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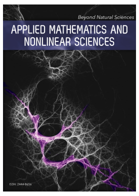

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On optimal system, exact solutions and conservation laws of the modified equal-width equation

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Abstract

In this paper we study the modified equal-width equation, which is used in handling simulation of a single dimensional wave propagation in nonlinear media with dispersion processes. Lie point symmetries of this equation are computed and used to construct an optimal system of one-dimensional subalgebras. Thereafter using an optimal system of one-dimensional subalgebras, symmetry reductions and new group-invariant solutions are presented. The solutions obtained are cnoidal and snoidal waves. Furthermore, conservation laws for the modified equal-width equation are derived by employing two different methods; the multiplier method and Noether approach.

Keywords: modified equal-width equation, Lie symmetries, optimal system of one-dimensional subalgebras, cnoidal and snoidal waves, conservation laws.
AMS 2010 codes: 35C07, 35L65.

1 Introduction

In this paper we study the third-order modified equal-width (MEW) equation

$$u_t + 3\alpha u^2 u_x - \beta u_{txx} = 0,\ \alpha \neq 0,\ \beta \neq 0, \tag{1}$$

where α and β are non-zero real parameters. Equation (1) is used in handling the simulation of a single dimensional wave propagation in nonlinear media with dispersion processes [1]. Some researchers have used different techniques and methods to construct travelling wave solutions of (1). Recently MEW equation (1) was

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investigated in [1], where the researchers employed extended simple equation method and also the $\exp(-\varphi(\xi))$ expansion method to generate travelling wave solutions of the equation. In [2], dynamical system technique for integer order was used and travelling wave solutions of the MEW equation were found, which comprised of solitary, periodic waves and also kink and anti-kink wave solutions. Homotopy perturbation method was applied to (1) and numerical solution of the MEW equation was obtained in [3].

In our study we use an entirely different approach to obtain new exact travelling wave solutions, namely cnoidal and snoidal wave solutions of MEW equation (1). Moreover, for the first time we derive conservation laws of the MEW equation by employing both the Noether approach as well as the multiplier approach.

2 Exact solutions of (1) constructed on optimal system

In this section, we first compute Lie point symmetries of (1) and then use them to construct an optimal system of one-dimensional subalgebras. Subsequently, we utilise this optimal system of one-dimensional subalgebras to obtain symmetry reductions and group-invariant solutions of (1) [4–8].

2.1 Lie point symmetries of (1)

The vector field

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{2}$$

where τ, ξ and η depend on t, x and u is a Lie point symmetry of equation (1) if

$$\mathrm{pr}^{(3)}X\Delta|_{\Delta=0} = 0, \tag{3}$$

where

$$\Delta \equiv u_t + 3\alpha u^2 u_x - \beta u_{txx}$$

and $\mathrm{pr}^{(3)}X$ is the third prolongation [6] of (2) defined as

$$\mathrm{pr}^{(3)}X = X + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tx}\frac{\partial}{\partial u_{tx}} + \zeta_{txx}\frac{\partial}{\partial u_{txx}}. \tag{4}$$

Here ζ_t, ζ_x, ζ_{tx} and ζ_{txx} are determined by

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),\\
\zeta_x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\
\zeta_{tx} &= D_x(\zeta_t) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi),\\
\zeta_{txx} &= D_x(\zeta_{tx}) - u_{ttx} D_x(\tau) - u_{txx} D_x(\xi),
\end{aligned} \tag{5}$$

where the total derivatives D_t and D_x are defined as

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots,\\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + \cdots.
\end{aligned} \tag{6}$$

Expanding (3) and splitting on derivatives of u yields an overdetermined system of linear homogeneous partial differential equations (PDEs). Solving these equations we obtain the values of τ, ξ and η, which lead to three Lie point symmetries of (1) given by

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.$$

The infinitesimal generator X_3 represents scaling symmetry whereas the one-parameter groups generated by X_1 and X_2 demonstrate time and space-invariance of the MEW equation.

2.2 Optimal system of one-dimensional subalgebras

We now calculate an optimal system of one-dimensional subalgebras by using Lie point symmetries of (1) obtained in the previous subsection. We employ the method given in [6]. We first construct the commutator table. Thereafter we compute adjoint representation using the Lie series

$$\mathrm{Ad}(\exp(\varepsilon X_i))X_j = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(\mathrm{ad}X_i)^n(X_j) = X_j - \varepsilon[X_i, X_j] + \frac{\varepsilon^2}{2!}[X_i, [X_i, X_j]] - \cdots,$$

where ε is a real number and $[X_i, X_j]$ denotes the commutator defined by

$$[X_i, X_j] = X_i X_j - X_j X_i.$$

The table of commutators of Lie point symmetries of equation (1) and adjoint representations of the symmetry group of (1) on its Lie algebra are presented in Table 1 and Table 2, respectively. Consequently, Table 1 and Table 2 are used to compute an optimal system of one-dimensional subalgebras for equation (1).

Table 1. Lie brackets for equation (1)

[,]	X_1	X_2	X_3
X_1	0	0	$2X_1$
X_2	0	0	0
X_3	$-2X_1$	0	0

Table 2. Adjoint representation of subalgebras

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$-2\varepsilon X_1 + X_3$
X_2	X_1	X_2	X_3
X_3	$e^{2\varepsilon}X_1$	X_2	X_3

Thus following [6] and utilising Tables 1 and 2 we can obtain an optimal system of one-dimensional subalgebras, which is given by $\{X_1 + cX_2, X_3 + aX_2\}$, where c and a are arbitrary constants.

2.3 Solutions and symmetry reductions

We now utilise the optimal system of one-dimensional subalgebras obtained above in the previous subsection and find group-invariant solutions and symmetry reductions for equation (1).

Consider the first operator $X_1 + cX_2$ of the optimal system. This operator has two invariants

$$\xi = x - ct \text{ and } U = u,$$

which give the group-invariant solution $U = U(\xi)$. Using ξ as our new independent variable, equation (1) is transformed into the nonlinear ordinary differential equation (ODE)

$$c\beta U'''(\xi) + 3\alpha U^2(\xi)U'(\xi) - cU'(\xi) = 0. \tag{7}$$

We now use the extended Jacobi elliptic function expansion method [9] to obtain travelling wave solutions of (1). We assume that solutions of the third-order nonlinear ODE (7) can be expressed in the form

$$U(\xi) = \sum_{i=-M}^{M} A_i H(\xi)^i, \tag{8}$$

where M is a positive integer obtained by the balancing procedure and A_i are constants to be determined. Here $H(\xi)$ satisfies the nonlinear first-order ODE

$$H'(\xi) = -\sqrt{(1 - H^2(\xi))(1 - \omega + \omega H^2(\xi))} \tag{9}$$

or

$$H'(\xi) = \sqrt{(1-H^2(\xi))(1-\omega H^2(\xi))}. \tag{10}$$

We recall that the Jacobi cosine-amplitude function

$$H(\xi) = \mathrm{cn}(\xi|\omega) \tag{11}$$

is a solution to (9), whereas the Jacobi sine-amplitude function

$$H(\xi) = \mathrm{sn}(\xi|\omega) \tag{12}$$

is a solution to (10). Here ω is a parameter such that $0 \leq \omega \leq 1$ [9, 10].

We note that when $\omega \to 1$, then $\mathrm{cn}(\xi|\omega) \to \mathrm{sech}(\xi)$ and $\mathrm{sn}(\xi|\omega) \to \tanh(\xi)$. Also, when $\omega \to 0$, then $\mathrm{cn}(\xi|\omega) \to \cos(\xi)$ and $\mathrm{sn}(\xi|\omega) \to \sin(\xi)$.

2.3.1 Cnoidal wave solutions

Considering the nonlinear ODE (7), the balancing procedure yields $M = 1$. Thus (8) takes the form

$$U(\xi) = A_{-1}H^{-1}(\xi) + A_0 + A_1H(\xi). \tag{13}$$

Substitution of U from (13) into (7) and utilising (9) we obtain

$$\begin{aligned}
&H(\xi)^4\beta cA_{-1} - H(\xi)^6\beta cA_1 + H(\xi)^4\beta cA_1 - 7H(\xi)^2\beta cA_{-1} - 12\beta c\omega A_{-1}\\
&+6\beta c\omega^2A_{-1} + 3H(\xi)^{10}\alpha\omega A_1{}^3 - 6H(\xi)^8\alpha\omega A_1{}^3 - H(\xi)^8c\omega A_1\\
&+6H(\xi)^7\alpha A_0A_1{}^2 + 3H(\xi)^6\alpha\omega A_1{}^3 - 3\alpha\omega A_{-1}{}^3 - 3H(\xi)^6\alpha A_1{}^3 + 3H(\xi)^8\alpha A_1{}^3\\
&-H(\xi)^6cA_1 + H(\xi)^4cA_1 + H(\xi)^4cA_{-1} - 3H(\xi)^2\alpha A_{-1}{}^3 - H(\xi)^2cA_{-1}\\
&-10H(\xi)^6\beta c\omega^2A_1 - 2H(\xi)^6\beta c\omega^2A_{-1} - 7H(\xi)^8\beta c\omega A_1 + 14H(\xi)^8\beta c\omega^2A_1\\
&-6H(\xi)^{10}\beta c\omega^2A_1 + 21H(\xi)^2\beta c\omega A_{-1} - 14H(\xi)^2\beta c\omega^2A_{-1} - 3H(\xi)^4\beta c\omega A_1\\
&-10H(\xi)^4\beta c\omega A_{-1} + 2H(\xi)^4\beta c\omega^2A_1 + 10H(\xi)^4\beta c\omega^2A_{-1} + 10H(\xi)^6\beta c\omega A_1\\
&+H(\xi)^6\beta c\omega A_{-1} + 3H(\xi)^8\alpha\omega A_0{}^2A_1 + 3H(\xi)^8\alpha\omega A_{-1}A_1{}^2 + 6H(\xi)^9\alpha\omega A_0A_1{}^2\\
&-6H(\xi)\alpha\omega A_{-1}{}^2A_0 - 3H(\xi)^2\alpha\omega A_{-1}A_0{}^2 - 3H(\xi)^2\alpha\omega A_{-1}{}^2A_1 + 12H(\xi)^3\alpha\omega A_{-1}{}^2A_0\\
&+3H(\xi)^4\alpha\omega A_0{}^2A_1 + 3H(\xi)^4\alpha\omega A_{-1}A_1{}^2 + 6H(\xi)^4\alpha\omega A_{-1}A_0{}^2 + 6H(\xi)^4\alpha\omega A_{-1}{}^2A_1\\
&+6H(\xi)^5\alpha\omega A_0A_1{}^2 - 6H(\xi)^5\alpha\omega A_{-1}{}^2A_0 - 6H(\xi)^6\alpha\omega A_0{}^2A_1 - 6H(\xi)^6\alpha\omega A_{-1}A_1{}^2\\
&-3H(\xi)^6\alpha\omega A_{-1}A_0{}^2 - 3H(\xi)^6\alpha\omega A_{-1}{}^2A_1 - 12H(\xi)^7\alpha\omega A_0A_1{}^2 + 6H(\xi)\alpha A_{-1}{}^2A_0\\
&+3H(\xi)^2\alpha A_{-1}A_0{}^2 + H(\xi)^2c\omega A_{-1} + 3H(\xi)^2\alpha A_{-1}{}^2A_1 + 6H(\xi)^2\alpha\omega A_{-1}{}^3\\
&-H(\xi)^4c\omega A_1 - 6H(\xi)^3\alpha A_{-1}{}^2A_0 - 2H(\xi)^4c\omega A_{-1} - 3H(\xi)^4\alpha A_{-1}A_1{}^2\\
&-3H(\xi)^4\alpha A_0{}^2A_1 - 3H(\xi)^4\alpha A_{-1}{}^2A_1 - 3H(\xi)^4\alpha A_{-1}A_0{}^2 + 2H(\xi)^6c\omega A_1\\
&-6H(\xi)^5\alpha A_0A_1{}^2 - 3H(\xi)^4\alpha\omega A_{-1}{}^3 + 3H(\xi)^6\alpha A_{-1}A_1{}^2 + 3H(\xi)^6\alpha A_0{}^2A_1\\
&+H(\xi)^6c\omega A_{-1} + 6\beta cA_{-1} + 3\alpha A_{-1}{}^3 = 0.
\end{aligned}$$

The above equation can be separated on like powers of $H(\xi)$ to obtain an overdetermined system of eleven algebraic equations

$$\begin{aligned}
&A_0A_1{}^2 = 0,\\
&A_{-1}{}^2A_0 - \omega A_{-1}{}^2A_0 = 0,\\
&2\omega A_{-1}{}^2A_0 - A_{-1}{}^2A_0 = 0,
\end{aligned}$$

$$
\begin{aligned}
&A_0{A_1}^2-2\omega A_0{A_1}^2=0,\\
&\alpha {A_1}^3-2\beta c\omega A_1=0,\\
&\omega A_0{A_1}^2-\omega {A_{-1}}^2A_0-A_0{A_1}^2=0,\\
&2\beta c\omega^2A_{-1}-\alpha\omega {A_{-1}}^3+\alpha {A_{-1}}^3-4\beta c\omega A_{-1}+2\beta cA_{-1}=0,\\
&3\alpha\omega A_{-1}{A_1}^2+3\alpha\omega {A_0}^2A_1-6\alpha\omega {A_1}^3+14\beta c\omega^2A_1+3\alpha {A_1}^3-7\beta c\omega A_1-c\omega A_1=0,\\
&6\alpha\omega {A_{-1}}^3-3\alpha\omega {A_{-1}}^2A_1-3\alpha\omega A_{-1}{A_0}^2-14\beta c\omega^2A_{-1}-3\alpha {A_{-1}}^3+3\alpha {A_{-1}}^2A_1\\
&+3\alpha A_{-1}{A_0}^2+21\beta c\omega A_{-1}-7\beta cA_{-1}+c\omega A_{-1}-cA_{-1}=0,\\
&3\alpha\omega {A_1}^3-3\alpha\omega {A_{-1}}^2A_1-3\alpha\omega A_{-1}{A_0}^2-6\alpha\omega A_{-1}{A_1}^2-6\alpha\omega {A_0}^2A_1-2\beta c\omega^2A_{-1}\\
&+3\alpha A_{-1}{A_1}^2-10\beta c\omega^2A_1+3\alpha {A_0}^2A_1-3\alpha {A_1}^3+\beta c\omega A_{-1}+10\beta c\omega A_1-\beta cA_1+c\omega A_{-1}\\
&+2c\omega A_1-cA_1=0,\\
&6\alpha\omega {A_{-1}}^2A_1-3\alpha\omega {A_{-1}}^3+6\alpha\omega A_{-1}{A_0}^2+3\alpha\omega A_{-1}{A_1}^2+3\alpha\omega {A_0}^2A_1+10\beta c\omega^2A_{-1}\\
&+2\beta c\omega^2A_1-3\alpha {A_{-1}}^2A_1-3\alpha A_{-1}{A_0}^2-3\alpha A_{-1}{A_1}^2-3\alpha {A_0}^2A_1-10\beta c\omega A_{-1}\\
&-3\beta c\omega A_1+\beta cA_{-1}+\beta cA_1-2c\omega A_{-1}-c\omega A_1+cA_{-1}+cA_1=0.
\end{aligned}
$$

Solving the above system of equations we obtain

$$\omega=\frac{8\beta+3k-1}{16\beta},\ A_0=0,\ A_1=\pm\sqrt{\frac{c(3k+8\beta-1)}{8\alpha}},\ A_{-1}=-\frac{3\beta\pm k}{\beta+1}A_1$$

with $k=\sqrt{8\beta^2+1}$.

Thus reverting to the original variables the solutions of (1) are

$$u(t,x)=\pm\sqrt{\frac{c(3k+8\beta-1)}{8\alpha}}\left\{\mathrm{cn}\,(\xi\,|\omega)-\left(\frac{3\beta\pm k}{\beta+1}\right)\mathrm{nc}\,(\xi\,|\omega)\right\},\tag{14}$$

where $\mathrm{nc}=1/\mathrm{cn}$.

2.3.2 Snoidal wave solutions

We now obtain snoidal wave solutions for equation (1). Here again $M=1$. Substituting the value of U from (13) into (7) and making use of (10) we obtain

$$
\begin{aligned}
&3H(\xi)^6\alpha {A_1}^3-H(\xi)^4cA_1+H(\xi)^2cA_{-1}+3H(\xi)^2\alpha {A_{-1}}^3+3H(\xi)^4\alpha {A_{-1}}^2A_1\\
&+3H(\xi)^4\alpha A_{-1}{A_0}^2+H(\xi)^6\beta cA_1-H(\xi)^4\beta cA_{-1}-H(\xi)^4\beta cA_1+7H(\xi)^2\beta cA_{-1}\\
&+3H(\xi)^4\alpha A_{-1}{A_1}^2+3H(\xi)^4\alpha {A_0}^2A_1+3H(\xi)^{10}\alpha\omega {A_1}^3-3H(\xi)^8\alpha\omega {A_1}^3-H(\xi)^8c\omega A_1\\
&-6H(\xi)^7\alpha A_0{A_1}^2-3H(\xi)^6\alpha A_{-1}{A_1}^2-3H(\xi)^6\alpha {A_0}^2A_1+H(\xi)^6c\omega A_{-1}+H(\xi)^6c\omega A_1\\
&+6H(\xi)^5\alpha A_0{A_1}^2-3H(\xi)^4\alpha\omega {A_{-1}}^3-3\alpha {A_{-1}}^3-6H(\xi)^7\alpha\omega A_0{A_1}^2-3H(\xi)^6\alpha\omega {A_{-1}}^2A_1\\
&-3H(\xi)^6\alpha\omega A_{-1}{A_0}^2-3H(\xi)^6\alpha\omega A_{-1}{A_1}^2-3H(\xi)^6\alpha\omega {A_0}^2A_1-6H(\xi)^5\alpha\omega {A_{-1}}^2A_0\\
&+3H(\xi)^4\alpha\omega {A_{-1}}^2A_1+3H(\xi)^4\alpha\omega A_{-1}{A_0}^2+6H(\xi)^9\alpha\omega A_0{A_1}^2+3H(\xi)^8\alpha\omega A_{-1}{A_1}^2\\
&+3H(\xi)^8\alpha\omega {A_0}^2A_1-3H(\xi)^8\alpha {A_1}^3+H(\xi)^6cA_1-H(\xi)^4cA_{-1}-6\beta cA_{-1}\\
&+7H(\xi)^2\beta c\omega A_{-1}-H(\xi)^4\beta c\omega A_1-8H(\xi)^4\beta c\omega A_{-1}-H(\xi)^4\beta c\omega^2A_{-1}+8H(\xi)^6\beta c\omega A_1\\
&+H(\xi)^6\beta c\omega A_{-1}+H(\xi)^6\beta c\omega^2A_1+H(\xi)^6\beta c\omega^2A_{-1}-7H(\xi)^8\beta c\omega A_1-7H(\xi)^8\beta c\omega^2A_1\\
&-3H(\xi)^2\alpha A_{-1}{A_0}^2-6H(\xi)\alpha {A_{-1}}^2A_0+3H(\xi)^2\alpha\omega {A_{-1}}^3-3H(\xi)^2\alpha {A_{-1}}^2A_1\\
&-H(\xi)^4c\omega A_{-1}+6H(\xi)^3\alpha {A_{-1}}^2A_0+6H(\xi)^{10}\beta c\omega^2A_1+6H(\xi)^3\alpha\omega {A_{-1}}^2A_0=0.
\end{aligned}
$$

Splitting on powers of $H(\xi)$ yields the following overdetermined system of algebraic equations:

$$
\begin{aligned}
&A_{-1}{}^2 A_0 = 0,\\
&A_0 A_1{}^2 = 0,\\
&\alpha A_{-1}{}^3 + 2\beta c A_{-1} = 0,\\
&\omega A_{-1}{}^2 A_0 + A_{-1}{}^2 A_0 = 0,\\
&\omega A_0 A_1{}^2 + A_0 A_1{}^2 = 0,\\
&\alpha A_1{}^3 + 2\beta c\omega A_1 = 0,\\
&\omega A_{-1}{}^2 A_0 - A_0 A_1{}^2 = 0,\\
&3\alpha\omega A_{-1} A_1{}^2 + 3\alpha\omega A_0{}^2 A_1 - 3\alpha\omega A_1{}^3 - 7\beta c\omega^2 A_1 - 3\alpha A_1{}^3 - 7\beta c\omega A_1 - c\omega A_1 = 0,\\
&3\alpha\omega A_{-1}{}^3 + 3\alpha A_{-1}{}^3 - 3\alpha A_{-1}{}^2 A_1 - 3\alpha A_{-1} A_0{}^2 + 7\beta c\omega A_{-1} + 7\beta c A_{-1} + c A_{-1} = 0,\\
&\beta c\omega^2 A_{-1} - 3\alpha\omega A_{-1}{}^2 A_1 - 3\alpha\omega A_{-1} A_0{}^2 - 3\alpha\omega A_{-1} A_1{}^2 - 3\alpha\omega A_0{}^2 A_1 + \beta c\omega^2 A_1\\
&\quad + 3\alpha A_1{}^3 - 3\alpha A_{-1} A_1{}^2 - 3\alpha A_0{}^2 A_1 + \beta c\omega A_{-1} + 8\beta c\omega A_1 + \beta c A_1 + c\omega A_{-1} + c\omega A_1 + c A_1 = 0,\\
&3\alpha\omega A_{-1}{}^2 A_1 - 3\alpha\omega A_{-1}{}^3 + 3\alpha\omega A_{-1} A_0{}^2 - \beta c\omega^2 A_{-1} + 3\alpha A_{-1}{}^2 A_1 + 3\alpha A_{-1} A_0{}^2 + 3\alpha A_{-1} A_1{}^2\\
&\quad + 3\alpha A_0{}^2 A_1 - 8\beta c\omega A_{-1} - \beta c\omega A_1 - \beta c A_{-1} - \beta c A_1 - c\omega A_{-1} - c A_{-1} - c A_1 = 0.
\end{aligned}
$$

Solving the above system of equations we get

$$\beta = -\frac{1}{1+\omega},\ A_{-1} = A_0 = 0,\ A_1 = \pm\sqrt{\frac{2c(\beta+1)}{\alpha}}.$$

Reverting to original variables we obtain solutions of (1) as

$$u(t,x) = \pm\sqrt{\frac{2c(\beta+1)}{\alpha}}\,\mathrm{sn}\,(\xi\,|\omega)\,. \tag{15}$$

We now consider the second operator $X_3 + aX_2$ of the optimal system. This symmetry operator yields two invariants $J_1 = e^x t^{-a/2}$ and $J_2 = ut^{1/2}$. Thus $J_2 = f(J_1)$ provides a group-invariant solution to (1). That is

$$u = t^{-1/2} f(e^x t^{-a/2}).$$

Substituting the above value of u in (1), we obtain the third-order nonlinear ODE

$$a\beta z^3 f'''(z) + \beta\,(3a+1)\,z^2 f''(z) + (a\beta - a + \beta)zf'(z) + 6\alpha z f(z)^2 f'(z) - f(z) = 0,$$

where $z = e^x t^{-a/2}$.

3 Conservation laws of the modified equal width equation (1)

In the section we derive conservation laws for (1) by employing two different techniques, namely the multiplier method and Noether approach.

Conservation laws have several important uses in the study of partial differential equations, especially for determining conserved quantities and constants of motion, detecting integrability and linearizations, finding potentials and nonlocally-related systems, as well as checking the accuracy of numerical solution methods [11–17].

3.1 Conservation laws of (1) using multiplier approach

We look for zeroth-order multiplier $\Lambda = \Lambda(t,x,u)$. Thus, the determining equation for this multiplier is stated as

$$\frac{\delta}{\delta u}\Big\{\Lambda(t,x,u)\Big(u_t + 3\alpha u^2 u_x - \beta u_{txx}\Big)\Big\} = 0, \tag{16}$$

where $\delta/\delta u$ is the Euler-Lagrange operator defined as

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} - D_t D_x^2 \frac{\partial}{\partial u_{txx}}$$

and the total derivatives D_t and D_x are defined as in (6). The above equation yields

$$u_t\Lambda_u + 3\alpha u^2 u_x \Lambda_u - \beta u_{txx}\Lambda_u + 6\alpha u u_x \Lambda - D_t(\Lambda) - D_x(3\alpha u^2\Lambda) + \beta D_x^2 D_t(\Lambda) = 0,$$

which on expanding gives

$$\begin{aligned}
&u_t\Lambda_u + 3\alpha u^2 u_x\Lambda_u - \beta u_{xxt}\Lambda_u + 6\alpha u u_x\Lambda - \Lambda_t - u_t\Lambda_u - 3\alpha u^2\Lambda_x - 6\alpha u u_x\Lambda - 3\alpha u^2 u_x\Lambda_u + \beta\Lambda_{txx}\\
&+ \beta u_t\Lambda_{uxx} + \beta u_x\Lambda_{tux} + \beta u_t u_x\Lambda_{uux} + \beta u_{tx}\Lambda_{ux} + \beta u_x\Lambda_{tux} + \beta u_t u_x\Lambda_{uux} + \beta u_x^2\Lambda_{tuu} + \beta u_t u_x^2\Lambda_{uuu}\\
&+ \beta u_{tx}u_x\Lambda_{uu} + \beta u_{tx}\Lambda_{ux} + \beta u_{tx}u_x\Lambda_{uu} + \beta u_{xx}\Lambda_{tu} + \beta u_{xx}u_t\Lambda_{uu} + \beta u_{xxt}\Lambda_u = 0.
\end{aligned}$$

Splitting the above equation on derivatives of u, we obtain

$$\Lambda_{uu} = 0,\ \Lambda_{ux} = 0,\ \Lambda_{tu} = 0,\ \beta\Lambda_{txx} - 3\alpha u^2\Lambda_x - \Lambda_t = 0.$$

By solving the above equations we get two multipliers given by

$$\Lambda_1(t,x,u) = u$$

and

$$\Lambda_2(t,x,u) = 1.$$

Corresponding to these two multipliers, we obtain the following two conservation laws:

$$\begin{aligned}
T_1^t &= \frac{1}{2}u^2 + \frac{1}{2}\beta u_x^2,\\
T_1^x &= \frac{3}{4}\alpha u^4 - \beta u u_{tx}
\end{aligned}$$

and

$$\begin{aligned}
T_2^t &= u,\\
T_2^x &= \alpha u^3 - \beta u_{tx}.
\end{aligned}$$

3.2 Conservation laws of (1) using Noether's theorem

In this subsection we derive conservation laws for the modified equal-width equation (1) using Noether's theorem [18, 19]. This equation as it is does not have a Lagrangian. In order to apply Noether's theorem we transform equation (1) to a fourth-order equation which will have a Lagrangian. Thus using the transformation $u = V_x$, equation (1) becomes

$$V_{tx} + 3\alpha V_x^2 V_{xx} - \beta V_{txxx} = 0. \tag{17}$$

It can readily be verified that a Lagrangian for equation (17) is given by

$$\mathscr{L} = -\frac{1}{2}V_x V_t - \frac{1}{4}\alpha V_x^4 - \frac{1}{2}\beta V_{xx}V_{tx} \tag{18}$$

because $\delta\mathscr{L}/\delta V = 0$ on (17). Here $\delta/\delta V$ is the Euler-Lagrange operator defined as

$$\frac{\delta}{\delta V} = \frac{\partial}{\partial V} - D_t\frac{\partial}{\partial V_t} - D_x\frac{\partial}{\partial V_x} + D_x^2\frac{\partial}{\partial V_{xx}} + D_tD_x\frac{\partial}{\partial V_{tx}}.$$

Consider the vector field

$$X = \tau(t,x,V)\frac{\partial}{\partial t} + \xi(t,x,V)\frac{\partial}{\partial x} + \eta(t,x,V)\frac{\partial}{\partial V}, \tag{19}$$

where τ, ξ and η depend on t, x and V. To determine Noether point symmetries X of (17) we insert the value of $\mathscr{L}$ from (18) in the determining equation

$$\mathrm{pr}^{[2]}X(\mathscr{L}) + \mathscr{L}\left[D_t(\tau) + D_x(\xi)\right] = D_t\left(B^t\right) + D_x\left(B^x\right), \tag{20}$$

where $B^t = B^t(t,x,V)$ and $B^x = B^x(t,x,V)$ are gauge terms and $\mathrm{pr}^{[2]}X$ is the second prolongation of X defined as

$$\mathrm{pr}^{[2]}X = X + \zeta_t\frac{\partial}{\partial V_t} + \zeta_x\frac{\partial}{\partial V_x} + \zeta_{xx}\frac{\partial}{\partial V_{xx}} + \zeta_{tx}\frac{\partial}{\partial V_{tx}} \tag{21}$$

with ζ_t, ζ_x, ζ_{xx} and ζ_{tx} as defined in (5). Expansion of equation (20) and separating with respect to derivatives of V yields an overdetermined system of linear PDEs. Thereafter solving these PDEs we obtain the following Noether point symmetries together with their gauge functions:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\ B^t = 0,\ B^x = 0,\\
X_2 &= \frac{\partial}{\partial x},\ B^t = 0,\ B^x = 0,\\
X_f &= f(t)\frac{\partial}{\partial V},\ B^t = 0,\ B^x = -\frac{1}{2}f'(t)V.
\end{aligned}$$

Next, we use the above results to compute conserved vectors of the fourth-order equation (17). Using formulae for the conserved vector (T^t, T^x) [20]

$$F^k = \mathscr{L}\tau^k + (\xi^\alpha - \psi^\alpha_{x^j}\tau^j)\left(\frac{\partial\mathscr{L}}{\partial\psi^\alpha_{x^k}} - \sum_{l=1}^{k}D_{x^l}\left(\frac{\partial\mathscr{L}}{\partial\psi^\alpha_{x^lx^k}}\right)\right) + \sum_{l=k}^{n}(\eta^\alpha_l - \psi^\alpha_{x^lx^j}\tau^j)\frac{\partial\mathscr{L}}{\partial\psi^\alpha_{x^kx^l}} - f^k$$

we obtain three conserved vectors associated with three Noether point symmetries X_1, X_2 and X_f. Then reverting to the original variable u, we have

$$\begin{aligned}
T_1^t &= -\frac{1}{4}\alpha u^4 - \frac{1}{2}\beta u_xu_t - \frac{1}{2}\beta u_{xx}\int u_t dx,\\
T_1^x &= \frac{1}{2}\left(\int u_t dx\right)^2 + \alpha u^3\int u_t dx - \frac{1}{2}\beta u_{xt}\int u_t dx + \frac{1}{2}\beta u_t^2 + \frac{1}{2}\beta u_x\int u_{tt} dx;
\end{aligned}$$

$$\begin{aligned}
T_2^t &= \frac{1}{2}u^2 - \frac{1}{2}\beta uu_{xx},\\
T_2^x &= \frac{3}{4}\alpha u^4 - \frac{1}{2}\beta uu_{xt} + \frac{1}{2}\beta u_xu_t;
\end{aligned}$$

$$\begin{aligned}
T_f^t &= -\frac{1}{2}f(t)u + \frac{1}{2}\beta f(t)u_{xx},\\
T_f^x &= -\frac{1}{2}f(t)\int u_t dx - \alpha f(t)u^3 + \frac{1}{2}\beta f(t)u_{xt} - \frac{1}{2}\beta f'(t)u_x + \frac{1}{2}f'(t)\int u dx.
\end{aligned}$$

Remark: It should be noted that due to the presence of arbitrary function $f(t)$ we have infinitely many nonlocal conservation laws.

4 Conclusions

In this paper we studied the modified equal-width equation (1). For the first time, Lie point symmetries of (1) were computed and used to construct an optimal system of one-dimensional subalgebras. Thereafter utilising this optimal system of one-dimensional subalgebras, symmetry reductions and new group-invariant solutions of (1) were presented. The solutions obtained were cnoidal and snoidal waves. Again for the first time, we computed the conservation laws for (1) by employing two different methods; the multiplier method and Noether approach.

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