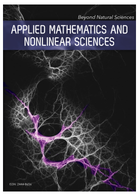
Applied Mathematics and Nonlinear Sciences 4(1) (2019) 35–42



Applied Mathematics and Nonlinear Sciences

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Numerical Solutions with Linearization Techniques of the Fractional Harry Dym Equation

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Submission Info

Communicated by Juan Luis García Guirao
Received 14th January 2019
Accepted 21st February 2019
Available online 27th March 2019

Abstract

In this study, numerical solutions of the fractional Harry Dym equation are investigated. Linearization techniques are utilized for non-linear terms existing in the fractional Harry Dym equation. The error norms L_2 and L_∞ are computed. Stability of the finite difference method is studied with the aid of Von Neumann stabity analysis.

Keywords: Harry Dym equation, finite difference method, von Neumann stability analysis.

1 Introduction

Fractional analysis; is the generalization of the classic analysis of integration and differentiation of process (noninteger) order. This issue is an old issue as much as differential calculus. G. W. Leibniz and Marquis de L′ Hospital correspondence in 1695, is known as the first exit point of fractional calculus. Leibniz expressed fractional order derivatives of noninteger order $f(x) = e^{mx}$, $m \in R$, as follows:

$$\frac{d^n e^{mx}}{dx^n} = m^n e^{mx},$$

here, n is a value, where i is the noninteger.

Later, many scientists, such as Liouville, Riemann, Weyl, Lacroix, Leibniz, Grunward and Letnikov (cf. [4]), expanded the range of this derivative.

Since the beginning of the definition of fractional order derivatives first handled by Leibniz, fractional partial differential equations have attracted the attention of many scientists and have also shown a progressive development.(cf. [1–9, 11, 13–19]).

ISSN 2444-8656 doi:10.2478/AMNS.2019.1.00004

Beside these facts, the third order fractional Harry Dym partial differential equation is studied in mathematics and especially in the theory of solitons. This equation is given as

$$\frac{\partial^\gamma u}{\partial t^\gamma} = u^3 \frac{\partial^3 u}{\partial x^3}. \tag{1.1}$$

Here, $0 < \gamma \leq 1$ is the order of fractional derivative and $u(x,t)$ is a function of x and t.

The Harry Dym equation first appeared in a study by Kruskal [10]. Harry Dym equation represents a system in which dispersion and nonlinearity were coupled. Furthermore, the Harry Dym equation is a completely integrable nonlinear evolution that may be solved by means of the inverse scattering transform. It does not possess the Painlevé property.

This paper is organized as follows:

In the second section, some basic facts dealing with the finite difference method are mentioned and three Linearization techniques are presented. In the third section, stability analysis of the proposed method is investigated and it is shown that the Harry Dym equation is stable under which conditions. Also, numerical examples are given. In the fourth section, conclusions obtained throughout the paper are discussed.

2 Finite Difference Methods

In this section, we first need to define a set of grid points in a domain D to obtain a numerical solution to Eq. (1.1) using finite difference methods as follows:

Let $\Delta x(h) = \frac{b-a}{N}$ (N is an integer) denotes a state step size and Δt denotes a time step size. Draw a set of horizontal and vertical line across D, and get all intersection points (x_j, t_n) or simply (j,n) where $x_j = a + j\Delta x$, $j = 0, 1, 2, \cdots, N$, and $t_n = n\Delta t$, $n = 0, 1, \ldots, M$. If we write $D = [a,b] \times [0,T]$ then we may choose $\Delta t = \frac{T}{M}$ (M is an integer) and $t_n = n\Delta t$, $n = 0, 1, \ldots, M$.

Then, an appropriate finite difference approximation is given in Eq. (1.1) instead of derivatives and its variable. In this case, the solution problem of Eq. (1.1) is reduced to solution problem of algebraic differential systems of linear and nonlinear equations consisting of finite difference equation. But when applied to non-linear problems, it normally leads to nonlinear system of equations and they cannot be solved directly. Therefore, we use three linearization techniques for a nonlinear term as given in Eq. (1.1).

2.1 Linearization 1

First, we use the Caputo fractional derivative approximation for $\frac{\partial^\gamma u}{\partial t^\gamma}$ defined by

$$\frac{\partial^\gamma u}{\partial t^\gamma} \simeq \frac{1}{h^2} \begin{cases} \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \sum\limits_{k=0}^{n} \left[u_m^{n+1-k} - u_m^{n-k}\right] \left[(k+1)^{1-\gamma} - k^{1-\gamma}\right] & n \geq 1 \\ \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \left(u_m^1 - u_m^0\right) & n = 0, \end{cases}$$

and Crank–Nicolson derivative approximation given by

$$u = \frac{u^{n+1} + u^n}{2},$$

in Eq. (1.1) at the nodal point $(m, n+1)$ [12]. Then, if we discretize time derivative of the fractional Harry Dym equation by using Caputo fractional derivative formula between two successive time levels n and $n+1$, Crank Nicolson derivative formula and usual finite difference formula between two successive time levels n and $n+1$, respectively, we obtain

$$\frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n}\left[u_m^{n+1-k}-u_m^{n-k}\right]\left[(k+1)^{1-\gamma}-k^{1-\gamma}\right] \tag{2.1}$$
$$=\left[\frac{\left(u^3\right)_m^{n+1}+\left(u^3\right)_m^{n}}{2}\right]\left[\frac{u_{m+2}^n-2u_{m+1}^n+2u_{m-1}^n-u_{m-2}^n}{2h^3}\right].$$

The nonlinear term in the above mentioned equation is linearized by using the following equation:

$$\left(u^3\right)_m^{n+1}=3\left(u^2\right)^n u^{n+1}-2\left(u^3\right)^n.$$

If the necessary arrangements are made in Eq. (2.1), we have the following equation:

$$\frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n}\left[u_m^{n+1-k}-u_m^{n-k}\right]\left[(k+1)^{1-\gamma}-k^{1-\gamma}\right] \tag{2.2}$$
$$=\frac{1}{4h^3}\left[3\left(u^2\right)_m^n u_m^{n+1}-\left(u^3\right)^n\right]\left[u_{m+2}^n-2u_{m+1}^n+2u_{m-1}^n-u_{m-2}^n\right].$$

2.1.1 Linearization 2

Let us use the Caputo fractional derivative approximation for $\frac{\partial^\gamma u}{\partial t^\gamma}$., Crank–Nicolson derivative approximation and usual finite difference approximation for U_{xxx} given by

$$U_{xxx}\simeq\frac{1}{2h^3}\left(u_{m+2}^n-2u_{m+1}^n+2u_{m-1}^n-u_{m-2}^n\right),$$

in Eq. (1.1) at the nodal point $(m,n+1)$, respectively [12]:

If we use the following linearization technique for the non-linear term U^3U_{xxx}:

$$U^3U_{xxx}\simeq\left(U_m^n\right)^3\frac{1}{2h^3}\left(u_{m+2}^n-2u_{m+1}^n+2u_{m-1}^n-u_{m-2}^n\right),$$

then we have the following system of algebraic equation:

$$\frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n}\left[u_m^{n+1-k}-u_m^{n-k}\right]\left[(k+1)^{1-\gamma}-k^{1-\gamma}\right] \tag{2.3}$$
$$=\left(U_m^n\right)^3\frac{1}{2h^3}\left(u_{m+2}^{n+1}-2u_{m+1}^{n+1}+2u_{m-1}^{n+1}-u_{m-2}^{n+1}\right).$$

Eq. (2.3) can be solved using an approximate algorithm.

2.1.2 Linearization 3

Let us use the Caputo fractional derivative approximation for $\frac{\partial^\gamma u}{\partial t^\gamma}$ and usual finite difference approximation for U_{xxx} in Eq. (1.1) at the nodal point $(m,n+1)$ respectively [12]: If we use the following linearization technique for the nonlinear term U^3U_{xxx}, then we have

$$U^3U_{xxx}\simeq\left(\frac{U_m^n+U_{m+1}^n}{2}\right)^3\frac{1}{2h^3}\left(u_{m+2}^n-2u_{m+1}^n+2u_{m-1}^n-u_{m-2}^n\right).$$

Thus, we get the following system of algebraic equation:.

$$\frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n-1}\left[u_m^{n-k}-u_m^{n-k-1}\right]\left[(k+1)^{1-\gamma}-k^{1-\gamma}\right] \tag{2.4}$$
$$=\left(\frac{U_m^n+U_{m+1}^n}{2}\right)^3\frac{1}{2h^3}\left(u_{m+2}^n-2u_{m+1}^n+2u_{m-1}^n-u_{m-2}^n\right).$$

Eq. (2.4) can be solved using an approximate algorithm.

3 Stability analysis

In this section, we investigate whether this method is stable based on von-Neumann analysis. If the Fourier method analyzes the stability, then the growth factor of a typical Fourier mode is defined as:

$$u_m^n = \xi^n e^{\ell m\varphi}, \quad \ell = \sqrt{-1}, \tag{3.1}$$

where ξ^n is considered as the amplification factor. First, by substituting the Fourier mode (3.1) into the recurrence relationship (2.3), one can obtain

$$\begin{aligned} &\frac{4h^3(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n-1}\left[(k+1)^{1-\gamma}-k^{1-\gamma}\right]\left[\xi^{n-k+1}-\xi^{n-k}\right]e^{\ell m\varphi} \\ &= (\xi^n e^{\ell m\varphi})^3\left(\xi^{n+1}e^{\ell(m+2)\varphi}-2\xi^{n+1}e^{\ell(m+1)\varphi}+2\xi^{n+1}e^{\ell(m-1)\varphi}-\xi^{n+1}e^{\ell(m-2)\varphi}\right). \end{aligned} \tag{3.2}$$

Next, if we assume that $\xi^{n+1} = \zeta\xi^n$ and $\zeta = \zeta(\varphi)$ are independent of time, we can easily obtain the following expression:

$$\begin{aligned} &\frac{4h^3(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n-1}\left[(k+1)^{1-\gamma}-k^{1-\gamma}\right]\left[\zeta^{-k-3n+1}-\xi^{-k-3n}\right] \\ &= \xi e^{3\ell m\varphi}\left(e^{2\ell\varphi}-2e^{\ell\varphi}+2e^{-\ell\varphi}-e^{-2\ell\varphi}\right). \end{aligned} \tag{3.3}$$

Hence, we get

$$\xi = \frac{X_1+iX_2}{Y_1+iY_2}, \tag{3.4}$$

$$\begin{aligned} X_1 &= \frac{2(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n-1}\left[(k+1)^{1-\gamma}-k^{1-\gamma}\right]\left[\zeta^{1-3n-k}-\zeta^{-3n-k}\right], \\ X_2 &= 0, \\ Y_1 &= -24\cos^2[m\gamma]\sin[m\varphi]\sin^2\left[\frac{\varphi}{2}\right]\sin[\varphi]+8\sin^3[m\varphi]\sin^2\left[\frac{\varphi}{2}\right]\sin[\varphi], \\ Y_2 &= -24\cos[m\gamma]\sin^2[m\varphi]\sin^2\left[\frac{\varphi}{2}\right]\sin[\varphi]+8\cos^3[m\varphi]\sin^2\left[\frac{\varphi}{2}\right]\sin[\varphi], \end{aligned} \tag{3.5}$$

which shows that

$$|\zeta| = \left|\frac{X_1+X_2}{Y_1+Y_2}\right|. \tag{3.6}$$

For the Fourier stability definition and for the examined scheme to be stable, the condition $|\xi| \leq 1$ must be gratified. Therefore, if the following inequality is provided, the schema is unconditionally stable.

$$\frac{X_1^2}{Y_1^2+Y_2^2} \leq 1. \tag{3.7}$$

Other schemes can be studied by following a similar way.

3.1 L_2 and L_∞ error norms

The equation of the numerical results was obtained for the test problem used in this study and all computations have been run on using double precision arithmetic. To show how accurate the results, both the error norm L_2

$$L_2 = \left\| U^{exact} - U_N \right\|_2 = \sqrt{h \sum_{J=0}^{N} \left| U_j^{exact} - (U_N)_j \right|^2},$$

and L_∞

$$L_\infty = \left\| U^{exact} - U_N \right\|_\infty = \max_j \left| U_j^{exact} - (U_N)_j \right|,$$

are going to be computed and presented.

3.2 Test Problem

The analytical solution of the fractional Harry Dym equation is given as follows [11].

$$u(x,t) = \left(4 - \frac{3}{2}(x+t)\right)^{\frac{2}{3}}, \qquad t \geq t_0, \qquad 0 \leq x \leq 1. \tag{3.8}$$

In our computations, for the numerical solution of the test problem three different linearization techniques have been applied. The values of the error norms L_2 and L_∞ have been computed at $t = 1$ for different values of Δt, h. The comparison of the error norms L_2 and L_∞ obtained by the linearization techniques is summarized in Table 1. As summarized in Table 1, it is obvious that the obtained results using linearization 1 are better than the obtained results using other linearizations. As shown in Figure 1, one can compare the exact and approximate solution of the collocation method using four radial basis functions for $h = 0.01$. Similarly, as shown in Figures 2 and 3 the exact and approximate solutions can be compared for linearization 2 and linearization 3 successively.

Table 1 Comparison of the error norms L_2 and L_∞ that are obtained using the linearization techniques at $t = 1, \alpha = 0.9$ for different values of Δt, h.

	Lin. I		Lin. II		Lin. III	
	$L_2 \times 10^2$	$L_\infty \times 10^2$	$L_2 \times 10^2$	$L_\infty \times 10^2$	$L_2 \times 10^2$	$L_\infty \times 10^2$
$\triangle t = h = 0.001$	0.00002	0.00002	0.00566	0.05664	0.00565	0.05659
$\triangle t = h = 0.05$	0.05254	0.08344	0.64556	0.94172	0.58516	0.84499
$\triangle t = h = 0.04$	0.02957	0.05128	0.50690	0.82106	0.47224	0.76100
$\triangle t = h = 0.03$	0.01418	0.02773	0.36800	0.68391	0.35113	0.65086
$\triangle t = h = 0.02$	0.00506	0.01186	0.23124	0.52313	0.22502	0.50855
$\triangle t = h = 0.01$	0.00088	0.00285	0.10161	0.32318	0.10046	0.31945
$\triangle t = h = 0.2$	2.44568	0.00285	3.12756	3.78842	0.80756	1.29370
$\triangle t = h = 0.1$	3.27620	0.41645	1.31211	1.41287	0.94027	0.94563

4 Conclusions

Finite difference methods based on using three different linearization techniques have been proposed for the numerical solutions of the fractional Harry Dym equation. In addition, numerical results were obtained by using three different linearization techniques. The proposed method has been tested on a problem and demonstrated how effective it is. The error norms L_2 and L_∞ have been calculated and given. The third linearization technique, as shown in Figure 3, yielded better results. Because the third linearization technique gives better results, this technique can be suggested in the next problems and the situation of the problem should be considered. The obtained results show that the error norms are sufficiently small during all computer runs. It has been observed that the considered method is a power numerical scheme to solve the fractional Harry Dym equation.

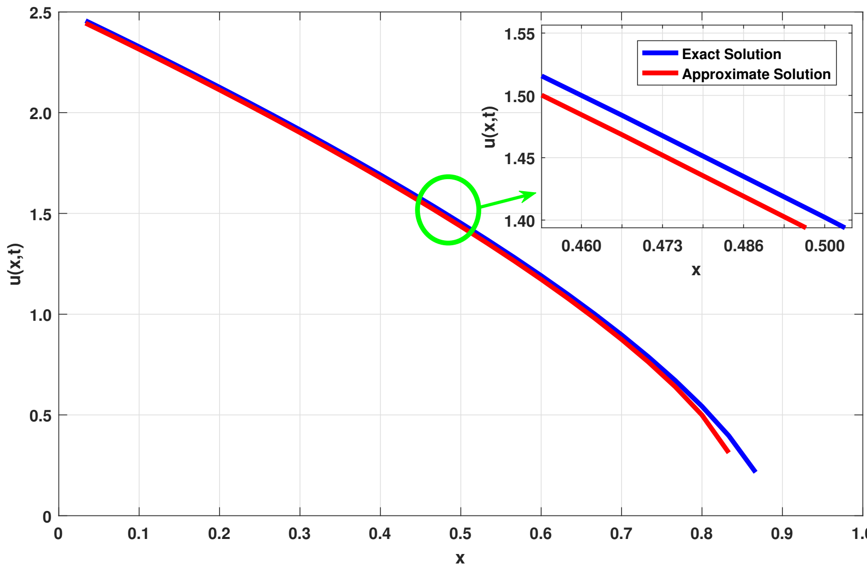

Fig. 1 Comparison of the exact and approximate solution for linearization 2 at $h = 0.01$

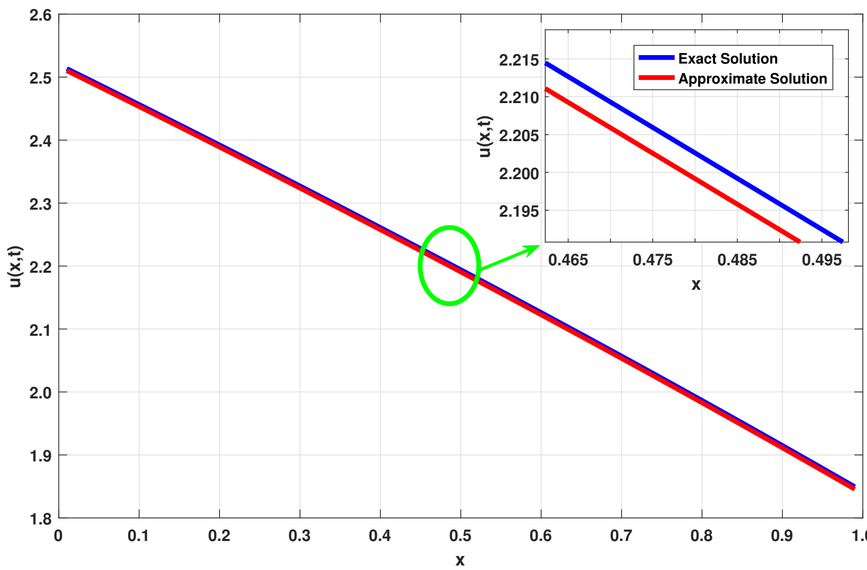

Fig. 2 Comparison of the exact and approximate solution for linearization 3 at h=0.01

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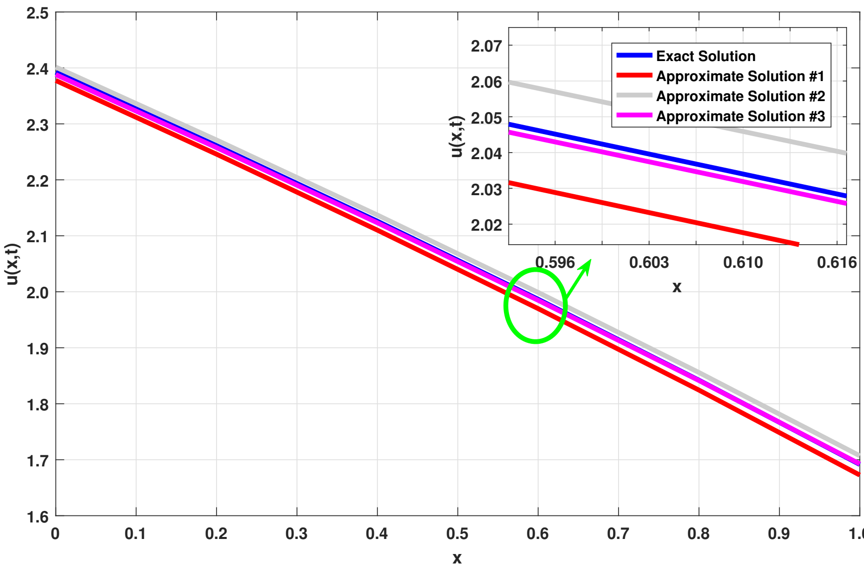

Fig. 3 Comparison of the exact and approximate solution for linearization 1, linearization 2, and linearization 3 at $h = 0.01$

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