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Perfect phase-coded pulse trains generated by Talbot effect

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Abstract

A perfect phase sequence is a finite and ordered set of constant-amplitude complex numbers whose periodic autocorrelation vanishes at any non-zero time shift. They find multiple applications in science an engineering as phase-coded waveforms, where the sequence defines the relative phases within a burst of electromagnetic or acoustic pulses. We show how a physical propagation effect, the so-called fractional Talbot phenomenon, can be used to generate pulse trains coded according to these sequences. The mathematical description of this effect is first reviewed and extended, showing its close relationship with Gauss perfect phase sequences. It is subsequently shown how it leads to a construction of Popović's Generalized Chirp-Like (GCL) sequences. Essentially, a set of seed pulses with prescribed amplitude and phase levels, cyclically feeds a linear and dispersive medium. At particular values of the propagation length, multiple pulse-to-pulse interference induced by dispersion passively creates the sought-for pulse trains composed of GCL sequences, with the additional property that its repetition rate has been increased with respect to the seed pulses. This observation constitutes a novel representation of GCL sequences as the result of dispersive propagation of a seed sequence, and a new route for the practical implementation of perfect phase-coded pulse waveforms using Talbot effect.

Keywords and phrases: Perfect sequences; Phase sequences; CAZAC sequences; Talbot effect
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1 Introduction

A sequence of L complex numbers $\boldsymbol{x} = (x_0 x_1 \ldots x_{L-1})$ is *perfect* or *zero-autocorrelation* (ZAC) if its cyclic or periodic autocorrelation $R_{\boldsymbol{x}}(n)$ vanishes for any non-zero delay shift [1]. Formally,

$$R_{\boldsymbol{x}}(n) = \sum_{k=0}^{L-1} \bar{x}_k x_{k+n\,(\mathrm{mod}\ L)} = E_{\boldsymbol{x}}\, \delta_{n,0\ (\mathrm{mod}\ L)} \tag{1}$$

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with $E_{\boldsymbol{x}} = \sum_{k=0}^{L-1} |x_k|^2$ the energy of the sequence and $\delta_{a,b}$ the Kronecker's delta. When the sequence is composed of phases, $|x_k| = 1$, it is said *unimodular* or of *constant amplitude* (CA). Accordingly, phase sequences which are also perfect are said CAZAC sequences.

Their basic characterization follows from the (unitary) Discrete Fourier Transform (DFT) of the autocorrelation,

$$\frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} R_{\boldsymbol{x}}(n) e^{-j2\pi nm/L} = \sqrt{L}\, |X_m|^2, \tag{2}$$

with $m = 0, \ldots, L-1$, $\boldsymbol{X} = \mathrm{DFT}(\boldsymbol{x}) = (X_0 X_1 \ldots X_{L-1})$ the DFT of sequence $\boldsymbol{x}$, and $j = \sqrt{-1}$. A sequence $\boldsymbol{x}$ is thus perfect or ZAC iff its DFT $\boldsymbol{X}$ is CA, and it is CAZAC iff $\boldsymbol{X}$ is also CAZAC. Therefore, CAZAC sequences always come in a DFT pair of CA sequences, $\boldsymbol{x} \xrightarrow{\mathrm{DFT}} \boldsymbol{X}$, and for this reason they are also referred to as *biunimodular*.

Transformation matrices provide a second characterization of CAZAC sequences. Given a sequence $\boldsymbol{x}$ of length L, we define an associated normalized circulant matrix as:

$$\mathscr{T}_{\boldsymbol{x}} = \frac{1}{\sqrt{L}} \begin{pmatrix} x_0 & x_1 & \cdots & x_{L-1} \\ x_{L-1} & x_0 & \cdots & x_{L-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_0 \end{pmatrix} \tag{3}$$

Then, it follows from the general diagonalization of circulant matrices in terms of $\boldsymbol{X}$ that $\boldsymbol{x}$ is ZAC iff $\mathscr{T}_{\boldsymbol{x}}$ is a unitary matrix, and it is CAZAC iff $\mathscr{T}_{\boldsymbol{x}}$ is a circulant complex Hadamard matrix, i. e., unitary, circulant, and with entries of constant amplitude.

These sequences have found practical application in several contexts. To name a few, the unitary transformation (3) can be employed to construct equalizers or multiplexing schemes where the power of each of the L input channels is smeared among the L output channels [2,3]. ZAC or CAZAC sequences may be used to define coded waveforms, so that the perfect autocorrelation (1) allows for its decorrelation in noisy environments. This fact has been extensively employed in pulse-compression radar, and in frame synchronization, channel equalization and estimation, and cell search in wireless communication systems [4–6]. Finally, property (2) permits the design of coded waveforms with uniform spectrum, with application as direct-sequence spread-spectrum codes or acoustic diffusers [7].

In these applications, the generation of coded waveforms typically uses a two-step method, where the sequence is first constructed and stored, and then transferred to the domain of application by use of a suitable modulation device. In this paper, we describe how these coded waveforms can be generated passively by a physical propagation effect, the Talbot self-imaging phenomenon [8]. First, it is shown the close relationship between Talbot effect and the simplest example of perfect phase sequences, namely Gauss sequences of quadratic phase. Using this background material, pulse sequences with phases following Popović's Generalized Chirp-Like (GCL) sequences [9–11] are constructed from trains coded with a restricted number of phase and amplitude levels, and subsequent propagation in a linear, time-invariant medium with quadratic phase transfer function. As is shown in Section 2, Popović's GCL sequences can be understood as multiparameter generalizations of Gauss sequences for specific values of the sequence length. In Sections 3 and 4 we review the physical basis of the Talbot phenomenon together with its mathematical description. The main result is presented in Section 5, and we end in Section 6 with our conclusions.

2 Gauss and GCL perfect phase sequences

The simplest examples of CAZAC sequences are Gauss (or Wiener [1]) sequences, which can be defined for any sequence length L as follows. Let us first denote by $\mathbb{Z}_{2L}^{\times} = \{1, 2, \ldots, 2L-1\}$ the non-zero residues mod $2L$.

Definition 1. Given any $P \in \mathbb{Z}_{2L}^{\times}$, coprime and of opposite parity to that of L, a Gauss sequence $\boldsymbol{g}$ of length L is the quadratic phase sequence with entries:

$$g_k = \exp\left(j\pi \frac{P}{L} k^2\right) \qquad k = 0,\ldots,L-1. \tag{4}$$

According to this definition, when L is even P must be odd and coprime with L, and when L is odd, P must be of the form $P = 2\tilde{P}$ with $\tilde{P}$ coprime with L. The resulting twofold expression constitutes the standard definition of Gauss sequences [1, 12], and corresponds to the Zadoff-Chu sequence [13, 14] for even length, or to the Ipatov sequence [15] for odd length.

A number of transformations preserve the perfect phase property of sequences of given length [16]: (a) projectivity or multiplication by a constant phase, $x_k \to x_k e^{j\phi}$; (b) complex conjugation, $x_k \to \bar{x}_k$; (c) linear phase shifts $x_k \to x_k \xi^{km}$, with m integer and ξ a primitive root of unity of order L; (c) cyclic shifts of the running index, $x_k \to x_{k+a \pmod{L}}$ for arbitrary integer a, and (d) modular dilation, $x_k \to x_{Dk \pmod{L}}$ with D and L coprime. Using these transformations, for instance, the range of definition of integer P in (4) can be restricted to $0 < P < L$, since the map $P \to P' = 2L - P$ leads to the complex conjugated sequence, $\boldsymbol{g} \to \bar{\boldsymbol{g}}$. Other forms of Gauss sequences obtained by cyclic and linear phase shifts are the Zadoff-Chu sequence for odd length, the P3 and P4 codes [17], and the Golomb sequence [18].

As for the classification of perfect phase sequences, only partial results are known. If the length L is prime, there only exists a finite number of CAZAC sequences [16, 19], but the exact number and construction is only known for the lowest integers [12]. When the sequence length L is not proportional to the square of an integer, it is believed that the number of CAZAC sequences is also finite [20, 21]. In turn, when L is of the form $L = N^2M$ with M and N arbitrary positive integers, two alternative constructions provide examples of sequences indexed by a set of N phases, and so the number of perfect phase sequences is (uncountably) infinite [9, 20]. In particular, Popović's construction goes as follows.

Theorem 1. [9] *Given any Gauss sequence $\boldsymbol{g}$ of length $L = N^2M$, consider any phase sequence $\boldsymbol{\sigma}$ of length N, here referred to as the seed sequence. Let us denote by $\boldsymbol{p_\sigma}$ its periodic extension of length $L = N^2M$: $(p_\sigma)_k = \sigma_{k \pmod{N}}$ for $k = 1,\ldots,N^2M-1$. Then, the pointwise multiplication $\boldsymbol{b} = \boldsymbol{g} \cdot \boldsymbol{p_\sigma}$, with entries:*

$$b_k = g_k \cdot (p_\sigma)_k = g_k \cdot \sigma_{k \pmod{N}} \qquad k = 0,\ldots,N^2M-1, \tag{5}$$

constitutes a CAZAC or perfect phase sequence, referred to as a GCL sequence.

This simple construction is sufficiently broad to accommodate other known families of CAZAC sequences as particular cases of the seed sequence [10, 11]. The Gauss sequences, for instance, correspond to an all-ones seed.

3 The Talbot effect

In 1836, Henry F. Talbot reported on the observation of what is now called the Talbot or periodic self-imaging phenomenon [22]. He described that, at certain distances after an amplitude diffraction grating illuminated by a pencil of sunlight, he could observe images of the diffraction grating by use of an magnifying lens, even if the grating was out of the focus of the lens. Almost half a century later, Lord Rayleigh repeated the experiment with quasi-monochromatic light, and explained this effect using electromagnetic theory [23]. In modern language, monochromatic light propagation in the paraxial region, i. e., close to the propagation axis, can be described by an optical transfer function which is of quadratic phase,

$$H(u) \sim \exp\left(-j\pi\lambda z u^2\right),$$

where λ is the illumination wavelength, z the propagation distance, and u the spatial frequency. Since a periodic object with spatial period a has its Fourier content concentrated in a comb of spatial frequencies $u_m = m/a$, with m integer, at propagation distances multiples of $z = 2a^2/\lambda$ all the Fourier components are again in phase after propagation, and the original grating is reproduced in the Fresnel region. In addition, at distances $z = (2k+1)a^2/\lambda$ for integer k the grating is reproduced with a half-interval lateral shift. Later on, the effect was generalized to fractional values of these distances [24], and Montgomery described the most general objects that can be self-imagined in the paraxial region, which are not necessarily periodic [25].

Restricting to the periodic case, the basic elements of the Talbot phenomenon are therefore the peridiocity of the wavefront, its coherent character, and the existence of a propagation medium whose frequency response is of quadratic phase, at least in the comb of spatial frequencies. As a simple wave propagation effect, Talbot-like phenomena have been described and applied in a myriad of scenarios, ranging from atom optics to acoustics [26]. Within linear optics, it has been demonstrated in both spatial and temporal domains, together with its Fourier duals. In the temporal domain, dispersion and periodic temporal modulation play the analogue roles to paraxial propagation and diffraction gratings. We use this formalism to present the fractional generalization of the effect.

Let us consider a modulated optical wave described by an electric field in complex form given by $\mathscr{E}(t) = E(t)\exp(j\omega_0 t)$, with ω_0 the angular frequency of the optical carrier and $E(t)$ its optical complex envelope. The envelope is assumed periodic with period T, and represented as the repetition of a certain period $w(t)$:

$$E(t) = \sum_{n=-\infty}^{+\infty} w(t - nT). \tag{6}$$

Field $\mathscr{E}(t)$ feeds a delay line with lowest-order dispersion. The optical transfer function, assuming constant or negligible attenuation, is $H_{opt}(\omega) = \exp(-j\Phi(\omega))$. If the input radiation field is narrowband, the phase can be expanded as:

$$\Phi(\omega) = \Phi(\omega_0) + \tau_g\,(\omega - \omega_0) + \frac{1}{2}\phi\,(\omega - \omega_0)^2,$$

with τ_g the group delay at the carrier frequency and ϕ the lowest-order dispersion coefficient. Neglecting the constant phase and the linear frequency term, which simply accounts for a constant propagation delay, the transfer function acting over the envelope is again of quadratic phase, $H(\omega) = \exp(-j\phi\,\omega^2/2)$.

In the temporal domain, the fractional generalization of the Talbot effect manifests itself at certain values of dispersion ϕ characterized by the period T and two positive and mutually coprime integers p and q, given by:

$$2\pi|\phi| = \frac{p}{q}T^2. \tag{7}$$

At these values of dispersion the optical envelope consists of a weighted and coherent superposition of shifted replicas of the waveform $w(t)$.

Theorem 2. [27–31] *After dispersive propagation at values of dispersion (7), the output optical envelope is:*

$$E'(t) = \sum_{k=-\infty}^{+\infty} \delta\left(t - kT - e_{pq}\frac{T}{2}\right) * \frac{1}{\sqrt{q}}\sum_{n=0}^{q-1} e^{j\sigma_\phi \xi_n} w\left(t - n\frac{T}{q}\right), \tag{8}$$

where $*$ stands for convolution, e_x represents the parity of integer x, so that $e_x = 0$ when x is even and $e_x = 1$ when x is odd, $\sigma_\phi = \pm 1$ is the sign of dispersion ϕ, $\delta(t)$ is Dirac's delta distribution, and the phases, which are periodic with period q, are given by a quadratic Gauss sum:

$$e^{j\xi_n} = e^{j\xi_{n+q}} = \frac{1}{\sqrt{q}}\sum_{m=0}^{q-1}(-1)^{pqm} e^{-j\pi\frac{p}{q}m^2} e^{j2\pi\frac{nm}{q}} = \frac{1}{\sqrt{q}}\sum_{m=0}^{q-1} e^{-j\pi(p+qe_{pq})m^2/q} e^{j2\pi\frac{nm}{q}}. \tag{9}$$

In the last expression of (9), we have used that $(-1)^{pqm} = \exp(-j\pi e_{pq} m^2)$. According to (8), the output periodic train is shifted by half a period when the product $p \cdot q$ is odd. When $q = 1$, the initial periodic structure is reproduced at the output. The resulting wave distribution is referred to as an *integer* Talbot image of order or index p of the input train. In general, trains with $q > 1$ are referred to as *fractional* Talbot images of order p/q. The periodicity of the Talbot weighting phases (9) allows for its identification with a sequence $\boldsymbol{t}$ of length q, referred to as the Talbot sequence. This also permits the restriction of p to values within the range $0 < p < 2q$, or within the non-zero residues mod $2q$, $\mathbb{Z}_{2q}^{\times}$. The following results lead to the solution of the Gauss sum as a DFT pair.

Theorem 3. [31] *Given two positive and coprime integers, p and q, with $0 < p < 2q$, there exists a unique positive integer s, in the range $0 < s < 2q$, such that:*

(a) s is a solution of the congruence: $sp = 1 + qe_q \pmod{2q}$,

(b) s has opposite parity to q.

In addition, integer s is coprime with q.

The solution of the congruence under the restriction imposed by (b) gives a representation of integer s as $s = [1/p]_{2q}$ for q even, and $s = 2[1/(2p)]_q$ for q odd [31], where expression $[1/a]_b$ stands for the modular inverse of integer a mod b. A table with the values of s as a function of $p, q \leq 10$ can be consulted in [30, 31]. The explicit DFT of the Talbot phases, shown below, follows from the above result.

Proposition 4. [31] *Given two positive and coprime integers, p and q, with $0 < p < 2q$, and s the integer constructed in Theorem 3, the DFT pair corresponding to the Talbot sequence is given by:*

$$t_n \equiv e^{j\sigma_\phi \xi_n} = e^{j\sigma_\phi \xi_0} \exp\left(j\pi\sigma_\phi s n^2/q\right) \quad \xrightarrow{\text{DFT}} \quad T_m \equiv \exp\left(-j\pi\sigma_\phi (p + qe_{pq}) m^2/q\right), \tag{10}$$

with:

$$q \text{ odd}: \quad e^{j\xi_0} = \left(\frac{s}{q}\right) e^{j\frac{\pi}{4}(q-1)} = \left(\frac{p}{q}\right) e^{j\frac{\pi}{4}(q-1)}$$
$$q \text{ even}: \quad e^{j\xi_0} = \left(\frac{q}{s}\right) e^{-j\frac{\pi}{4}s} = \left(\frac{q}{p}\right) e^{-j\frac{\pi}{4}p}$$

and $\left(\frac{a}{b}\right)$ the Jacobi symbol of arbitrary integer a and an odd and positive integer b.

Notice that, regardless the relative parity of coprimes p and q, both integers s and $p + qe_{pq}$ in (10) are relatively prime and of opposite parity to q. This means that Talbot sequences, $\boldsymbol{t}$ and its dual $\boldsymbol{T}$, are Gauss sequences $\boldsymbol{g}$ of the form given by (4).

We now analyze the connection of integers p and q defining the DFT pair $\boldsymbol{t} \to \boldsymbol{T}$, with the coprime integers of opposite parity, P and L, that define $\boldsymbol{g}$. This problem is of practical interest since one may wish to construct the complete set of Gauss sequences by use of a minimum set of fractional Talbot orders. The connection will be established through the (dual) Talbot sequence $\boldsymbol{T}$, and restricted, without loss of generality, to negative dispersion, $\sigma_\phi = -1$. We pose the problem in terms of the index sets that define both types of sequences. For a given sequence length, $q = L$, the Gauss sequences are associated to the following index set:

$$\mathsf{G}_q = \{P \in \mathbb{Z}_{2q}^{\times} \,|\, \gcd(P, q) = 1,\ e_{qP} = 0\},$$

so that $g_m = \exp(j\pi P m^2/q)$, and the Talbot orders, and thus the Talbot sequences, are associated to:

$$\mathsf{T}_q = \{p \in \mathbb{Z}_{2q}^{\times} \,|\, \gcd(p, q) = 1\},$$

so that $T_m = \exp(j\pi(p + qe_{pq})m^2/q)$. The problem is therefore the determination of subsets $\mathsf{S} \subset \mathsf{T}_q$ such that the map $\varphi_q : p \in \mathsf{S} \mapsto \langle p + qe_{pq} \rangle_{2q} \in \mathsf{G}_q$ is bijective, $\langle \cdot \rangle_{2q}$ denoting the residue mod $2q$. In that case, we write $\mathsf{S} \sim \mathsf{G}_q$. Moreover, if the product $p \cdot q$ is even for any $p \in \mathsf{S}$, then $\mathsf{S} \sim \mathsf{G}_q$ is equivalent to $\mathsf{S} = \mathsf{G}_q$ as index sets, since then φ_q becomes the identity map.

We introduce the partition $\mathsf{T}_q = \mathsf{T}_q^{\text{even}} \cup \mathsf{T}_q^{\text{odd}}$, composed of the subsets of T_q with even and odd integers q, respectively. For q odd we define a second partition, $\mathsf{T}_q = \mathsf{T}'_q \cup \mathsf{T}''_q$ composed of the set of integers p in the range $0 < p < q$, denoted by T'_q, and the set of integers p in the range $q < p < 2q$, denoted by T''_q.

Example 5. *The index set of the Gauss sequences, G_q, for $q = 5$ is composed of integers $\{2,4,6,8\}$, as shown in the table below, together with the index set T_q of fractional Talbot orders with $q = 5$ and its partitions. The map $p \mapsto \langle p+qe_{pq}\rangle_{2q}$ in T_5 leaves invariant the even integers, and for odd integers gives $1 \mapsto 6$, $3 \mapsto 8$, $7 \mapsto 2$ and $9 \mapsto 4$.*

G_5		2		4	6		8	
T_5	1	2	3	4	6	7	8	9
$\mathsf{T}_5^{\text{even}}$		2		4	6		8	
$\mathsf{T}_5^{\text{odd}}$	1		3			7		9
T'_5	1	2	3	4				
T''_5					6	7	8	9

Proposition 6.
(a) If q is even, $\mathsf{G}_q = \mathsf{T}_q = \mathsf{T}_q^{odd}$.
(b) If q is odd, $\mathsf{G}_q = \mathsf{T}_q^{even} \sim \mathsf{T}_q^{odd} \sim \mathsf{T}'_q \sim \mathsf{T}''_q$.

Proof. Part (a) follows from the observation that, when q is even, $\mathsf{T}_q^{even} = \varnothing$ and $\mathsf{G}_q = \mathsf{T}_q = \mathsf{T}_q^{odd}$, since any integer p coprime with q is necessarily odd, and so of opposite parity to q. As for part (b), $\mathsf{G}_q = \mathsf{T}_q^{even}$ follows from the same argument, since again p and q are not only coprime but of opposite parity.

To show that $\mathsf{G}_q \sim \mathsf{T}_q^{odd}$, we notice that the map φ_q reads in this case $p \in \mathsf{T}_q^{odd} \mapsto \langle p+qe_{pq}\rangle_{2q} = \langle p+q\rangle_{2q}$, which is trivially injective. To show that it is surjective, take an even integer $P \in \mathsf{G}_q \subset \mathbb{Z}_{2q}^{\times}$, coprime with q, so that it can be written in a unique way in terms of an odd integer $p \in \mathbb{Z}_{2q}^{\times}$, either as $P = p+q$ if $q < P < 2q$, or as $P = p-q$ if $0 < P < q$. Integer p constructed this way is also coprime with q and therefore belongs to T_q^{odd}. Finally, since $\varphi_q(p) = P$, map φ_q is onto.

As for $\mathsf{G}_q \sim \mathsf{T}'_q$, the map $p \in \mathsf{T}'_q \mapsto \langle p+qe_{pq}\rangle_{2q}$ is surjective, since any even $P \in \mathsf{G}_q$ is the image of an even integer $p_e = P \in \mathsf{T}'_q$ if $0 < P < q$, and of an odd integer $p_o = P-q \in \mathsf{T}'_q$ if $q < P < 2q$. This construction of the pre-images p_e and p_o is unique, so the map is also injective. The same type of arguments applies to $\mathsf{G}_q \sim \mathsf{T}''_q$. □

This result characterizes different subsets of fractional Talbot orders p/q that generate the complete set of Gauss sequences. Notice also that the twofold redundancy found for odd q, associated to the partition $\mathsf{T}_q = \mathsf{T}_q^{\text{even}} \cup \mathsf{T}_q^{\text{odd}}$, is lifted by the half-period shift in the dispersed field (8), so that the inequivalent forms of this field are in correspondence with Talbot orders. Finally, the equalities $\mathsf{G}_q = \mathsf{T}_q^{odd}$ for q even, and $\mathsf{G}_q = \mathsf{T}_q^{even}$ for q odd, give rise to a bijection within G_q defined by the assignment in Theorem 3, which is also involutive as is immediate to show:

Corollary 7. *For any positive integer q, the map $\psi_q : p \in \mathsf{G}_q \mapsto \psi_q(p) = s \in \mathsf{G}_q$ given by $s = [1/p]_{2q}$ if q is even and $s = 2[1/(2p)]_q$ if q is odd, is an involution in G_q.*

This involution describes, in particular, the integers involved in the exponents of the DFT pair of Gauss sequences. The relationship of Talbot sequences with Gauss sequences, together with the repetitive or cyclic structure of the output wave, permits the definition of a discrete-time transformation based on Talbot effect, as explained in the following section.

4 Talbot circulant discrete signal processing and generation of Gauss-coded pulse sequences

Let us consider a periodic pulse train with period T/q. The train is modulated periodically with a set of q phase and amplitude levels $\boldsymbol{a} = (a_0 a_1 \ldots a_{q-1})$, so that the resulting period is T. The optical envelope is described by (6) with a period expressed as:

$$w(t) = \sum_{k=0}^{q-1} a_k \, p\left(t - k\frac{T}{q}\right), \tag{11}$$

with $p(t)$ the pulse envelope with support contained in $[0, T/q]$. The fractional Talbot self-image of this input pulse sequence is a pulse train with the same period T, described by the following proposition.

Proposition 8. [32] *The result of the propagation of the pulse train described by (11) in a Talbot dispersive line of order p/q is given by the following complex envelope:*

$$E'(t) = \sum_{k=-\infty}^{+\infty} w'\left(t - kT - e_{pq}\frac{T}{2}\right), \tag{12}$$

where the output period is $w'(t) = \sum_{n=0}^{q-1} b_n \, p(t - nT/q)$ with an output pulse sequence $\boldsymbol{b}$ given by:

$$b_n = \frac{e^{j\sigma_\phi \xi_0}}{\sqrt{q}} \sum_{k=0}^{q-1} \exp\left(j\pi\sigma_\phi \frac{s}{q}(n-k)^2\right) a_k. \tag{13}$$

Sequence $\boldsymbol{b}$ can therefore be represented as the circulant or periodic convolution of length q, denoted as ⓠ, between the input sequence and the Talbot sequence or, equivalently, as the image of the unitary and circulant transformation (3) associated to $\boldsymbol{t}$:

$$\boldsymbol{b} = \frac{1}{\sqrt{q}} \, \boldsymbol{t} \circledcirc \boldsymbol{a} = \mathscr{T}_{\boldsymbol{t}}(\boldsymbol{a}) \tag{14}$$

An immediate consequence of this representation, for instance, is that $\boldsymbol{b}$ is ZAC iff $\boldsymbol{a}$ is ZAC, since $\mathscr{T}_{\boldsymbol{t}}$ is unitary and circulant, and thus preserves the circular autocorrelation. Notice also that the inverse transform is that generated by the complex conjugated sequence, $\mathscr{T}_{\boldsymbol{t}}^{+} = \mathscr{T}_{\bar{\boldsymbol{t}}}$.

The simplest application of transform (13, 14) is the passive generation of ultrashort pulse trains with increased repetition rate [33]. A pulse train with period T is described by an input sequence $\boldsymbol{a} = \boldsymbol{\delta}$ with a single nonzero level, $a_k = \delta_{k,0}$. Using (13) we find $\mathscr{T}_{\boldsymbol{t}}(\boldsymbol{\delta}) = \boldsymbol{t}/\sqrt{q}$, and thus the output period is composed of a sequence of q pulses within the same period, with phase levels given by the Talbot sequence $\boldsymbol{t}$ and a common amplitude factor $1/\sqrt{q}$. The perfect autocorrelation properties of these Talbot-imagined pulse trains, inherited from the Gauss sequences, were first noticed in [34].

The inverse transformation uses an input sequence with phase levels complex conjugated to the Talbot sequence, $\bar{\boldsymbol{t}}$. This leads to $\mathscr{T}_{\boldsymbol{t}}(\bar{\boldsymbol{t}}) = \sqrt{q}\,\mathscr{T}_{\boldsymbol{t}}(\mathscr{T}_{\bar{\boldsymbol{t}}}(\boldsymbol{\delta})) = \sqrt{q}\,\boldsymbol{\delta}$, so the unique pulse in the output gathers all the energy from the input period. This transformation describes the process of coherent addition in Talbot array illuminators [31, 35] and noiseless pulse amplification [36]. Transform $\mathscr{T}_{\boldsymbol{t}}$, restricted to Talbot orders of the form $1/q$, has been used to define orthogonal multiplexing schemes [2], and its combination with modulation has also led to the design of compact processors for the passive, analog construction of the DFT of repetitive pulse sequences [32].

5 Talbot generation of GCL-coded pulse sequences

Our main result describes the realization of GCL sequences as the passive output of a fractional Talbot line fed by the upsampled DFT dual of the seed sequence in (5). We first recall the definition of upsampled sequence: given a sequence $\boldsymbol{\sigma}$ of length L, the upsampled sequence $\boldsymbol{u_\sigma}$ of length $L' = SL\,(S > 1)$ is constructed by adjoining $S-1$ zeros after each element of $\boldsymbol{\sigma}$, so that the entries of $\boldsymbol{u_\sigma}$ are $(u_\sigma)_k = \sum_{\mu=0}^{L-1} \sigma_\mu \, \delta_{k,S\mu}$ with $k = 0, \ldots, SL-1$. The following result, to be used below, is well known in digital signal processing.

Lemma 9. *Given a sequence $\boldsymbol{\sigma}$ of length L and with DFT $\boldsymbol{\Sigma}$, the DFT of the upsampled sequence $\boldsymbol{u_\sigma}$ and of its periodic extension $\boldsymbol{p_\sigma}$, both of length $L' = SL$, with $S > 1$, are given by:*
(a) $\mathrm{DFT}(\boldsymbol{p_\sigma}) = \sqrt{S}\,\boldsymbol{u_\Sigma}$.
(b) $\mathrm{DFT}(\boldsymbol{u_\sigma}) = \frac{1}{\sqrt{S}}\,\boldsymbol{p_\Sigma}$.

The main result of the paper provides the sought-for representation of Popović's GCL sequences in terms of the transformation induced by Talbot effect.

Theorem 10. *Given any perfect (ZAC) sequence of length N, $\boldsymbol{\sigma}$, with DFT $\boldsymbol{\Sigma}$, then the fractional Talbot image $\boldsymbol{b}$ of the upsampled sequence $\boldsymbol{u_\sigma}$ of length N^2M, given by:*

$$\boldsymbol{b} = \sqrt{NM}\,\mathscr{T}_{\boldsymbol{t}}(\boldsymbol{u_\sigma}) \tag{15}$$

is a GCL perfect phase (CAZAC) sequence for any Talbot order p/q with $q = N^2M$. Moreover, all GCL sequences can be represented in this form, as the Talbot image of an upsampled ZAC sequence.

Proof. We first prove that $\boldsymbol{b}$ is CAZAC. To show that it is ZAC, we compute the DFT as the circular convolution (14) and show that is CA. Using Lemma 9 with $S = NM$:

$$\boldsymbol{B} = \mathrm{DFT}(\boldsymbol{b}) = \sqrt{NM}\,\boldsymbol{T}\cdot\mathrm{DFT}(\boldsymbol{u_\sigma}) = \boldsymbol{T}\cdot\boldsymbol{p_\Sigma}. \tag{16}$$

Since $\boldsymbol{\sigma}$ is ZAC, then both $\boldsymbol{\Sigma}$ and $\boldsymbol{p_\Sigma}$ are CA. This implies that $\boldsymbol{B}$ is CA, since $\boldsymbol{T}$ is also CA.

To show that $\boldsymbol{b}$ is CA, we compute directly from (13), assuming $\sigma_\phi = 1$ since the other sign can be treated in a similar manner:

$$\begin{aligned} b_n =& \sqrt{NM}\frac{e^{j\xi_0}}{\sqrt{N^2M}}\sum_{k=0}^{N^2M-1}\exp\left(j\pi\frac{s}{N^2M}(n-k)^2\right)\sum_{\mu=0}^{N-1}\sigma_\mu\delta_{k,NM\mu} \\ =& \frac{e^{j\xi_0}}{\sqrt{N}}\exp\left(j\pi\frac{sn^2}{N^2M}\right)\sum_{\mu=0}^{N-1}\sigma_\mu\exp\left(-j2\pi\frac{sn\mu}{N}\right)\exp\left(j\pi sM\mu^2\right) \\ =& \frac{e^{j\xi_0}}{\sqrt{N}}\exp\left(j\pi\frac{sn^2}{N^2M}\right)\sum_{\mu=0}^{N-1}\sigma_\mu\exp\left(-j2\pi\frac{s\mu}{N}(n-NM/2)\right) = t_n\cdot\Sigma_{sn-sNM/2\ (\mathrm{mod}\,N)}. \end{aligned}$$

The shift $\mu_0 = sNM/2 \pmod N$ is meaningful since, according to Theorem 2.(b), s has opposite parity to $q = N^2M$, and therefore to NM. Denoting $\Sigma'_\mu = \Sigma_{s\mu-\mu_0}$, with $\mu = 0,\ldots,N-1$, we have thus found:

$$\boldsymbol{b} = \boldsymbol{t}\cdot\boldsymbol{p_{\Sigma'}}, \tag{17}$$

which is manifestly CA. Moreover, $\boldsymbol{t} \xrightarrow{\mathrm{DFT}} \boldsymbol{T}$ are Gauss sequences, and therefore the DFT pair (16) and (17) is composed of GCL sequences. Finally, this representation describes all possible DFT pairs of GCL sequences, since $\boldsymbol{\Sigma}$ or $\boldsymbol{\Sigma}'$ are arbitrary seed phase sequences of length N and, according to Proposition 6, Talbot sequences of a given length exhaust all possible Gauss sequences of that length. □

Particular cases of this construction can be analyzed as follows. The standard pulse trains with phases given by Gauss sequences are recovered for $N = 1$, since in this case $\boldsymbol{u_\sigma} = \boldsymbol{\delta}$. Also, Theorem 10 can be specialized to sequences $\boldsymbol{\sigma}$ of length N which are not only ZAC but also CAZAC. In this case, (15) maps CAZAC sequences $\boldsymbol{\sigma}$ of length N into CAZAC sequences $\boldsymbol{b}$ of length $q = N^2M$. This observation is of practical relevance since in this case $\boldsymbol{\sigma}$ represents a sequence of phase levels, and therefore it is not required the use of the, more complex, arbitrary phase and amplitude modulation.

From the practical point of view, the physical generation of GCL sequences is simplified by its construction in terms of an upsampled sequence. In this regard, let us recall that the upsampled sequence $\boldsymbol{u_\sigma}$ contains N pulses per period T, and so it can be implemented by use of a pulse source with repetition rate N/T. The output, however, contains N^2M pulses per period, and thus its repetition rate has been increased by a factor NM.

6 Conclusions

We have presented the connection existing between the theory of CAZAC or perfect phase sequences and the fractional Talbot or periodic self-imaging phenomenon, and in particular with Gauss sequences of quadratic phase and also with its multiparameter generalization, Popović's GCL sequences. Our main result has shown that fractional Talbot propagation of order $p/q = p/(N^2M)$, for positive integers N and M, of repetitive pulse train modulated according to N ZAC or CAZAC levels, leads in a natural way to pulse sequences of length $L = N^2M$ whose phase levels are GCL sequences, with the additional practical benefit of an NM-fold increase in repetition rate of the feeding pulse source. This result represents a passive method for creating perfect phase-coded pulse trains by use of fractional Talbot effect from a seed sequence of pulses at lower repetition rate.

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