CrossMark

sciendo Applied Mathematics and Nonlinear Sciences 4(1) (2019) 113–128

Applied Mathematics and Nonlinear Sciences

https://www.sciendo.com

Some results on D-homothetic deformation of $(LCS)_{2n+1}$-manifolds

Sunil Kumar Yadav†

Department of Mathematics, Poornima college of Engineering, ISI-6, RIICO, Institutional Area, Sitapura, Jaipur-302022, Rajasthan, India

Submission Info

Communicated by Juan Luis García Guirao
Received February 13th 2019
Accepted April 26th 2019
Available online June 24th 2019

Abstract

The present paper deals with the study of a D-homothetic deformation of an extended generalized ϕ-recurrent $(LCS)_{2n+1}$-manifolds their geometrical properties are discussed. Finally, we construct an example of an extended generalized ϕ-recurrent $(LCS)_3$-manifolds that are neither ϕ-recurrent nor generalized ϕ-recurrent under such deformation is constructed.

Keywords: Generalized recurrent $(LCS)_{2n+1}$-manifolds, extended generalized ϕ-recurrent $(LCS)_{2n+1}$-manifolds, concircular curvature tensor, ϕ-sectional curvature and D-homothetic deformation.
AMS 2010 codes: 53C10, 53C25, 53C40.

1 Introduction

Matsumoto et al.([1], [2]) introduced the idea of a Lorentzian para Sasakian manifold (briefly *LP*-Sasakian manifold) in 1988. Shaikh in 2003, gave the notion of a Lorentzian concircular structure manifolds (briefly *LCS*-manifold) [3], which is the generalization of an *LP*-Sasakian manifold. Since then, many geometers studied the properties of this manifold, for instance ([4], [5], [6], [7], [8]). The notion of local symmetry of a Riemannian manifold has been studied by many author in several ways to a different structures. As a weaker version of local symmetry, Takahashi [22] introduced the notion of a local ϕ-symmetry on a Sasakian manifold. Generalizing the notion of a local ϕ-symmetry of Takahashi [22], De et al. [10] introduced the idea of ϕ-recurrent for the Sasakian manifolds. Locally symmetric and ϕ-symmetric LP-Sasakian manifolds were studied by Shaikh and Baishya [21]. The properties of the locally ϕ-symmetric and the locally ϕ-recurrent $(LCS)_n$-manifolds were, respectively, studied in [4] and [5]. The notion of a generalized recurrent manifold has been introduced by Dubey et al. [12] and then studied by others. Again, the notion of a generalized Ricci-recurrent manifold has been introduced and studied by De et al. [11].

A Riemannian manifold (M^n, g), $(n > 2)$, is called a generalized recurrent manifold [12], if its non-vanishing

†Email address: prof_-sky16@yahoo.com

sciendo ISSN 2444-8656 doi:10.2478/AMNS.2019.1.00012 sciendo

curvature tensor R satisfies

$$\nabla R = A \otimes R + B \otimes G, \tag{1.1}$$

where A and B are non-vanishing 1-forms such that $A(.) = g(., \rho_1)$, $B(.) = g(., \rho_2)$ and the tensor G is defined by

$$G(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{1.2}$$

for $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the collection of all smooth vector fields of M and ∇ denotes the operator of covariant differentiation with respect to the metric g. The 1-forms A and B are called the associated 1-forms of M.

A Riemannian manifold (M^n, g), $(n > 2)$, is said to be a generalized Ricci-recurrent manifold [11], if its Ricci tensor S of type $(0,2)$ satisfies

$$\nabla S = A \otimes S + B \otimes g, \tag{1.3}$$

where A and B are non vanishing 1-forms defined as (1.1).

In 2007, *Ö*zg*ü*r [15] studied generalized recurrent Kenmotsu manifolds. Generalizing the notion of *Ö*zg*ü*r [15], Basari and Murathan [9] introduced the notion of the generalized ϕ-recurrent Kenmotsu manifolds. In addition, the properties of the generalized ϕ-recurrent Sasakian, *LP*-Sasakian, Lorentzian α-Sasakian, Kenmotsu manifolds, generalized Sasakian space-forms and $(LCS)_{2n+1}$-manifolds are, respectively, studied in [7], [16], [17], [19]. The properties of the extended generalized ϕ-recurrent β-Kenmotsu, Sasakian and $(LCS)_{2n+1}$-manifolds have been studied in [20], [18] and [7], respectively. As a continuation of above studies, we characterize the $(LCS)_{2n+1}$-manifolds under D-homothetic deformation. The outline of this paper is as follows:

After introduction in Section 1, we brief the known results of the $(LCS)_{2n+1}$-manifolds in Section 2. In Section 3, we prove our main results in the form of theorems and corollaries. It is proved that the structure tensor of the manifold commutes with the Ricci tensor under the D-homothetic deformation. This section also covers the properties of extended generalized ϕ-recurrent, ϕ-sectional curvature tensor, locally ϕ-Ricci symmetric, η-parallel Ricci tensor and extended generalized concircularly ϕ-recurrent $(LCS)_{2n+1}$-manifolds. In the last section, we give a non-trivial example of an extended generalized ϕ-recurrent $(LCS)_{2n+1}$-manifold under D-homothetic deformation and validate our results.

2 Preliminaries

A Lorentzian manifold M of dimension $(2n+1)$ is a smooth connected para-contact Hausdorff manifold with the Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \Re$ is a non degenerate inner product of signature $(-,+,...,+)$, where T_pM denotes the tangent space of M at p and $\Re$ are the real number space. A non-zero vector field $V \in T_pM$ is said to be time like (respectively, non-space like, null, and space like) if it satisfies $g_p(V,V) < 0$ (respectively, $\leq 0, = 0, > 0$) ([1], [2]).

Definition 2.1. *A vector field ρ on (M,g) defined by $g(X,\rho) = A(X)$, $\forall X \in \chi(M)$ is said to be a concircular vector field if*

$$(\nabla_X A)(Y) = \alpha\{g(X,Y) + \omega(X)\omega(Y)\},$$

where α is the non-zero scalar and ω is the closed 1-form [14].

Let M is a Lorentzian manifold admitting a unit time like concircular vector field ξ, which is called the generator of the manifold. Then we have

$$g(\xi,\xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field on M and therefore there exists a non-zero 1-form η such that

$$g(X,\xi) = \eta(X), \tag{2.2}$$

which satisfies

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\} \ (\alpha \neq 0), \tag{2.3}$$

for all the vector fields, X and Y, where α is the non-zero scalar function that satisfies

$$(\nabla_X \alpha) = X\alpha = d\alpha(X) = \rho\eta(X). \tag{2.4}$$

Here, ρ is the certain scalar function such that $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{2.5}$$

then from (2.3) and (2.5), we have

$$\phi X = X + \eta(X)\xi, \tag{2.6}$$

from which it follows that ϕ is a tensor field of type $(1,1)$, which is called the structure tensor of M. Thus M together with the unit timelike concircular vector field ξ, its associated 1-form η and $(1,1)$-tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly (LCS)-manifold) [3]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [2]. Thus, we can say that the (LCS)-manifold is the generalization of the LP-Sasakian manifold. In the present paper, we consider the LCS-manifold of dimension $(2n+1)$. We have the following basic results of $(LCS)_{2n+1}$-manifold as:

$$\begin{aligned} &\eta(\xi) = -1, \quad \phi\xi = 0, \quad \phi^2 X = X + \eta(X)\xi, \quad \eta(\phi X) = 0 \\ &\textit{and} \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \end{aligned} \tag{2.7}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}, \tag{2.8}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)\{\eta(Y)X - \eta(X)Y\}, \tag{2.9}$$

$$R(\xi,X)Y = (\alpha^2 - \rho)\{g(X,Y)\xi - \eta(Y)X\}, \tag{2.10}$$

$$(\nabla_X \phi)(Y) = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.11}$$

$$S(X,\xi) = 2n(\alpha^2 - \rho)\eta(X), \tag{2.12}$$

$$S(\phi X, \phi Y) = S(X,Y) + 2n(\alpha^2 - \rho)\eta(X)\eta(Y), \tag{2.13}$$

$$X\rho = d\rho(X) = \beta\eta(X) \tag{2.14}$$

for all the vector fields X, Y, Z on M [3].

Definition 2.2. *A Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be a locally ϕ-Ricci manifold symmetric if*

$$\phi^2((\nabla_X Q)(Y)) = 0, \tag{2.15}$$

where Q denotes the Ricci operator defined by $S(X,Y) = g(QX,Y)$ and X, Y are the vector fields orthogonal to ξ.

The notion of η-parallelism on a Sasakian manifold was introduced by Kon [13]. An $(LCS)_{2n+1}$-manifold is said to be η-parallel if its Ricci tensor S satisfies

$$(\nabla_X S)(\phi Y, \phi Z) = 0, \tag{2.16}$$

for $X, Y, Z \in \chi(M^{2n+1})$.

If $M(\phi, \xi, \eta, g)$ is an almost contact metric manifold of dimension $(2n+1)$ $(i.e., dim M = m = 2n+1)$, then the equation $\eta = 0$ defines an $(m-1)$-dimensional distribution D on M [24], and if we change the structure tensors of an almost contact metric manifold by

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\phi} = \phi, \quad \bar{g} = ag + a(a-1)\eta \otimes \eta,$$

where a is the non-zero positive constant. Then such transformation is known as the $(m-1)$-homothetic deformation or D-homothetic deformation [23]. The study of D-homothetic deformation has been noticed in ([26], [27]). If $M(\phi, \xi, \eta, g)$ is an almost contact metric structure with contact form η, then $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is also an almost contact metric structure [23]. If we denote the difference $\bar{\Gamma}^i_{jk} - \Gamma^i_{jk}$ of Christoffel symbols by V^i_{jk}, then we have

$$V(X,Y) = (1-a)\{\eta(Y)\phi X + \eta(X)\phi Y\} + \frac{1}{2}(1-\frac{1}{a})\{(\nabla_X \eta)(Y) + (\nabla_Y \eta)(X)\}\xi \tag{2.17}$$

for $X, Y \in \chi(M)$ [23]. If R and $\bar{R}$ denote, respectively, the curvature tensors of the manifolds $M(\phi, \xi, \eta, g)$ and $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$, then it is related to the expression

$$\bar{R}(X,Y)Z = R(X,Y)Z + (\nabla_X V)(Z,Y) - (\nabla_Y V)(Z,X) + V(V(Z,Y),X) - V(V(Z,X),Y), \tag{2.18}$$

for arbitrary vector fields X, Y, Z [23].

A plane section in the tangent space $T_p(M)$ is called a ϕ-section if there exists a unit vector X in $T_p(M)$ orthogonal to ξ such that $\{X, \phi X\}$ is an orthonormal basis of the plane section. A sectional curvature of the form

$$K(X, \phi X) = g(R(X, \phi X)X, \phi X)$$

is known as a ϕ-sectional curvature in $T_p(M)$. A para contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be of constant ϕ-sectional curvature if at each point of the manifold, the sectional curvature $K(X, \phi X)$ is independent of the choice of non-zero vector $X \in D_p$, where D denotes the contact distribution of the para contact metric manifold defined by the equation $\eta = 0$.

3 Main Results

In this section, we study the extended generalized ϕ-recurrent, ϕ-sectional curvature, locally ϕ-Ricci symmetric, η-parallel Ricci tensor and extended generalized concircularly ϕ-recurrent $(LCS)_{2n+1}$-manifolds under D-homothetic deformation. In consequence of (2.3) and (2.17), we get

$$V(X,Y) = (1-a)\{\eta(Y)\phi X + \eta(X)\phi Y\} + \alpha(1-\frac{1}{a})\{g(X,Y) + \eta(X)\eta(Y)\}\xi, \tag{3.1}$$

In view of (2.3), (2.4), (2.7) and (2.11), (3.1) it yields

$$\begin{aligned}(\nabla_Z V)(X,Y) = {} & \alpha(1-a)\{g(Y,Z)\phi X + g(Z,X)\phi Y + \eta(Y)\eta(Z)\phi X \\ & +\eta(X)\eta(Z)\phi Y + g(X,Z)\eta(Y)\xi + g(Z,Y)\eta(X)\xi \\ & +2\eta(X)\eta(Y)Z + 4\eta(X)\eta(Y)\eta(Z)\xi\} \\ & +\alpha^2(1-\frac{1}{a})[\{g(Z,X)\eta(Y) + \eta(Z)g(X,Y) \\ & +g(Z,Y)\eta(X) + 3\eta(X)\eta(Y)\eta(Z)\}\xi + g(\phi X, \phi Y)Z] \\ & -(\frac{a-1}{a})(\xi\alpha)\{g(X,Y) + \eta(X)\eta(Y)\}\eta(Z)\xi.\end{aligned} \tag{3.2}$$

Using (3.1) and (3.2) in (2.18) and then by virtue of (2.3) and (2.9), we obtain

$$\begin{aligned}\bar{R}(X,Y)Z = {} & R(X,Y)Z - \alpha(1-a)\{g(Y,Z)\phi X + \eta(Y)\eta(Z)\phi X - g(X,Z)\phi Y \\ & -\eta(X)\eta(Z)\phi Y + g(Y,Z)\eta(X)\xi + 2\eta(Z)\eta(X)Y - \eta(Y)g(Z,X)\xi \\ & -2\eta(Z)\eta(Y)X\} + (\frac{a-1}{a})(\xi\alpha)\{\eta(Y)g(Z,X) - \eta(X)g(Z,Y)\}\xi \\ & +(1-a)^2\{\eta(X)\eta(Z)\phi^2 Y - \eta(Y)\eta(Z)\phi^2 X\} \\ & -\alpha^2(\frac{a-1}{a})\{g(Z,X)Y + \eta(Z)\eta(X)Y - g(Z,Y)X - \eta(Z)\eta(Y)X\} \\ & -\alpha\frac{(1-a)^2}{a}\{g(Z,Y)\phi X + \eta(Z)\eta(Y)\phi X - g(Z,X)\phi Y \\ & -\eta(Z)\eta(X)\phi Y + [\eta(X)g(\phi Z,Y) - g(\phi Z,X)\eta(Y)]\xi\}. \end{aligned} \tag{3.3}$$

Taking $Y = Z = \xi$ in (3.3) and then use of (2.7) it gives

$$\bar{R}(X,\xi)\xi = R(X,\xi)\xi + (1-a)(2\alpha + a - 1)\phi^2 X. \tag{3.4}$$

Let $\{e_i, \phi e_i, \xi\}$, $i = 1,2,...,n$, be an orthonormal frame at any point of the tangent space $T(M)$ of the manifold M. Then replacing $Y = Z = e_i$ in (3.3), taking summation over i, $1 \leq i \leq n$, and using $\eta(e_i) = 0$, we obtain

$$\begin{aligned}\sum_{i=1}^{n}\varepsilon_i\overline{R}(X,e_i)e_i = {} & \sum_{i=1}^{n}\varepsilon_i R(X,e_i)e_i - n(\frac{a-1}{a})(\xi\alpha)\eta(X)\xi \\ & +\frac{\alpha(a-1)}{a}[(a+1)n-1]\eta(X)\xi + \frac{\alpha(a-1)(\alpha+1)}{a}(n-1)X, \end{aligned} \tag{3.5}$$

where $\varepsilon_i = g(e_i,e_i)$. Again, taking $Y = Z = \phi e_i$ in (3.3) and taking summation over i, $1 \leq i \leq n$, and using the fact that $\eta \circ \phi = 0$, we get

$$\begin{aligned}\sum_{i=1}^{n}\varepsilon_i\overline{R}(X,\phi e_i)\phi e_i = {} & \sum_{i=1}^{n}\varepsilon_i R(X,\phi e_i)\phi e_i - n(\frac{a-1}{a})(\xi\alpha)\eta(X)\xi \\ & +\frac{\alpha(a-1)}{a}[(a+1)n-1]\eta(X)\xi + \frac{\alpha(a-1)(\alpha+1)}{a}(n-1)X. \end{aligned} \tag{3.6}$$

In view of (3.5) and (3.6), we have

$$\begin{aligned}\overline{Q}X - \overline{R}(X,\xi)\xi = {} & QX - R(X,\xi)\xi - 2n(\frac{a-1}{a})(\xi\alpha)\eta(X)\xi \\ & +\frac{2\alpha(a-1)}{a}[(a+1)n-1]\eta(X)\xi + \frac{2\alpha(a-1)(\alpha+1)}{a}(n-1)X. \end{aligned} \tag{3.7}$$

In consequence of (2.7) and (3.4), (3.7) it yields

$$\begin{aligned}\overline{S}(X,Y) = {} & S(X,Y) + (\frac{a-1}{a})\{2\alpha(\alpha+1)(n-1) - a(2\alpha + a - 1)\}g(X,Y) \\ & +(\frac{a-1}{a})\{2\alpha(a+1)(n-1) - a(a-1) - 2n(\xi\alpha)\}\eta(X)\eta(Y), \end{aligned} \tag{3.8}$$

which implies that

$$\begin{aligned}\overline{Q}X = {} & QX + (\frac{a-1}{a})\{2\alpha(\alpha+1)(n-1) - a(2\alpha + a - 1)\}X \\ & +(\frac{a-1}{a})\{2\alpha(a+1)(n-1) - a(a-1) - 2n(\xi\alpha)\}\eta(X)\xi. \end{aligned} \tag{3.9}$$

Applying $\overline{\phi} = \phi$ on both sides of (3.9) and using (2.7), we have

$$\overline{\phi}\,\overline{Q}X = \phi QX + (\frac{a-1}{a})\{2\alpha(\alpha+1)(n-1) - a(2\alpha+a-1)\}\phi X. \tag{3.10}$$

Again applying $\overline{\phi}X = \phi X$ in (3.9) and using (2.7) give

$$\overline{Q}\,\overline{\phi}X = Q\phi X + (\frac{a-1}{a})\{2\alpha(\alpha+1)(n-1) - a(2\alpha+a-1)\}\phi X. \tag{3.11}$$

By virtue of (3.10) and (3.11), we obtain

$$\{\overline{\phi}\,\overline{Q} - \overline{Q}\,\overline{\phi}\}X = \{\phi Q - Q\phi\}X. \tag{3.12}$$

The Ricci operator Q commutes with the structure tensor ϕ in an $(LCS)_{2n+1}$-manifold [6]. Thus we can state the following theorem as:

Theorem 3.1. *In a $(2n+1)$-dimensional Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g)$, the Ricci operator $\overline{Q}$ and structure vector field $\overline{\phi}$ are commuted with respect to the D-homothetic deformation.*

3.1 Extended generalized Φ-recurrent $(LCS)_{2n+1}$-manifolds

In this subsection, we study the properties of the extended generalized ϕ-recurrent $(LCS)_{2n+1}$-manifolds under D-homothetic deformation.

Definition 3.2. *A Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, is said to be an extended generalized ϕ-recurrent $(LCS)_{2n+1}$-manifold under D-homothetic deformation if its curvature tensor $\overline{R}$ satisfies*

$$\phi^2((\nabla_W\overline{R})(X,Y)Z) = A(W)\phi^2(\overline{R}(X,Y)Z) + B(W)\phi^2(G(X,Y)Z), \tag{3.13}$$

for X, Y, Z, $W \in \chi(M^{2n+1})$, where A and B are non-vanishing 1-forms such that $A(X) = g(X,\rho_1)$, $B(X) = g(X,\rho_2)$ and G is a tensor field of type $(1,3)$ defined as (1.2). The 1-forms A and B are called the associated 1-forms of the manifold.

Let us suppose that the Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, is an extended generalized ϕ-recurrent under D-homothetic deformation. Then from (2.7) and (3.13), we have

$$\begin{aligned}(\nabla_W\overline{R})(X,Y)Z + \eta((\nabla_W\overline{R})(X,Y)Z)\xi &= A(W)\{\overline{R}(X,Y)Z\\ &+\eta(\overline{R}(X,Y)Z)\xi\} + B(W)\{G(X,Y)Z + \eta(G(X,Y)Z)\xi\},\end{aligned} \tag{3.14}$$

from which it follows that

$$\begin{aligned}g((\nabla_W\overline{R})(X,Y)Z,U) + \eta((\nabla_W\overline{R})(X,Y)Z)\eta(U) &= A(W)\{g(\overline{R}(X,Y)Z,U)\\ &+\eta(\overline{R}(X,Y)Z)\eta(U)\} + B(W)\{g(G(X,Y)Z,U) + \eta(G(X,Y)Z)\eta(U)\}.\end{aligned} \tag{3.15}$$

Let $\{e_i; i=1,2,...,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Replacing $X=U=e_i$ in (3.15) and taking summation over i, $1 \le i \le 2n+1$, and then using (2.9), we have

$$\begin{aligned}(\nabla_W\overline{S})(Y,Z) + g((\nabla_W\overline{R})(\xi,Y)Z,\xi) &= A(W)\{\overline{S}(Y,Z)\\ &+\eta(\overline{R}(\xi,Y)Z) + B(W)\{(2n-1)g(Y,Z) - \eta(Y)\eta(Z)\}.\end{aligned} \tag{3.16}$$

In consequence of (2.1), (2.2), (2.4), (2.6), (2.7), (2.8) and (3.3), we can find that

$$\eta(\overline{R}(X,Y)Z) = \frac{\alpha(2a-1)(\alpha+1-a)+\rho}{a}\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\},$$

which becomes

$$\eta(\overline{R}(\xi,Y)Z) = \frac{\alpha(2a-1)(a-\alpha-1)+\rho}{a}\{g(Y,Z)+\eta(Y)\eta(Z)\}.$$

The covariant derivative of the above equation along the vector field W, after long calculations, gives

$$g((\nabla_W\overline{R})(\xi,Y)Z,\xi) = \left(\frac{(2a-1)(a-2\alpha+1)d\alpha(W)+d\rho(W)}{a}\right)\{g(Y,Z)+\eta(Y)\eta(Z)\}.$$

In view of the above relations, it follows from (3.16) that

$$\begin{aligned}(\nabla_W\overline{S})(Y,Z) = {} & A(W)\overline{S}(Y,Z)+B(W)g(Y,Z)\\ & +\left(\frac{\alpha(2a-1)(a-\alpha-1)+\rho}{a}\right)A(W)\{g(Y,Z)+\eta(Y)\eta(Z)\}\\ & -\left(\frac{(2a-1)(a-2\alpha+1)d\alpha(W)+d\rho(W)}{a}\right)\{g(Y,Z)+\eta(Y)\eta(Z)\}\\ & +B(W)\{2(n-1)g(Y,Z)-\eta(Y)\eta(Z)\}.\end{aligned} \tag{3.17}$$

Analogous to the definition of (1.3), we can define:

Definition 3.3. *A Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, is said to be a generalized Ricci-recurrent manifold under D-homothetic deformation if its non-vanishing Ricci tensor $\tilde{S}$ satisfies the relation*

$$(\nabla_W\tilde{S})(Y,Z) = A(W)\tilde{S}(Y,Z)+B(W)g(Y,Z),$$

for all vector fields W, X, $Y \in \chi(M^{2n+1})$, where the 1-forms A and B are defined in (1.1).

From equation (3.17) and the above definition, it follows that an extended generalized ϕ-recurrent $(LCS)_{2n+1}$-manifold under D-homothetic deformation is a generalized Ricci-recurrent manifold if and only if

$$\begin{aligned}&\left(\frac{\alpha(2a-1)(a-\alpha-1)+\rho}{a}\right)A(W)\{g(Y,Z)+\eta(Y)\eta(Z)\}\\ &-\left(\frac{(2a-1)(a-2\alpha+1)d\alpha(W)+d\rho(W)}{a}\right)\{g(Y,Z)+\eta(Y)\eta(Z)\}\\ &+B(W)\{2(n-1)g(Y,Z)-\eta(Y)\eta(Z)\}=0.\end{aligned} \tag{3.18}$$

This leads to the following:

Theorem 3.4. *An extended generalized ϕ-recurrent Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, under D-homothetic deformation is generalized Ricci-recurrent manifold if and only if the relation (3.18) holds.*

Let $\{e_i; i=1,2,\ldots,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Setting $Y=Z=e_i$ in (3.18) and taking summation over i, $1\le i\le 2n+1$, we have

$$\left(\frac{2n(2a-1)(a-\alpha-1)}{a}\right)\{\alpha A(W)-d\alpha(W)\}+\frac{2n}{a}\{\rho A(W)-d\rho(W)\}+(4n^2-2n-1)B(W)=0.$$

In particular, if we suppose that α is constant, then last the expression becomes

$$a_1A(W)+b_1B(W)=0,$$

where $a_1=2n\alpha(2a-1)(a-\alpha-1)\neq 0$ and $b_1=a(4n^2-2n-1)\neq 0$. This shows that the 1-forms are in opposite directions.

Corollary 3.5. *If an extended generalized ϕ-recurrent Lorentzian concircular structure manifold M^{2n+1}, $n>1$, under D-homothetic deformation is a generalized Ricci-recurrent manifold, then the 1-forms A and B tend to be in opposite directions, provided α is constant.*

3.2 ϕ-sectional curvature of $(LCS)_{2n+1}$-manifolds

In this section we consider the ϕ-sectional curvature of a $(2n+1)$-dimensional *LCS*-manifold under *D*-homothetic deformation. In view of (2.7) and (3.3), we have

$$
\begin{aligned}
&{}'\bar{R}(X,Y,Z,W) = {}'R(X,Y,Z,W) - \alpha(1-a)\{g(Y,Z)g(\phi X,W)\\
&+\eta(Y)\eta(Z)g(\phi X,W) - g(X,Z)g(\phi Y,W) - \eta(X)\eta(Z)g(\phi Y,W)\\
&+g(Y,Z)\eta(X)\eta(W) + 2\eta(Z)\eta(X)g(Y,W) - \eta(Y)g(Z,X)\eta(W) - 2\eta(Z)\eta(Y)g(X,W)\}\\
&-(\frac{a-1}{a})(\xi\alpha)\{\eta(Y)g(Z,X) - \eta(X)g(Z,Y)\}\eta(W) + (1-a)^2\{\eta(X)\eta(Z)g(\phi^2 Y,W)\\
&-\eta(Y)\eta(Z)g(\phi^2 X,W)\} - \alpha^2(\frac{a-1}{a})\{g(Z,X)g(Y,W) + \eta(Z)\eta(X)g(Y,W)\\
&-g(Z,Y)g(X,W) - \eta(Z)\eta(Y)g(X,W)\} + \alpha\frac{(1-a)^2}{a}\{g(Z,Y)g(\phi X,W)\\
&+\eta(Z)\eta(Y)g(\phi X,W) - g(Z,X)g(\phi Y,W) - \eta(Z)\eta(X)g(\phi Y,W) + [\eta(X)g(\phi Z,Y)\\
&+g(\phi X,Y)\eta(Z) - g(\phi Z,X)\eta(Y) - g(\phi Y,X)\eta(Z)]\eta(W)\}.
\end{aligned}
$$

Replacing Y by ϕX, Z by X and W by ϕX in the above equation, we find that

$$
\overline{K}(X,\phi X) - K(X,\phi X) = \frac{\alpha(\alpha+1)(1-a)}{a}. \tag{3.19}
$$

This leads to the following:

Theorem 3.6. *The ϕ-sectional curvature of an $(LCS)_{2n+1}$-manifold is not an invariant property under D-homothetic deformation.*

If a Lorentzian concircular structure manifold $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ of dimension $(2n+1)$ satisfies $R(X,Y)\xi = 0$ for arbitrary vector fields X and Y, then the ϕ-sectional curvature of the manifold $M(\phi,\xi,\eta,g)$ vanishes *i.e.*, $K(X,\phi X) = 0$. This shows that the ϕ-sectional curvature $\overline{K}(X,\phi X)$ is not vanishing and therefore we can state the following:

Corollary 3.7. *There exists a $(2n+1)$-dimensional LCS-manifold $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ with non-zero non constant ϕ-sectional curvature.*

3.3 Locally ϕ-Ricci symmetric $(LCS)_{2n+1}$-manifolds

This subsection deals with the study of locally ϕ-Ricci symmetric $(LCS)_{2n+1}$-manifolds under *D*-homothetic deformation. Differentiating (3.9) covariantly with respect to W, and using (2.3), (2.5) and (2.11), we get

$$
\begin{aligned}
(\nabla_W\overline{Q})(X) = (\nabla_W Q)(X) &+ \frac{a-1}{a}\{a(2\alpha+a-1) + 2\alpha[a(3n-2)-(n-1)]\\
&+2n(\xi\alpha)\}\{(\nabla_W\eta)(X)\xi + \eta(X)W + \eta(X)\eta(W)\xi\}\\
&+\frac{2(a-1)}{a}[\{a(3n-1)-(n-1)\}d\alpha(W) - nd\rho(W)]\,\eta(X)\xi\\
&+\frac{2(a-1)}{a}\{(2a+2\alpha-1)(n-1)+a\}d\alpha(W)X.
\end{aligned} \tag{3.20}
$$

Operating ϕ^2 on both side of (3.20) and suppose that X is an orthogonal vector to ξ, we find that

$$
\overline{\phi}^2(\nabla_W\overline{Q})(X) = \phi^2(\nabla_W Q)(X) + \frac{2(a-1)}{a}\{(2a+2\alpha-1)(n-1)+a\}d\alpha(W)X. \tag{3.21}
$$

Since $\alpha \neq constant$, in general, therefore we lead to the following:

Theorem 3.8. *The property of locally ϕ-Riccisymmetry on an $(LCS)_{2n+1}$-manifold is not invariant under the D-homothetic deformation.*

In particular, if we suppose that α is constant, then from equation (3.21) and Theorem 3.8, we can state the following:

Corollary 3.9. *The property of locally ϕ-Riccisymmetry on an $(LCS)_{2n+1}$-manifold is an invariant under the D-homothetic deformation if and only if α is constant.*

3.4 η-parallel Ricci tensor of $(LCS)_{2n+1}$-manifolds

In this subsection, we study the properties of η-parallelism of Ricci tensor on an $(LCS)_{2n+1}$-manifold under D-homothetic deformation. Differentiating (3.8) covariantly with respect to W and then using (2.3), we get

$$\begin{aligned}(\nabla_W\overline{S})(X,Y) = (\nabla_W S)(X,Y) + \frac{a-1}{a}\{a(2\alpha+a-1)+2\alpha[a(3n-2)-(n-1)]\\ +2n(\xi\alpha)\}\{(\nabla_W\eta)(X)\eta(Y)+(\nabla_W\eta)(Y)\eta(X)\}\\ +\frac{2(a-1)}{a}[\{a(3n-1)-(n-1)\}d\alpha(W)-nd\rho(W)]\,\eta(X)\eta(Y)\\ +\frac{2(a-1)}{a}\{(2a+2\alpha-1)(n-1)+a\}d\alpha(W)g(X,Y).\end{aligned} \tag{3.22}$$

Replacing the vector fields X by ϕX and Y by ϕY in (3.22) and then using (2.7), we obtain

$$\begin{aligned}(\nabla_W\overline{S})(\phi X,\phi Y) = (\nabla_W S)(\phi X,\phi Y)\\ +\frac{2(a-1)}{a}\{(2a+2\alpha-1)(n-1)+a\}d\alpha(W)g(\phi X,\phi Y).\end{aligned} \tag{3.23}$$

Thus we can state to the following:

Theorem 3.10. *The property of η-parallelism of the Ricci tensor on a $(LCS)_{2n+1}$-manifold is not invariant under D-homothetic deformation.*

If we suppose that α is constant, then with the help of (3.23) and Theorem 3.10 we can state the following:

Corollary 3.11. *The property of η-parallelism of the Ricci tensor on a $(LCS)_{2n+1}$-manifold is invariant under D-homothetic deformation if and only if α is constant.*

3.5 (EGC) ϕ-recurrent $(LCS)_{2n+1}$-manifolds

The properties of extended generalized concircularly (EGC) ϕ-recurrent $(LCS)_{2n+1}$-manifolds are studied in this subsection.

Definition 3.12. *A Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, is said to be an extended generalized concircularly ϕ-recurrent $(LCS)_{2n+1}$-manifold under D-homothetic deformation if its concircular curvature tensor $\overline{C}$ satisfies the condition*

$$\phi^2((\nabla_W\overline{C})(X,Y)Z) = A(W)\phi^2(\overline{C}(X,Y)Z)+B(W)\phi^2(G(X,Y)Z), \tag{3.24}$$

for all X, Y, Z, $W\in\chi(M^{2n+1})$, where A and B are non-vanishing 1-forms defined in (1.1) and G is the tensor of type $(1,3)$ defined in (1.2).

The concircular curvature tensor $\overline{C}$ [25] of type $(1,3)$ is given by

$$\overline{C}(X,Y)Z = \overline{R}(X,Y)Z - \frac{\overline{r}}{2n(2n+1)}G(X,Y)Z, \tag{3.25}$$

where $\bar{r}$ is the scalar curvature of the manifold under D-homothetic deformation. Let us consider an extended generalized concircularly ϕ-recurrent Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, under D-homothetic deformation. Taking covariant derivative of (3.25) along the vector field W, we have

$$(\nabla_W\overline{C})(X,Y)Z = (\nabla_W\overline{R})(X,Y)Z - \frac{d\tilde{r}(W)}{2n(2n+1)}G(X,Y)Z,$$

which is equivalent to

$$g((\nabla_W\overline{C})(X,Y)Z,U) = g((\nabla_W\overline{R})(X,Y)Z,U) - \frac{d\tilde{r}(W)}{2n(2n+1)}g(G(X,Y)Z,U).$$

Let $\{e_i; i=1,2,...,2n+1\}$ be a set of orthonormal basis of the tangent space at any point of the manifold. Replacing $X=U=e_i$ in the above equation and taking summation over i, $1\le i\le 2n+1$, we have

$$\sum_{i=1}^{2n+1} g((\nabla_W\overline{C})(e_i,Y)Z,e_i) = (\nabla_W\overline{S})(Y,Z) - \frac{d\tilde{r}(W)}{(2n+1)}g(Y,Z). \tag{3.26}$$

With the help of (2.7), (3.24) assumes the form

$$\begin{aligned}(\nabla_W\overline{C})(X,Y)Z+\eta((\nabla_W\overline{C})(X,Y)Z)\xi &= A(W)\{\overline{C}(X,Y)Z\\ &+\eta(\overline{C}(X,Y)Z)\xi\}+B(W)\{\overline{G}(X,Y)Z+\eta(\overline{G}(X,Y)Z)\xi\},\end{aligned} \tag{3.27}$$

from which it follows that

$$\begin{aligned}&g((\nabla_W\overline{C})(X,Y)Z,U)+\eta((\nabla_W\overline{C})(X,Y)Z)\eta(U) = A(W)\{g(\overline{C}(X,Y)Z,U)\\ &+\eta(\overline{C}(X,Y)Z)\eta(U)\}+B(W)\{g(\overline{G}(X,Y)Z,U)+\eta(\overline{G}(X,Y)Z)\eta(U)\}.\end{aligned} \tag{3.28}$$

Putting $X=U=e_i$ in (3.28) and taking summation over i, $1\le i\le(2n+1)$, and then using (3.26), we have

$$\begin{aligned}&(\nabla_W\overline{S})(Y,Z)-\frac{d\tilde{r}(W)}{2n+1}g(Y,Z)+g((\nabla_W\overline{C})(\xi,Y)Z,\xi)\\ &=A(W)\{\overline{S}(Y,Z)-\frac{\tilde{r}}{2n+1}g(Y,Z)+\eta(\overline{C}(\xi,Y)Z)\}\\ &+B(W)\{(2n-1)g(Y,Z)-\eta(Y)\eta(Z)\}.\end{aligned} \tag{3.29}$$

In view of (1.2), (2.7), (2.10) and (3.3), equation (3.25) reduces to

$$\eta(\overline{C}(\xi,Y)Z) = \left(\frac{\bar{r}}{2n(2n+1)} - \alpha(\alpha+1-a)(\frac{2a-1}{a}) + \frac{\rho}{a}\right)\{g(Y,Z)+\eta(Y)\eta(Z)\}. \tag{3.30}$$

By virtue of (2.3), (2.4), (2.11), (3.25) and (3.30), it is obvious that

$$\begin{aligned}g((\nabla_W\overline{C})(\xi,Y)Z,\xi) = \{&\frac{d\bar{r}(W)}{2n(2n+1)}+\frac{(2a-1)(a-1-2\alpha)}{a}d\alpha(W)\\ &+\frac{d\rho(W)}{a}\}\{g(Y,Z)+\eta(Y)\eta(Z)\}.\end{aligned} \tag{3.31}$$

In view of (3.30) and (3.31), equation (3.29) takes the form

$$\begin{aligned}(\nabla_W\overline{S})(Y,Z) &= A(W)\overline{S}(Y,Z)+B(W)g(Y,Z)+\left(\frac{d\bar{r}(W)-\bar{r}A(W)}{2n+1}\right)g(Y,Z)\\ &+\{\left(\frac{\bar{r}}{2n(2n+1)}-\frac{\alpha(\alpha+1-a)(2a-1)-\rho}{a}\right)A(W)\\ &-\left(\frac{d\bar{r}(W)}{2n(2n+1)}+\frac{(2a-1)(a-1-2\alpha)}{a}d\alpha(W)+\frac{d\rho(W)}{a}\right)\}\{g(Y,Z)\\ &+\eta(Y)\eta(Z)\}+B(W)\{2(n-1)g(Y,Z)-\eta(Y)\eta(Z)\}.\end{aligned} \tag{3.32}$$

This leads to the following:

Theorem 3.13. *An extended generalized concircularly ϕ-recurrent Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, under D-homothetic deformation is generalized Ricci-recurrent if and only if the relation*

$$\begin{aligned}&\left(\frac{d\bar{r}(W)-\bar{r}A(W)}{2n+1}\right)g(Y,Z)+\{\left(\frac{\bar{r}}{2n(2n+1)}-\frac{\alpha(\alpha+1-a)(2a-1)-\rho}{a}\right)A(W)\\&-\left(\frac{d\bar{r}(W)}{2n(2n+1)}+\frac{(2a-1)(a-1-2\alpha)}{a}d\alpha(W)+\frac{d\rho(W)}{a}\right)\}\{g(Y,Z)\\&+\eta(Y)\eta(Z)\}+B(W)\{2(n-1)g(Y,Z)-\eta(Y)\eta(Z)\}=0,\end{aligned}\tag{3.33}$$

holds for all W, Y, $Z\in\chi(M^{2n+1})$.

If we replace the vector field Z by ξ in (3.33), then we can observe that $A=\lambda B$, provided that $\bar{r}$ is a non-zero constant. Here $\lambda=\frac{4n^2-1}{\bar{r}}(\neq 0)$ is a constant. With the help of the above discussion and Theorem 3.13, we can state the following:

Corollary 3.14. *Let an extended generalized concircularly ϕ-recurrent Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, with constant scalar curvature, under D-homothetic deformation is a generalized Ricci-recurrent manifold, then the associated 1-forms A and B are co-directional, i.e., $A=\lambda B$.*

4 Example

We consider a 3-dimensional manifold $M=\{(x,y,z)\in\Re^3 : z\neq 0\}$, where (x,y,z) are the standard coordinates in $\Re^3$. Let $\{E_1,E_2,E_3\}$ be linearly independent global frame on M given by

$$E_1=e^z\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right),\qquad E_2=e^z\frac{\partial}{\partial y},\qquad E_3=e^{2z}\frac{\partial}{\partial z}.$$

Let g be the Lorentzian metric defined by

$$g(E_1,E_3)=g(E_2,E_3)=g(E_1,E_2)=0,\ g(E_1,E_1)=g(E_2,E_2)=1,\ g(E_3,E_3)=-1.$$

Let η be the 1-form defined by $\eta(V)=g(V,E_3)$ for any $V\in\chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by $\phi E_1=E_1$, $\phi E_2=E_2$, $\phi E_3=0$. Then using the linearity of ϕ and g, we have

$$\eta(E_3)=-1,\ \phi^2V=V+\eta(V)E_3,\ g(\phi V,\phi W)=g(V,W)+\eta(V)\eta(W),$$

for any V, $W\in\chi(M)$. Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[E_1,E_2]=-e^zE_2,\qquad [E_1,E_3]=-e^{2z}E_1,\qquad [E_2,E_3]=-e^{2z}E_2.$$

For the Levi-Civita connection ∇ of the metric g, we have

$$2g(\nabla_XY,Z)=Xg(Y,Z)+Yg(X,Z)-Zg(X,Y)-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]).$$

This expression is known as Koszul's formula. Taking $E_3=\xi$ and using Koszula's formula for the Lorentzian metric g, we can easily calculate the following:

$$\begin{aligned}&\nabla_{E_1}E_3=-e^{2z}E_1,\quad \nabla_{E_1}E_1=-e^{2z}E_3,\quad \nabla_{E_1}E_2=0,\\&\nabla_{E_2}E_3=-e^{2z}E_2,\quad \nabla_{E_3}E_2=0,\quad \nabla_{E_2}E_1=e^zE_2,\\&\nabla_{E_3}E_3=0,\quad \nabla_{E_2}E_2=-e^{2z}E_3-e^zE_1,\quad \nabla_{E_3}E_1=0.\end{aligned}$$

From the above calculations it can be easily see that $E_3 = \xi$ is a unit timelike concircular vector field and hence (ϕ, ξ, η, g) is an $(LCS)_3$-structure on manifold M^3. Consequently $M^3(\phi, \xi, \eta, g)$ is an $(LCS)_3$-manifold with $\alpha = -e^{2z} \neq 0$ such that $(X\alpha) = \rho\eta(X)$, where $\rho = 2e^{4z}$. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor R as follows:

$$\begin{aligned}
&R(E_2,E_3)E_3 = -e^{4z}E_2, \quad R(E_1,E_3)E_3 = -e^{4z}E_1,\\
&R(E_1,E_3)E_1 = -e^{4z}E_3, \quad R(E_1,E_2)E_1 = -e^{2z}\{e^{2z}+1\}E_2,\\
&R(E_1,E_2)E_2 = \{e^{4z}-e^{2z}\}E_1, \quad R(E_2,E_3)E_2 = -e^{3z}(e^{z}E_3+E_1)
\end{aligned}$$

and the components that can be obtained from these by the symmetric properties. With the help of the above equations, we can find the Ricci tensors and scalar curvature as:

$$S(E_1,E_1) = g(R(E_1,E_2)E_2,E_1) + g(R(E_1,E_3)E_3,E_1) = -e^{2z}.$$

In the same fasion, we can find that

$$S(E_2,E_2) = e^{2z}, \quad S(E_3,E_3) = 2e^{4z}$$

and

$$r = S(E_1,E_1) + S(E_2,E_2) - S(E_3,E_3) = -2e^{4z}.$$

It is well known that in a three dimensional $(LCS)_3$-manifold, the curvature tensor R satisfies the relation

$$R(X,Y)Z = S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}.$$

In consequence of (2.1), (2.2), (2.9) and (2.12), we have

$$S(X,Y) = \{\frac{r}{2} - (\alpha^2-\rho)\}g(X,Y) + \{\frac{r}{2} - 3(\alpha^2-\rho)\}\eta(X)\eta(Y).$$

From the last expressions, we can find that

$$\begin{aligned}
R(X,Y)Z = &\{r - 2(\alpha^2-\rho)\}\{g(Y,Z)X - g(X,Z)Y\}\\
&+\{\frac{r}{2} - 3(\alpha^2-\rho)\}\{\eta(Y)X - \eta(X)Y\}\eta(Z)\\
&+\{\frac{r}{2} - 3(\alpha^2-\rho)\}\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}\xi.
\end{aligned} \tag{4.1}$$

which is equivalent to

$$\begin{aligned}
{}'R(X,Y,Z,U) = &\{r - 2(\alpha^2-\rho)\}\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\}\\
&+\{\frac{r}{2} - 3(\alpha^2-\rho)\}\{\eta(Y)g(X,U) - \eta(X)g(Y,U)\}\eta(Z)\\
&+\{\frac{r}{2} - 3(\alpha^2-\rho)\}\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}\eta(U).
\end{aligned} \tag{4.2}$$

where ${}'R(X,Y,Z,W) = g(R(X,Y)Z,W)$. In view of (4.2), the ϕ-sectional curvature of the manifold M^3 is given by

$$K(X,\phi X) = 2(\alpha^2-\rho) - r.$$

for any vector field X orthogonal to ξ. Also, in this example, we have $K(E_1,\phi E_1) = g(R(E_1,\phi E_1)E_1,\phi E_1) = 0$ and $K(E_2,\phi E_2) = 0$. Again from the above relations, we can find

$$\overline{K}(E_1,\phi E_1) - K(E_1,\phi E_1) = \frac{\alpha(\alpha+1)(1-a)}{a} = \overline{K}(E_2,\phi E_2) - K(E_2,\phi E_2).$$

Hence the Theorem 3.6 and Corollary 3.7 are verified.

Since $\{E_1, E_2, E_3\}$ forms a basic of the 3-dimensional $(LCS)_3$-manifold, therefore the vector fields X, Y, Z $\in \chi(M^3)$ can be written as

$$X = a_1E_1 + b_1E_2 + c_1E_3, \;\; Y = a_2E_1 + b_2E_2 + c_2E_3, \;\; Z = a_3E_1 + b_3E_2 + c_3E_3,$$

where a_i, b_i, $c_i \in \mathfrak{R}^+$ (the set of all positive real numbers), $i = 1,2,3$. Then

$$\begin{aligned}\overline{R}(X,Y)Z = {} & [e^{4z}\{(b_2b_3 - c_2c_3)a_1 + a_2(c_1c_3 - b_1b_3) - b_1c_2b_3\} \\ & + e^{3z}(c_1b_2 - b_1c_2)b_3 + e^{2z}(b_1a_2 - a_1b_2)b_3]E_1 + [e^{4z}\{(b_1a_2 - a_1b_2)a_3 \\ & + (c_1b_2 - b_1c_2)c_3\} + e^{2z}(a_2b_1 - a_1b_2)a_3]E_2 \\ & + [e^{4z}\{(c_1a_2 - a_1c_2)a_3 + (c_1b_2 - b_1c_2)b_3\}]E_3,\end{aligned} \tag{4.3}$$

$$\begin{aligned}G(X,Y)Z = {} & (a_2a_3 + b_2b_3 - c_2c_3)(a_1E_1 + b_1E_2 + c_1E_3) \\ & - (a_1a_3 + b_1b_3 - c_1c_3)(a_2E_1 + b_2E_2 + c_2E_3).\end{aligned} \tag{4.4}$$

With the help of the above results, we can find the following after a long calculation

$$\begin{aligned}(\nabla_{E_1}\overline{R})(X,Y)Z = {} & \{e^{5z}(a_1b_2 - b_1a_2)b_3 + e^{4z}(b_1c_2 - c_1b_2)b_3\}E_1 \\ & + \{b_1c_2b_3e^{6z} + e^{5z}(b_1c_2 - c_1b_2)b_3 + e^{4z}(a_1b_2 - b_1a_2)b_3\}E_3 \\ & + e^{4z}\{(b_1c_2 - c_1b_2)a_3 + (b_1a_2 - a_1b_2)c_3\}E_2,\end{aligned} \tag{4.5}$$

$$\begin{aligned}(\nabla_{E_2}\overline{R})(X,Y)Z = {} & [e^{5z}\{c_1b_2b_3 + c_2c_3(a_1 - b_1)\} + e^{4z}\{(b_1c_2 - c_1b_2)a_3 \\ & + (b_1a_2 - a_1b_2)c_3 - a_1c_2b_3\} + 2e^{3z}(a_1b_2 - a_2b_1)a_3]E_1 \\ & + [e^{5z}\{2c_1a_2c_3 - c_2c_3(a_1 + b_1)\} + e^{4z}\{(c_1a_2 - a_1c_2)a_3 \\ & + (c_1b_2 - b_1c_2)b_3\} + 2e^{3z}(b_1a_2 - a_1b_2)b_3]]E_2 \\ & + [e^{5z}\{(a_3 - b_3)a_1c_2 + e^{4z}(a_1b_2 - a_2b_1)a_3]E_3\end{aligned} \tag{4.6}$$

and

$$(\nabla_{E_3}\overline{R})(X,Y)Z = 0. \tag{4.7}$$

In view of (4.3) and (4.4), we get

$$\phi^2(\overline{R}(X,Y)Z) = l_1E_1 + m_1E_2, \quad \phi^2(G(X,Y)Z) = l_2E_1 + m_2E_2, \tag{4.8}$$

where

$$l_1 = e^{4z}\{(b_2b_3 - c_2c_3)a_1 + a_2(c_1c_3 - b_1b_3) - b_1c_2b_3\} + e^{3z}(c_1b_2 - b_1c_2)b_3 + e^{2z}(b_1a_2 - a_1b_2)b_3,$$

$$m_1 = e^{4z}\{(b_1a_2 - a_1b_2)a_3 + (c_1b_2 - b_1c_2)c_3\} + e^{2z}(a_2b_1 - a_1b_2)a_3,$$

$$l_2 = (a_1b_2 - a_2b_1)b_3 - (a_1c_2 - a_2c_1)c_3,$$

$$m_2 = (a_2b_1 - a_1b_2)a_3 - (b_1c_2 - b_2c_1)c_3.$$

With the help of equations (4.5)-(4.7), we can observe that

$$\phi^2((\nabla_{E_i}\overline{R})(X,Y)Z) = p_iE_1 + q_iE_2, \;\; i = 1,2,3,$$

where

$$p_1 = e^{5z}(a_1b_2 - b_1a_2)b_3 + e^{4z}(b_1c_2 - c_1b_2)b_3,$$

$$q_1 = e^{4z}\{(b_1c_2 - c_1b_2)a_3 + (b_1a_2 - a_1b_2)c_3\},$$
$$p_2 = e^{5z}\{c_1b_2b_3 + c_2c_3(a_1 - b_1)\} + e^{4z}\{(b_1c_2 - c_1b_2)a_3 + (b_1a_2 - a_1b_2)c_3 - a_1c_2b_3\} + 2e^{3z}(a_1b_2 - a_2b_1)a_3,$$
$$q_2 = e^{5z}\{2c_1a_2c_3 - c_2c_3(a_1 + b_1)\} + e^{4z}\{(c_1a_2 - a_1c_2)a_3 + (c_1b_2 - b_1c_2)b_3\} + 2e^{3z}(b_1a_2 - a_1b_2)b_3,$$
$$p_3 = 0 \ and \ q_{3=0}.$$

Now, we consider the 1-forms A and B as follows:

$$A(E_i) = \frac{m_2p_i - l_2q_i}{l_1m_2 - m_1l_2}, \ B(E_i) = \frac{l_1q_i - m_1p_i}{l_1m_2 - m_1l_2}, \tag{4.9}$$

for $i = 1,2,3$ such that $l_1m_2 - m_1l_2 \neq 0$, $m_2p_i - l_2q_i \neq 0$ and $l_1q_i - m_1p_i \neq 0$, $i = 1,2,3$. From (3.13), we have

$$\phi^2((\nabla_{E_i}\overline{R})(X,Y)Z = A(E_i)\phi^2(\overline{R}(X,Y)Z) + B(E_i)\phi^2(G(X,Y)Z), \tag{4.10}$$

where $i = 1,2,3$. From (4.8) and (4.9), it can be easily show that the manifold satisfies the relation (4.10). Hence the manifold under consideration is an extended generalized ϕ-recurrent $(LCS)_3$-manifold under D-homothetic deformation, which is neither ϕ-recurrent nor generalized ϕ-recurrent. Therefore, we have the following:

Theorem 4.1. *There exists an extended generalized ϕ-recurrent $(LCS)_3$-manifold $M^3(\phi,\xi,\eta,g)$, under D-homothetic deformation which is neither ϕ-recurrent nor generalized ϕ-recurrent.*

References

[1] K. Matsumoto and I. Mihai, On a certain transformation in a Lorentzian para Sasakian manifold, Tensor N. S. 47 (1988) 189-197.
[2] K. Matsumoto, On Lorentzian almost para-contact manifolds, Bull. Yamagata Univ. Nature. Sci. 12 (1989) 151-156.
[3] A. A. Shaikh, On Lorentzian almost para-contact manifolds with a structure of the con-circular type, Kyungpook Math. J. 43 (2003) 305-314.
[4] A. A. Shaikh, T. Basu and S. Eyasmin, On locally ϕ-symmetric $(LCS)_n$-manifolds, Int. J. Pure Appl. Math. 41 (8) (2007) 1161-1170.
[5] A. A. Shaikh, T. Basu and S. Eyasmin, On the existence of ϕ-recurrent $(LCS)_{2n+1}$-manifolds, Extr. Math. 23 (1) (2008) 71-83.
[6] A. A. Shaikh, Some results on $(LCS)_n$-manifolds, J. Korean Math. Soc. 46 (3) (2009) 449-461.
[7] S. Yadav, D. L. Suthar and M. Hailu, On extended generalized ϕ-recurrent $(LCS)_{2n+1}$-manifolds, Bol. Soc. Pran. Mat. (3s.) 37 (2) (2019) 9-21.
[8] S. K. Hui, On ϕ-pseudosymmetries of $(LCS)_n$-manifolds, Kyungpook Math. J., 53 (2013), 285-294.
[9] A. Basari and C. Murathan, On generalized ϕ-recurrent Kenmotsu manifolds, Fen Derg. 3 (1) (2008) 91-97.
[10] U. C. De, A. A. Shaikh and S. Biswas, On ϕ-recurrent Sasakian manifolds, Novi Sad J. Math. 33 (2003) 13-48.
[11] U. C. De, N. Guha and D. Kamilya, On generalized Ricci recurrent manifolds, Tensor N. S. 56 (1995) 312-317.
[12] R. S. D. Dubey, Generalized recurrent spaces, Indian J. Pure Appl. Math. 10 (12) (1979) 1508-1513.
[13] M. Kon, Invariant sub-manifolds in Sasakian manifolds, Math. Ann. 219 (1976) 277-290.
[14] J. A. Schouten, Ricci Calculus, Springer-Verlag, 2nd edition, (1954).
[15] C. *Özgü*r, On generalized recurrent Kenmotsu manifolds, World Applied Sci. J. 2 (1) (2007) 9-33.
[16] D. A. Patil, D. G. Prakasha and C. S. Bagewadi, On generalized ϕ-recurrent Sasakian manifolds, Bull. Math. Anal. Appl. 1 (3) (2009) 42-48.
[17] D. G. Prakasha and A. Yildiz, Generalized ϕ-recurrent Lorentzian α-Sasakian manifolds, Commun. Fac. Sci. Univ. Ank. Ser. A1 59 (1) (2010) 53-65.
[18] D. G. Prakasha, On extended generalized ϕ-recurrent Sasakian manifolds, Journal of the Egyptian Mathematical Society 21 (2013) 25-31.
[19] A. A. Shaikh, D. G. Prakasha and H. Ahmad, On generalized ϕ-recurrent LP-Sasakian manifolds, Journal of the Egyptian Mathematical Society 23 (2015) 161-166.
[20] A. A. Shaikh and S. K. Hui, On extended generalized ϕ-recurrent β-Kenmotsu manifolds, Publ. De. L鈥檌nstitut Mathematique, Nouvelle series tome 89 (103) (2011) 77-88.
[21] A. A. Shaikh and K. K. Baishya, On ϕ-symmetric LP-Sasakian manifolds, Yokohama Math. J. 52 (2006) 97-112.
[22] T. Takahashi, Sasakian ϕ-symmetric spaces, Tohoku Math. J. 29 (1977) 91-113.

[23] S. Tanno, The topology of contact Riemannian manifolds, J. Tohoku Math. 12 (1968) 700-717.
[24] S. Tanno, Partially conformal transformations with respect to $(m-1)$-dimensional distribution of m-dimensional Riemannian manifolds, J. Tohoku Math. 2 (17) (1965) 358-409.
[25] K. Yano, Concircular geometry, I, Proc. Imp. Acad. Tokyo, 16 (1940) 195-200.
[26] K. De, *D*-homothetic deformation of *LP*-Sasakian manifolds, Novi Sad J. Math. 45 (2) (2015) 113-123.
[27] Eric Boeckx and J. T. Cho, *D*-homothetic transforms of ϕ-symmetric spaces, Mediterranean J. of Math. 11 (2) (2014) 745-753.