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Noether's theorems for the relative motion systems on time scales

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Abstract

This paper propose Noether symmetries and the conserved quantities of the relative motion systems on time scales. The Lagrange equations with delta derivatives on time scales are presented for the system. Based upon the invariance of Hamilton action on time scales, under the infinitesimal transformations with respect to the time and generalized coordinates, the Hamilton' s principle, the Noether theorems and conservation quantities are given for the systems on time scales. Lastly, an example is given to show the application the conclusion.

Keywords: time scale; Noether theorem; the relative motion systems.

1 Introduction

A time scale is an arbitrary nonempty closed subset of the real numbers.The calculus of time scales were initiated by B.Aulbach and S.Hilger [1] in order to create a theory which can unify discrete and continuous analysis. A time scale is a model of time, and the new theory has found important applications in several fields which require simultaneous modeling of discrete and continuous data, in the calculus of variations, control theory, and optimal control [2–5]. The calculus of variations on time scales was initiated with the presentation of Euler-Lagrange equations on time scales was presented in 2004 [6]. But, Torres put forward the second Euler-Lagrange equations and researched the higher-order calculus of variations on time scales [7, 8]. The calculus of variations and control theory are disciplines in which there appears to be many opportunities for application of time scales [9–11].

In 1918, Noether proposed famous Noether symmetry theorems which could be used to deal with the invariance of the Hamilton action under the infinitesimal transformations: when a system exhibits a symmetry, then a conservation law can be obtained. Bartosiewicz and Torres showed that there existed a conserved quantity in Lagrangian system for each Noether symmetry [12] on time scales by the technique of time-re-parameterization. Using this technique, Cai and Fu studied the theories of Noether symmetries of the nonconservative and non-

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holonomic systems on time scales [13, 14]. Noether theory of the Hamilton systems on time scales was given by Zhang [15, 16]. It is worth mentioning that the dynamic systems on time scales with delta derivative have just started to originate.

With the development of modern science and technology, people pay attention to the dynamic of relative motion. Jet aircrafts, rockets, satellite, spacecraft and so on generally involve application of the relative motion systems. We also know that the movement of Mechanical systems is researched in either absolute coordinate system or moving coordinate system. The dynamic systems in the moving coordinate system is called the relative motion dynamic. In 1961, Lur'e introduced the equation of the relative motion systems for conservative systems [17]. In 1993, The Lagrange equation of relative motion dynamics for the general holonomic system was first studied by Liu [18]. In recent decades, a series of innovative research results about dynamics of relative motion have been obtained [19, 20, 22].

In this letter, we study the Noether symmetry of relative motion systems on time scales. The structure of this letter as follows: In Section 2, we review preparatory knowledge and properties of time scales. In Section 3, we establish the equations of the relative motion systems with delta derivatives. In Section 4, Noether theorems and conserved quantities for the relative motion systems are founded. Lastly, an example is used to illustrate the results.

2 Previous results of time scales

To begin with, we briefly present some main definitions and properties about times scales. More detailed theory of time scales can refer to [23–25].

Definition 1 A time scale T is an arbitrary nonempty closed subset of the set R of real number. For $t \in T$, we define the forward jump operator $\sigma : T \to T$ by

$$\sigma(t) = inf\{s \in T : s > t\}, \tag{1}$$

and the backward jump operator $\rho : T \to T$ by

$$\rho(t) = sup\{s \in T : s < t\}. \tag{2}$$

The graininess function $\mu : T \to [0, \infty]$ is defined by $\mu(t) = \sigma(t) - t$ for each $t \in T$.

A point is called right-dense, right-scattered, left-dense or left-scattered if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, respectively. We say $T^k = T - \{M\}$ if T has a left-scattered maximum M, otherwise $T^k = T$.

Definition 2 Assume $f : T \to R$ is a function and $t \in T^k$, we define $f^\Delta(t)$ to be the real number with the property with given any ε, there is neighborhood $U = (t - \delta, t + \delta) \bigcap T$ of such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s|, \tag{3}$$

is true for all $s \in U$, we call $f^\Delta(t)$ the delta derivative of f at t.

Definition 3 A function $f : T \to R$ is continuous at $t \in T^k$ and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}. \tag{4}$$

Furthermore, if f is differentiable at $t \in T^k$, then

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t). \tag{5}$$

Definition 4 Assume $f : T \to R$ is a regulated function, existing a function F with $F^\Delta(t) = f(t)$ is called a pre-antiderivative of f and in this case an integral of f from a to $b(a, b \in T)$ is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a), \tag{6}$$

We define the indefinite integral of f by

$$\int f(t)\Delta t = F(t) + C. \tag{7}$$

Where C is an arbitrary constant.

We shall often note $f^{\Delta}(t)$ by $\frac{\Delta}{\Delta t} f(t)$ if f is a composition of other functions. Furthermore, if f and g are both differentiable, the next formulate hold

$$\begin{aligned}
&(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t),\\
&(kf)^{\Delta}(t) = kf^{\Delta}(t),\\
&(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t) + f^{\Delta}(t)g^{\sigma}(t),
\end{aligned} \tag{8}$$

where we abbreviate $f \circ \sigma = f(\sigma(t))$ by f^{σ}.

Remark 1 we consider the two cases $T = R$ and $T = Z$.

(i) If $T = R$, $f : R \to R$ is delta differentiable at $t \in R$, then $f^{\Delta}(t) = f'(t)$.

(ii)If $T = Z$, $f : Z \to R$ is delta derivative at every $t \in Z$ with $f^{\Delta}(t) = f(t+1) - f(t)$.

Lemma 1 (Dubois-Reymond) [6] Let $g \in C_{rd}, g : [a,b] \to R^n$, if

$$\int_a^b g^T \eta^{\Delta}(t)\Delta t = 0, \tag{9}$$

for all $\eta \in C^1_{rd}$ with $\eta(a) = \eta(b) = 0$, then,

$$g(t) \equiv c, \forall t \in [a,b]^k forsomec \in R^n,$$

where C^1_{rd} means the set of differentiable functions with rd-continuous derivative.

We can also obtain the following conclusions about the time scales [25].

Assume that $\alpha : T \to R$ is strictly increasing and $T^* := \alpha(T)$ is a time scale, Let $\beta : T^* \to R$, then there exists t in the real interval $[t, \sigma(t)]$ with

$$\begin{aligned}
&(\alpha \circ \beta)^{\Delta}(t) = \beta^{\Delta^*}(\alpha(t))\alpha^{\Delta}(t),\\
&\frac{1}{\alpha^{\Delta}} = (\alpha^{-1})^{\Delta^*} \circ \alpha.
\end{aligned} \tag{10}$$

If $f : T \to R$ is an rd-continuous function and α is differentiable with rd-continuous derivative, then for $a, b \in T$,

$$\int_a^b f(t)\alpha^{\Delta}(t)\Delta t = \int_{\alpha(a)}^{\alpha(b)} \left(f \circ \alpha^{-1}\right)(t)\Delta^* t = \int_{\alpha(a)}^{\alpha(b)} f(t^*)\Delta^* t. \tag{11}$$

Let $\gamma = \alpha^{-1}$, then $q^*(t^*) := Q_{\varepsilon}(\gamma(t^*), q(\gamma(t^*)))$.

3 Lagrange equations for the relative motion systems on time scales

3.1 Equation of Chetaev constraint the relative motion systems

We know that the motion of a complex system may include the motion of a carrier, as well as the motion of a carried system relative to the carrier.

Suppose that the velocity of the base point in a carrier v_0 and its angular velocity is ω. We assume that the motion of N particles wouldn't change the motion rule of the carrier which is predetermined. N generalized coordinates q_s $(s=1,\cdots,n)$ determine the configuration of systems. If the movement of the systems are constrained by the double-sided ideal Chetaev nonholonomic constraints,

$$f_\beta(q_s,\dot{q}_s,t)=0(\beta=1,\cdots,g;s=1,\cdots,n).$$

The equations of the relative motion systems are[18]

$$\begin{cases} \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s}-\frac{\partial T}{\partial q_s}=Q_s-\frac{\partial}{\partial q_s}(V^0+V^\omega)+Q_s^\omega+\Gamma_s+\Lambda_s, \\ \\ \Lambda_s=\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}(s=1,\cdots,n), \end{cases}$$

where T is the kinetic energy of the relative motion function, λ_β is Lagrange multiplier, Q_s, V^0, V^ω, Q_s^ω, Γ_s are respectively the generalized forces, the potential energy of uniform force field, the potential energy of inertial centrifugal force field, generalized rotary inertia force, generalized gyroscopic force.
The Q_s can be divided into parts of potential and nonpotential

$$Q_s=Q_s'+Q_s'',Q_s'=\frac{\partial V}{\partial q_s}.$$

We construct Lagrangian of the relative motion system

$$L=T-V-V^0-V^\omega.$$

Equations can be written as follows

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s''+Q_s^\omega+\Gamma_s+\Lambda_s.$$

3.2 Lagrange equation for the relative motion systems with delta derivatives

Firstly, we introduce the following relationships [13]

$$\frac{\Delta}{\Delta t}(\delta q_s)=\delta\left(\frac{\Delta}{\Delta t}q_s\right)=\delta q_s^\Delta, \tag{12}$$

$$(\delta q_s)^\sigma=\delta q_s^\sigma. \tag{13}$$

The Hamilton principle for the relative motion system on time scales is written by

$$\int_{t_a}^{t_b}[\delta L+(Q_s''+Q_s^\omega+\Gamma_s)\delta q_s^\sigma]\Delta t=0, \tag{14}$$

where $(Q_s''+Q_s^\omega+\Gamma_s)\delta q_s^\sigma$ is the virtual work of generalized force.
We take total variation for Lagrange function L, then

$$\delta L=\frac{\partial L}{\partial q_s^\sigma}\delta q_s^\sigma+\frac{\partial L}{\partial q_s^\Delta}\delta q_s^\Delta. \tag{15}$$

By using of Eq. (15) and Eq. (14), we can obtain

$$\begin{aligned}
&\int_{t_a}^{t_b}\left[\frac{\partial L}{\partial q_s^\sigma}\delta q_s^\sigma+\frac{\partial L}{\partial q_s^\Delta}\delta q_s^\Delta+(Q_s''+Q_s^\omega+\Gamma_s)\delta q_s^\sigma\right]\Delta t\\
&=\int_{t_a}^{t_b}\left[\left(\frac{\partial L}{\partial q_s^\sigma}+Q_s''+Q_s^\omega+\Gamma_s\right)\delta q_s^\sigma+\frac{\partial L}{\partial q_s^\Delta}\delta q_s^\Delta\right]\Delta t\\
&=\int_{t_a}^{t_b}\left[\left(\frac{\partial L}{\partial q_s^\sigma}+Q_s''+Q_s^\omega+\Gamma_s\right)(\delta q_s)^\sigma+\frac{\partial L}{\partial q_s^\Delta}(\delta q_s)^\Delta\right]\Delta t\\
&=\int_{t_a}^{t_b}\left[\int_{t_a}^{t}\left(\frac{Q_s''+Q_s^\omega+\Gamma_s+\partial L(\tau,q_s^\sigma(\tau),q_s^\Delta(\tau))}{\partial q_s^\sigma(\tau)}\right)\Delta\tau\cdot\delta q_s\right]^\Delta\Delta t\\
&-\int_{t_a}^{t_b}\left[\int_{t_a}^{t}\left(\frac{Q_s''+Q_s^\omega+\Gamma_s+\partial L(\tau,q_s^\sigma(\tau),q_s^\Delta(\tau))}{\partial q_s^\sigma(\tau)}\right)\Delta\tau\right](\delta q_s)^\Delta\Delta t+\int_{t_a}^{t_b}\frac{\partial L}{\partial q_s^\Delta}(\delta q_s)^\Delta\Delta t\\
&=\int_{t_a}^{t_b}\left[\frac{\partial L}{\partial q_s^\Delta}-\int_{t_a}^{t}\left(Q_s''+Q_s^\omega+\Gamma_s+\frac{\partial L(\tau,q_s^\sigma(\tau),q_s^\Delta(\tau))}{\partial q_s^\sigma(\tau)}\right)\Delta\tau\right](\delta q_s)^\Delta\Delta t\\
&=0
\end{aligned}$$

Therefore, by Lemma 1

$$\frac{\partial L}{\partial q_s^\Delta}-\int_{t_a}^{t}\left(Q_s''+Q_s^\omega+\Gamma_s+\frac{\partial L(\tau,q_s^\sigma(\tau),q_s^\Delta(\tau))}{\partial q_s^\sigma(\tau)}\right)\Delta\tau\equiv const,t\in[t_a,t_b].$$

Hence

$$\frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_s^\Delta}-\frac{\partial L}{\partial q_s^\sigma}-(Q_s''+Q_s^\omega+\Gamma_s)=0 \tag{16}$$

Assuming the movement of the system is constrained by the double-sided ideal nonholonomic of Chetaev type with delta derivatives

$$f_\beta(t,q_s^\sigma,q_s^\Delta)=0,(\beta=1,2,\cdots,g). \tag{17}$$

Suppose the restrictions that constraints impose on the virtual displacements are

$$\frac{\partial f_\beta}{\partial q_s^\Delta}\delta q_s^\sigma=0,(s=1,2,\cdots,n;\beta=1,2,\cdots,g). \tag{18}$$

Multiplying δq_s^σ on both sides of Eq. (16)

$$\left((Q_s''+Q_s^\omega+\Gamma_s)-\frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_s^\Delta}+\frac{\partial L}{\partial q_s^\sigma}\right)\cdot\delta q_s^\sigma=0. \tag{19}$$

Introducing the Lagrange multiplier λ and multiplying λ on both sides of Eq.(18),

$$\lambda\frac{\partial f_\beta}{\partial q_s^\Delta}\delta q_s^\sigma=0. \tag{20}$$

Form Eq. (20) and Eq. (19), we get

$$\left((Q_s''+Q_s^\omega+\Gamma_s)-\frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_s^\Delta}+\frac{\partial L}{\partial q_s^\sigma}+\lambda\frac{\partial f_\beta}{\partial q_s^\Delta}\right)\cdot\delta q_s^\sigma=0. \tag{21}$$

Differential Eq. (21), we obtain the equation of the relative motion systems with Chetaev type constraints on time scales

$$\frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_s^\Delta}-\frac{\partial L}{\partial q_s^\sigma}=Q_s''+Q_s^\omega+\Gamma_s+\Lambda_s. \tag{22}$$

$$(\Lambda_s=\lambda\frac{\partial f_\beta}{\partial q_s^\Delta})$$

4 Noether theorem of the relative motion systems

4.1 Noether's theorem without transforming time

The Hamilton action with the delta derivative on time scales can be expressed as

$$S(\gamma) = \int_{t_a}^{t_b} L(t, q_s^\sigma, q_s^\Delta)\Delta t,$$

where γ is a curve.

Introducing the following single parameter infinitesimal transformations without transforming time:

$$\begin{aligned} &t^* = t, \\ &q_s^* = q_s + \varepsilon\xi_s(t,q) + o(\varepsilon), \end{aligned} \tag{23}$$

if and only if

$$\int_{t_a}^{t_b} L(t, q_s^\sigma, q_s^\Delta)\Delta t = \int_{t_a}^{t_b} L(t, q_s^{*\sigma}, q_s^{*\Delta})\Delta t. \tag{24}$$

Where ε is the infinitesimal parameter, $\xi_s : [a,b] \times R^n \to R$ is delta differentiable functions.

The relationship between the isochronous δ and the total variation Δ on time scale is as follows

$$\Delta q_s^\sigma = \delta q_s^\sigma + q_s^\Delta \Delta t.$$

According to Eq. (23), we have

$$\delta q_s^\sigma = \Delta q_s^\sigma - q_s^\Delta \Delta t = \varepsilon \xi_s^\sigma. \tag{25}$$

Substituting Eq. (25) into Eq. (18) has

$$\frac{\partial f_\beta}{\partial q_s^\Delta}\xi_s^\sigma = 0, (s = 1,2,\cdots,n; \beta = 1,2,\cdots,g). \tag{26}$$

Definition 5 The action S is said to be quasi invariant on U under the transformation groups (23), if and only if for any subinterval $[t_a,t_b] \in [a,b]$, any $\varepsilon, q \in U$

$$\int_{t_a}^{t_b} L(t, q_s^\sigma, q_s^\Delta)\Delta t = \int_{t_a}^{t_b} L(t, q_s^{*\sigma}, q_s^{*\Delta})\Delta t + \int_{t_a}^{t_b} \left(\frac{\Delta}{\Delta t}(\Delta G) + (Q_s'' + Q_s^\omega + \Gamma_s + \Lambda_s)\delta q_s^\sigma \right)\Delta t,$$

where the transformations satisfy the condition Eq.(26). we say that the invariance is called Noether generalized quasi-symmetry of the relative motion systems on time scales.

Theorem 1 If the action S is quasi-invariant on the infinitesimal transformations Eq.(23) then

$$\frac{\partial L}{\partial q_s^\sigma}\xi_s^\sigma + \frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta + (Q_s'' + Q_s^\omega + \Gamma_s + \Lambda_s)\xi_s^\sigma = -\frac{\Delta}{\Delta t}G, \tag{27}$$

where $\xi_s^\sigma(t,q) = \xi_s(\sigma(t), q(\sigma(t)))$, $\xi_s^\Delta(t,q) = \frac{\Delta}{\Delta t}\xi_s(t,q)$.

Proof. Consider the infinitesimal transformations (t, q_s^*) given by group(23) and Definition 5, we can obtain

$$\begin{aligned} \int_{t_a}^{t_b} L(t, q_s^\sigma, q_s^\Delta)\Delta t &= \int_{t_a}^{t_b} L(t, q_s^{*\sigma}, q_s^{*\Delta})\Delta t \\ &+ \int_{t_a}^{t_b} \left(\frac{\Delta}{\Delta t}(\Delta G) + (Q_s'' + Q_s^\omega + \Gamma_s + \Lambda_s)\delta q_s^\sigma \right)\Delta t \\ &= \int_{t_a}^{t_b} L(t, q_s^\sigma + \varepsilon\xi_s^\sigma + o(\varepsilon), q_s^\Delta + \varepsilon\xi_s^\Delta + o(\varepsilon))\Delta t \\ &+ \int_{t_a}^{t_b} \left(\frac{\Delta}{\Delta t}(\Delta G) + (Q_s'' + Q_s^\omega + \Gamma_s + \Lambda_s)\delta q_s^\sigma \right)\Delta t \end{aligned}$$

Taking into account any subinterval $[t_a,t_b] \in [a,b]$, we can get the following equivalent equation:

$$L(t,q_s^\sigma,q_s^\Delta) \ = L(t,q_s^\sigma+\varepsilon\xi_s^\sigma+o(\varepsilon),q_s^\Delta+\varepsilon\xi_s^\Delta+o(\varepsilon))+(Q_s''+Q_s^\omega+\Gamma_s+\Lambda_s)\Delta q_s^\sigma+\frac{\Delta}{\Delta t}(\Delta G). \qquad (28)$$

Differentiation both sides of Eq.(28)with respect to ε,

$$\frac{\partial L}{\partial t}\frac{\partial t}{\partial \varepsilon}+\frac{\partial L}{\partial q_s^\sigma}\frac{\partial q_s^{*\sigma}}{\partial \varepsilon}+\frac{\partial L}{\partial q_s^\Delta}\frac{\partial q_s^{*\Delta}}{\partial \varepsilon}+(Q_s''+Q_s^\omega+\Gamma_s+\Lambda_s)\frac{\partial \delta q_s^\sigma}{\partial \varepsilon}+\frac{\Delta}{\Delta t}G=0$$

Since

$$\frac{\partial q_s^{*\sigma}}{\partial \varepsilon}|_{\varepsilon=0}=\xi_s^\sigma,\frac{\partial q_s^{*\Delta}}{\partial \varepsilon}|_{\varepsilon=0}=\xi_s^\Delta.$$

Eq. (25) shows that

$$\frac{\partial \delta q_s^\sigma}{\partial \varepsilon}=\xi_s^\sigma.$$

Then, setting $\varepsilon=0$, we can obtain the Eq. (27).

Theorem 2 For Chetaev constraint the relative motion systems on time scales, if the infinitesimal transformations Eq. (23) satisfy the conditions Eq. (26), then the system Eq. (22) has conserved quantities of the form

$$I=\frac{\partial L}{\partial q_s^\Delta}\xi_s+G. \qquad (29)$$

Proof. It proves that Eq. (29) is equivalent to the proof of $I=const$, takeing the derivative of I with respect to t, then

$$\frac{\Delta}{\Delta t}I=\frac{\Delta}{\Delta t}\left[\frac{\partial L}{\partial q_s^\Delta}\xi_s+G\right]=\frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_s^\Delta}\xi_s^\sigma+\frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta+\frac{\Delta}{\Delta t}G.$$

Multiplying ξ_s^σ on both sides of Eq.(22)

$$\frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_s^\Delta}\xi_s^\sigma=\frac{\partial L}{\partial q_s^\sigma}\xi_s^\sigma+(Q_s''+Q_s^\omega+\Gamma_s+\Lambda_s)\xi_s^\sigma.$$

Observe that Eq. (27)

$$\frac{\Delta}{\Delta t}I=\frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta+\left(Q_s''+Q_s^\omega+\Gamma_s+\Lambda_s+\frac{\partial L}{\partial q_s^\sigma}\right)\xi_s^\sigma+\frac{\Delta}{\Delta t}G=0.$$

Namely

$$I=const.$$

4.2 Noether theorem with transforming time

Considering the following infinitesimal transformations with the time and the state variables:

$$\begin{aligned} t^* &= t+\varepsilon\xi_0(t,q)+o(\varepsilon), \\ q_s^* &= q_s+\varepsilon\xi_s(t,q)+o(\varepsilon). \end{aligned} \qquad (30)$$

Where $\xi_0,\xi_s:[a,b]\times R^n\to R$ are delta differentiable functions.

In this case, we assume the map $t \in [a,b] \mapsto \alpha(t) = t^* \in R$ is a strictly increasing C^1_{rd} function and its image is a new time scale $t^* = \alpha(t)$, whose forward jump operator and delta derivative are denote by σ^* and Δ^*. Following the arguments provided above,

$$\sigma^* \circ \alpha = \alpha \circ \sigma.$$

According to groups (30), we have

$$\delta q_s^\sigma = \Delta q_s^\sigma - q_s^\Delta \Delta t = \varepsilon(\xi_s^\sigma - q_s^{\sigma\Delta}\xi_0^\sigma). \tag{31}$$

Substituting Eq. (31) into Eq. (18)

$$f_{\beta s}(\xi_s^\sigma - q_s^{\sigma\Delta}\xi_0^\sigma) = 0 \tag{32}$$

Definition 6 The action S is called a generalized quasi invariant in the transformation groups (30) and if and only if for any subinterval $[t_a, t_b] \in [a,b]$

$$\begin{aligned}\int_{t_a}^{t_b} L(t, q_s^\sigma, q_s^\Delta)\Delta t &= \int_{\alpha(t_a)}^{\alpha(t_b)} L\left(t^*, q_s^{*\sigma^*}(t^*), q_s^{*\Delta^*}(t^*)\right)\Delta^* t^* \\ &+ \int_{t_a}^{t_b}\left(\frac{\Delta}{\Delta t}(\Delta G) + (Q_s'' + Q_s^\omega + \Gamma_s + \Lambda_s)\delta q_s^\sigma\right)\Delta t.\end{aligned}$$

Theorem 3 Noether theory points out that ξ_0, ξ_s satisfy the Noether identity if it exists the gauge $G = G(t, q_s^\sigma, q_s^\Delta)$. If the action S is generalized quasi-invariant in the infinitesimal transformations Eq. (30), then

$$\begin{aligned}&\frac{\partial L}{\partial t}\xi_0 + \frac{\partial L}{\partial q_s^\sigma}\xi_s^\sigma + \frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta + L\xi_0^\Delta - \frac{\partial L}{\partial q_s^\Delta}\xi_0^\Delta q_s^\Delta + (Q_s'' + Q_s^\omega \\ &+\Gamma_s + \Lambda_s) \times [\xi_s^\sigma - \xi_0^\sigma(q_s^\Delta + \mu(t)q_s^{\Delta\Delta})] = -\frac{\Delta}{\Delta t}G.\end{aligned} \tag{33}$$

Proof. Consider the Definition 6, we have

$$\begin{aligned}&\int_{t_a}^{t_b} L(t, q_s^\sigma, q_s^\Delta)\Delta t \\ &= \int_{\alpha(t_a)}^{\alpha(t_b)} L\left(t^*, q_s^{*\sigma^*}, q_s^{*\Delta^*}\right)\Delta^* t^* \\ &+ \int_{t_a}^{t_b}\left(\frac{\Delta}{\Delta t}(\Delta G) + (Q_s'' + Q_s^\omega + \Gamma_s + \Lambda_s)\delta q_s^\sigma\right)\Delta t \\ &= \int_{t_a}^{t_b} L\left(\alpha(t), (q_s^* \circ \sigma^* \circ \alpha)(t), q_s^{*\Delta^*}(\alpha(t))\right)\alpha^\Delta(t)\Delta t \\ &+ \int_{t_a}^{t_b}\left(\frac{\Delta}{\Delta t}(\Delta G) + (Q_s'' + Q_s^\omega + \Gamma_s + \Lambda_s)\delta q_s^\sigma\right)\Delta t.\end{aligned} \tag{34}$$

Since σ^* is the new forward jump operator and Δ^* is the new delta derivative, we can have by Eq. (10)

$$(q_s^* \circ \alpha)^\Delta(t) = q_s^{*\Delta^*}(\alpha(t))\alpha^\Delta(t).$$

Then

$$q_s^{*\Delta^*}(\alpha(t)) = \frac{(q_s^* \circ \alpha)^\Delta(t)}{\alpha^\Delta(t)}. \tag{35}$$

Gathering Eq. (34) and Eq. (35), we can derive

$$\begin{aligned}
&\int_{t_a}^{t_b} L(t, q_s^\sigma, q_s^\Delta)\Delta t \\
&= \int_{t_a}^{t_b} L\left(\alpha(t), (q_s^* \circ \sigma \circ \alpha)(t), \frac{(q_s^* \circ \alpha)^\Delta(t)}{\alpha^\Delta(t)}\right)\alpha^\Delta(t)\Delta t \\
&+ \int_{t_a}^{t_b} \left(\frac{\Delta}{\Delta t}(\Delta G) + (Q_s'' + Q_s^\omega + \Gamma_s + \Lambda_s)\delta q_s^\sigma\right)\Delta t.
\end{aligned}$$

Since $[t_a, t_b]$ is arbitrary subinterval

$$\begin{aligned}
L(t, q_s^\sigma, q_s^\Delta) \ &= L\left(\alpha(t), (q_s^* \circ \sigma \circ \alpha)(t), \frac{(q_s^* \circ \alpha)^\Delta(t)}{\alpha^\Delta(t)}\right)\alpha^\Delta(t) \\
&+ (Q_s'' + Q_s^\omega + \Gamma_s + \Lambda_s)\delta q_s^\sigma + \frac{\Delta}{\Delta t}(\Delta G).
\end{aligned} \tag{36}$$

According to group (30), differentiating both sides of Eq. (36) with respect to ε, setting $\varepsilon = 0$, we get

$$\begin{aligned}
0 \ &= \left[\frac{\partial L}{\partial t}\frac{\partial \alpha(t)}{\partial \varepsilon} + \frac{\partial L}{\partial q_s^\sigma}\frac{\partial (q_s^* \circ \alpha \circ \sigma)}{\partial \varepsilon} + \frac{\partial L}{\partial q_s^\Delta}\frac{\partial}{\partial \varepsilon}\left(\frac{(q_s^* \circ \alpha)^\Delta}{\alpha^\Delta}\right)\right]\alpha^\Delta + L\frac{\partial \alpha^\Delta}{\partial \varepsilon} \\
&+ (Q_s'' + Q_s^\omega + \Gamma_s + \Lambda_s)\frac{\partial \delta q_s^\sigma}{\partial \varepsilon} + \frac{\Delta}{\Delta t}G.
\end{aligned} \tag{37}$$

For the map $t \in [a,b] \mapsto \alpha(t) \equiv t^* \in R$, and $q_s^* \circ \alpha \circ \sigma(t) = q_s^*(\alpha(\sigma(t)))$

$$\begin{aligned}
&\frac{\partial t^*}{\partial \varepsilon}|_{\varepsilon=0} = \xi_0, \\
&\frac{\partial [(q_s^* \circ \alpha \circ \sigma)]}{\partial \varepsilon}|_{\varepsilon=0} = \xi_s^\sigma.
\end{aligned}$$

For $f(t) = t$, then $t^\Delta = 1$ by Remark 1, we can derive

$$\begin{aligned}
\frac{\partial}{\partial \varepsilon}\left(\frac{(q_s^* \circ \alpha)^\Delta}{\alpha^\Delta}\right)|_{\varepsilon=0} \ &= \frac{\partial}{\partial \varepsilon}\left(\frac{q_s^\Delta + \varepsilon \xi_s^\Delta + o(\varepsilon)}{t^\Delta + \varepsilon \xi_0^\Delta + o(\varepsilon)}\right) \\
&= \frac{\xi_s^\Delta (t^\Delta + \varepsilon \xi_0^\Delta) - \xi_0^\Delta (q_s^\Delta + \varepsilon \xi_s^\Delta)}{(t^\Delta + \varepsilon \xi_0^\Delta)^2}|_{\varepsilon=0} \\
&= \xi_s^\Delta - q_s^\Delta \xi_0^\Delta.
\end{aligned}$$

and

$$\frac{\partial t^{*^\Delta}}{\partial \varepsilon}|_{\varepsilon=0} = \xi_0^\Delta,$$

$$\frac{\partial \delta q_s^\sigma}{\partial \varepsilon}|_{\varepsilon=0} = \Delta q_s^\sigma - q_s^{\sigma\Delta}(\Delta t)^\sigma|_{\varepsilon=0} = \xi_s^\sigma - \xi_0^\sigma q_s^{\sigma\Delta}.$$

Considering Definition 3, we have

$$q_s^{\sigma\Delta} = (q_s^\Delta)^\sigma = q_s^\Delta + \mu(t) q_s^{\Delta\Delta}.$$

Then,

$$\xi_s^\sigma - \xi_0^\sigma q_s^{\Delta\sigma} = \xi_s^\sigma - \xi_0^\sigma (q_s^\Delta + \mu(t) q_s^{\Delta\Delta}). \tag{38}$$

All of the above equation, we obtain the Noether identity Eq. (33) of the relative motion systems with Chetaev type constraints on time scales.

Theorem 4 For Chetaev constraint the relative motion systems on time scales, if the infinitesimal transformations Eq. (30) satisfy the conditions Eq. (32), then the system Eq. (22) has conserved quantities of the form

$$I = \frac{\partial L}{\partial q_s^\Delta} + \left(L - \frac{\partial L}{\partial q_s^\Delta} q_s^\Delta - \frac{\partial L}{\partial t}\mu(t) \right) \xi_0^\sigma + G. \tag{39}$$

Proof. It proves that Eq. (39) is equivalent to the proof of $I = const$, takeing the derivative of I with respect to t, then

$$\begin{aligned} \frac{\Delta}{\Delta t} I &= \frac{\Delta}{\Delta t}\left[\frac{\partial L}{\partial q_s^\Delta} + \left(L - \frac{\partial L}{\partial q_s^\Delta} q_s^\Delta - \frac{\partial L}{\partial t}\mu(t) \right) \xi_0^\sigma + G \right] \\ &= \frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_s^\Delta}\xi_s^\sigma + \frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta + \frac{\Delta}{\Delta t}\left[L - \frac{\partial L}{\partial q_s^\Delta} q_s^\Delta - \frac{\partial L}{\partial t}\mu(t) \right] \xi_0^\sigma \\ &+ \left(L - \frac{\partial L}{\partial q_s^\Delta} q_s^\Delta - \frac{\partial L}{\partial t}\mu(t) \right) \xi_0^\sigma + \frac{\Delta}{\Delta t} G \end{aligned} \tag{40}$$

From Eq. (22) and (33), we can obtain

$$\begin{aligned} \frac{\Delta}{\Delta t} I &= \left(\frac{\partial L}{\partial q_s^\sigma} + (Q_s'' + Q_s^\omega + \Gamma_s + \Lambda_s) \right) \xi_s^\sigma + \frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta \\ &+ \left(\frac{\partial L}{\partial t} - (Q_s'' + Q_s^\omega + \Gamma_s + \Lambda_s) q_s^{\sigma\Delta} \right) \xi_0^\sigma + L\xi_0^\Delta \\ &- \frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta q_s^\Delta - \frac{\partial L}{\partial t}\mu(t)\xi_0^\Delta + \frac{\Delta}{\Delta t} G \\ &= \left(\frac{\partial L}{\partial q_s^\sigma} + (Q_s'' + Q_s^\omega + \Gamma_s + \Lambda_s) \right) \xi_s^\sigma \\ &+ \frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta + \left(\frac{\partial L}{\partial t} - (Q_s'' + Q_s^\omega + \Gamma_s + \Lambda_s) q_s^{\sigma\Delta} \right) (\xi_0^\sigma + \mu(t)\xi_0^\Delta) \\ &+ L\xi_0^\Delta - \frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta q_s^\Delta - \frac{\partial L}{\partial t}\mu(t)\xi_0^\Delta + \frac{\Delta}{\Delta t} G \\ &= \frac{\partial L}{\partial t}\xi_0 + \frac{\partial L}{\partial q_s^\sigma}\xi_s^\sigma + \frac{\partial L}{\partial q_s^\sigma}\xi_s^\sigma + L\xi_0^\Delta - \frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta q_s^\Delta + (Q_s'' \\ &+ Q_s^\omega + \Gamma_s + \Lambda_s)(\xi_s^\sigma - \xi_0^\sigma q_s^{\sigma\Delta}) + \frac{\Delta}{\Delta t} G \\ &= 0 \end{aligned} \tag{41}$$

According to the above proof, we can learn that Eq. (39) is called the Noether’s conserved quantities for the relative motion systems with Chetaev type constraints on time scales.

5 Examples

We first consider an example of the relative motion systems, the time scale is:

$$T = 2^n : n \in N \cup 0$$

Suppose lagrangian equation of the system is :

$$\begin{aligned} L &= \frac{1}{2}m\left((q_1^\Delta)^2 + (q_2^\Delta)^2 + (q_3^\Delta)^2 \right) + \frac{1}{2}m\omega^2\left((q_1^\sigma)^2 + (q_2^\sigma)^2 \right) \\ &- \frac{1}{2}k\left((q_1^\sigma)^2 + (q_2^\sigma)^2 + (q_3^\sigma)^2 \right). \end{aligned} \tag{42}$$

The generalized rotary inertia force and nonconservative force are respectively

$$Q_s'' = 0, Q_s^{\omega} = 0 (s = 1,2,3).$$

The generalized gyroscopic force are

$$\Gamma_1 = 2m\omega q_2^{\Delta}, \Gamma_2 = -2m\omega q_1^{\Delta}, \Gamma_3 = 0.$$

The Chetaev constraint is

$$f = q_2^{\Delta} - t q_1^{\Delta} = 0 \tag{43}$$

From Eq. (22), we can obtain

$$\begin{aligned} mq_1^{\Delta\Delta} &= -kq_1^{\sigma} + m\omega^2 q_1^{\sigma} + 2m\omega q_2^{\Delta} - \lambda t, \\ mq_2^{\Delta\Delta} &= -kq_2^{\sigma} + m\omega^2 q_2^{\sigma} + 2m\omega q_1^{\Delta} + \lambda, \\ mq_3^{\Delta\Delta} &= -kq_3^{\sigma}. \end{aligned} \tag{44}$$

Using Eq. (42) and Eq. (43), we have

$$\lambda = 2m\omega q_1^{\Delta} + \frac{mq_1^{\Delta} + (k - m\omega^2)(q_2^{\sigma} - q_1^{\sigma} t)}{1 + t^2}. \tag{45}$$

The differential equation of the relative motion systems

$$\begin{aligned} mq_1^{\Delta\Delta} &= -(k - m\omega^2)\tfrac{q_1^{\sigma} + tq_2^{\sigma}}{1+t^2} - \tfrac{mq_1^{\Delta} t}{1+t^2}, \\ mq_2^{\Delta\Delta} &= -(k - m\omega^2)\tfrac{q_1^{\sigma} t + t^2 q_2^{\sigma}}{1+t^2} + \tfrac{mq_1^{\Delta}}{1+t^2}, \\ mq_3^{\Delta\Delta} &= -kq_3^{\sigma}. \end{aligned} \tag{46}$$

From the Theorem 3, we obtain

$$\begin{aligned} &\xi_1^{\sigma}(m\omega^2 q_1^{\sigma} - kq_1^{\sigma}) + \xi_2^{\sigma}(m\omega^2 q_2^{\sigma} - kq_2^{\sigma}) - \xi_3^{\sigma} kq_3^{\sigma} + mq_1^{\Delta}(\xi_1^{\Delta} - \xi_0^{\Delta} q_1^{\Delta}) \\ &+ mq_2^{\Delta}(\xi_2^{\Delta} - \xi_0^{\Delta} q_2^{\Delta}) + mq_3^{\Delta}(\xi_1^{\Delta} - \xi_0^{\Delta} q_3^{\Delta}) + \left[\frac{1}{2} m \left((q_1^{\Delta})^2 + (q_2^{\Delta})^2 + (q_3^{\Delta})^2\right)\right. \\ &\left. + \frac{1}{2} m\omega^2 \left((q_1^{\sigma})^2 + (q_2^{\sigma})^2\right) - \frac{1}{2} k \left((q_1^{\sigma})^2 + (q_2^{\sigma})^2 + (q_3^{\sigma})^2\right)\right] \xi_0^{\Delta} \\ &+ \left[2m\omega q_2^{\Delta} - 2m\omega q_1^{\Delta} t - \frac{mq_1^{\Delta} + (k - m\omega^2)(q_2^{\sigma} - q_1^{\sigma} t)}{1+t^2} t\right] \times (\xi_1^{\sigma} - \xi_0^{\sigma} q_1^{\sigma\Delta}) \\ &+ \left[2m\omega q_1^{\Delta} - 2m\omega q_1^{\Delta} + \frac{mq_1^{\Delta} + (k - m\omega^2)(q_2^{\sigma} - q_1^{\sigma} t)}{1+t^2}\right] \times (\xi_2^{\sigma} - \xi_0^{\sigma} q_2^{\sigma\Delta}) \\ &= -\frac{\Delta}{\Delta t} G. \end{aligned} \tag{47}$$

Choosing the infinitesimal generators as:

$$\xi_0 = \xi_1 = \xi_2 = 0, \xi_3 = q_3^{\Delta}. \tag{48}$$

Using Eqs. (47-48), we have

$$G = \frac{k}{2} q_3^{\sigma} - \frac{1}{2} m (q_3^{\Delta})^2.$$

According to Theorem 4, the system has the Noether conserved quantity on time scale as

$$I = \frac{1}{2} m (q_3^{\Delta})^2 + \frac{1}{2} k (q_3^{\sigma})^2 = const. \tag{49}$$

6 Conclusion

In this paper, based on the theory of calculus on time scale and variational principle, we studied Noether symmetries and the conservation laws of relative motion systems on time scales. The results have shown significant approaches to seek conservation laws for these systems and provide a good method for solving the practical problems such asbiology, thermodynamics, engineering and so on.

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