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Proof without words: Periodic continued fractions

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Abstract

In this paper, We give a generalization the resut of Roger B. Nelsen, by giving a closed form expression for $x = [a_0, a_1, \cdots, a_k, \overline{b_1, \cdots, b_m}]$,

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1 Introduction

Let $x := x_0$ be a real number. Set $a_0 = [x]$, the greatest integer in x and $\dfrac{1}{x_0 - a_0}$ its complete quotients. Set $a_i = [x_i]$, the greatest integer in x_i and $x_{i+1} = \dfrac{1}{x_i - a_i}$ for all $i \geq 1$. Then,

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

The algorithm stops after finitely many steps if and only if x is rational. The above expansion is called The *simple continued fraction* of x. It is customarily written $x = [a_0, a_1, \cdots, a_n, \cdots]$.

We call convergents of x the reduced fractions difined by:

$\dfrac{p_0}{q_0} = a_0,$

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$$\frac{p_1}{q_1} = a_0 + \frac{1}{a_1},$$

$\cdots,$

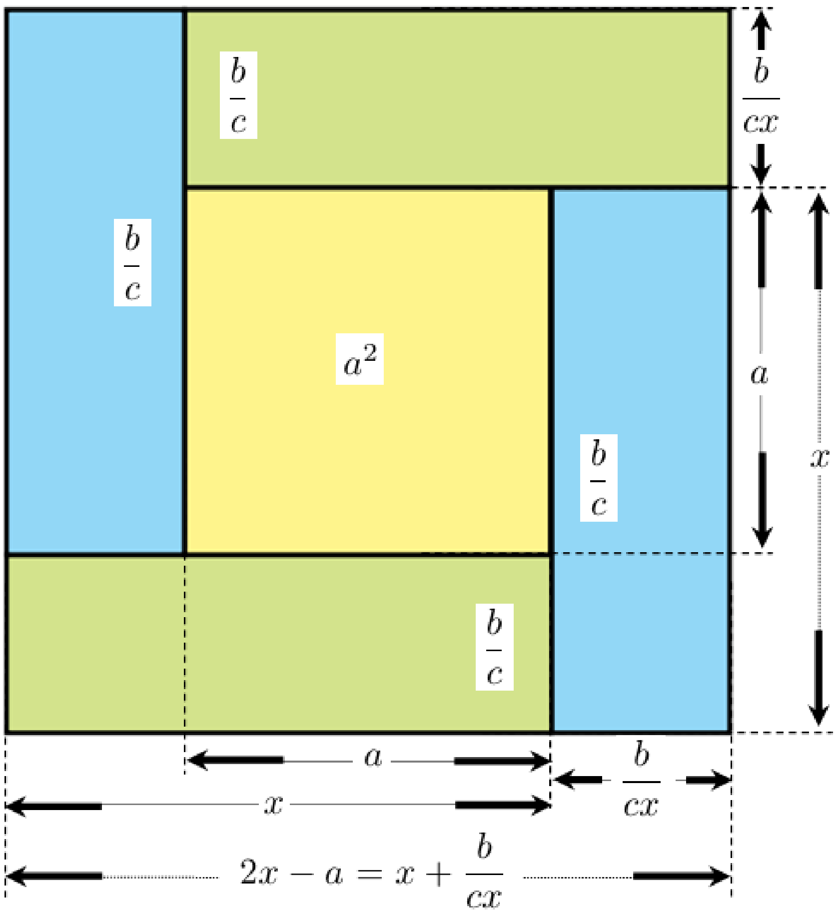
$$\frac{p_n}{q_n} = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_3+}\cdots\frac{1}{a_n}, \cdots.$$

If there exists $k \geq 0$ and $m > 0$ such that whenever $r > k$, we have $a_r = a_{r+m}$, the continued fraction is said periodic, with period $(b_1, \cdots, b_m) = (a_{k+1}, \cdots, a_{k+m})$ and pre-period $(a_0, a_1, \cdots, a_k)$, which can be written for simplicity $x = [a_0, a_1, \cdots, a_k, \overline{b_1, \cdots, b_m}]$. These so-called periodic continued fractions are precisely those that represent quadratic irrationalities.
We find a closed form expression for $x = [a_0, a_1, \cdots, a_k, \overline{b_1, \cdots, b_m}]$, which generalized a previous resut of Roger B. Nelsen.

2 Main result

Lemma 1. *Let $x > 0$ such that $x = a + \frac{b}{cx}$, then $x = \frac{1}{2}\left(a^2 + \sqrt{a^2 + 4\frac{b}{c}}\right)$*

Proof. Consider the Following figure:



We have $(2x-a)^2 = a^2 + 4\frac{b}{c}$, then $x = \frac{1}{2}\left(a^2 + \sqrt{a^2 + 4\frac{b}{c}}\right)$.

Lemma 2. *If $x = [\overline{a_0, a_1, \cdots, a_n}]$, then $x = \frac{p_n - q_{n-1}}{q_n} + \frac{p_{n-1}}{q_n x}$.*

Proof. We have $x = [\overline{a_0, a_1, \cdots, a_n}] = [a_0, a_1, \cdots, a_n, x] = \frac{p_n x - p_{n-1}}{q_n x - q_{n-1}}$. Then, $q_n x^2 = (p_n - q_{n-1}) + p_{n-1}$. which gives $x = \frac{p_n - q_{n-1}}{q_n} + \frac{p_{n-1}}{q_n x}$. Completing the proof.

Theorem 3. *The periodic continued fraction $[\overline{a_0, a_1, \cdots, a_n}]$ equals*

$$\frac{1}{2}\left[\left(\frac{p_n - q_{n-1}}{q_n}\right)^2 + \sqrt{\left(\frac{p_n - q_{n-1}}{q_n}\right)^2 + 4\frac{p_{n-1}}{q_n}}\right].$$

Corollary 4 (Theorem [1])**.** . *The periodic continued fraction $[\overline{a, b}]$ equals*

$$\frac{1}{2}\left(a^2 + \sqrt{a^2 + 4\frac{a}{b}}\right).$$

Corollary 5. *The periodic continued fraction $[\overline{a, b, c}]$ equals*

$$\frac{1}{2}\left[\left(a + \frac{c-b}{bc+1}\right)^2 + \sqrt{\left(a + \frac{c-b}{bc+1}\right)^2 + 4\frac{ab+1}{bc+1}}\right].$$

Example 6. *As examples, notice that* $[\overline{1}] = [\overline{1,1,1}] = \frac{1}{2}\left(1 + \sqrt{5}\right)$, $[\overline{a}] = [\overline{a,a}] = [\overline{a,a,a}] = \frac{1}{2}\left(a^2 + \sqrt{a^2+4}\right)$, $[\overline{3,1,2}] = \frac{1}{2}\left(\frac{100}{9} + \sqrt{\frac{148}{9}}\right)$.

Corollary 7. *Let $x = [a_0, a_1, \cdots, a_k, \overline{b_1, \cdots, b_m}]$, be a periodic continued fraction, with period $(b_1, \cdots, b_m)$ and pre-period $(a_0, a_1, \cdots, a_k)$.*
Note $\frac{p_i}{q_i} = [a_0, a_1, \cdots, a_i]$, for all $0 \le i \le k$ and $\frac{p'_j}{q'_j} = [b_1, \cdots, b_j]$ for all $0 \le j \le m$. Then,

$$x = \frac{p_k\left(\frac{1}{2}\left[\left(\frac{p'_m - q'_{m-1}}{q'_m}\right)^2 + \sqrt{\left(\frac{p'_m - q'_{m-1}}{q_n}\right)^2 + 4\frac{p'_{m-1}}{q'_m}}\right]\right) + p_{k-1}}{q_k\left(\frac{1}{2}\left[\left(\frac{p'_m - q'_{m-1}}{q'_m}\right)^2 + \sqrt{\left(\frac{p'_m - q'_{m-1}}{q_n}\right)^2 + 4\frac{p'_{m-1}}{q'_m}}\right]\right) + q_{k-1}}.$$

Example 8. *As examples, notice that*

$$[1,2,3,4,5,2,\overline{1,1,1,4}] = \frac{225\sqrt{7}+43}{157\sqrt{7}+30},$$
$$[1,2,2,n,\overline{1,2n}] = \frac{7\sqrt{n^2+2n}+3}{5\sqrt{n^2+2n}+2},$$
$$[1,2,2,1,4,n,\overline{n,2n}] = \frac{57\sqrt{n^2+2}+10}{33\sqrt{n^2+2}+7}.$$

3 Conclusions

We find a closed form expression for$x = [a_0, a_1, \cdots, a_k, \overline{b_1, \cdots, b_m}]$, which generalized a previous resut of Roger B. Nelsen.

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References

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