stochastic differential equations and semilinear PDEs*, Probability Theory and Related Fields, **98**, 209-227, (1994).
[6] S. Peng, Z. Yang: *Anticipated backward stochastic differential equations*, Ann. Probab, **37**, 877-902, (2009).
[7] Y. Wang, Z. Huang: *Backward stochastic differential equations with non Lipschitz coefficients equations*, Statistics and Probability Letters, **79**, 1438-1443, (2009).
[8] H. Wu, W. Wang, J. Ren: *Anticipated backward stochastic differential equations with non-Lipschitz coefficients*, Statistics and Probability Letters, **82**, 672-682, (2012).
[9] X. Xu: *Anticipated backward doubly stochastic differential equations*, Applied Mathematics and Computation, **220**, 53-62, (2013).
[10] F. Zhang: *Comparison theorems for anticipated BSDEs with non-lipschitz coefficients*, J. Math. Appl, **416**, 768-782, (2014).