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Attractors for a nonautonomous reaction-diffusion equation with delay

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Abstract

In this paper, we discuss the existence and uniqueness of solutions for a non-autonomous reaction-diffusion equation with delay, after we prove the existence of a pullback $\mathscr{D}$-asymptotically compact process. By a priori estimates, we show that it has a pullback $\mathscr{D}$-absorbing set that allow us to prove the existence of a pullback $\mathscr{D}$-attractor for the associated process to the problem.

Keywords: pullback attractors, nonclassical reaction-diffusion equations, critical exponent, delay term
AMS 2010 codes: 35B41, 35B40; Secondary 37L30,35Q35.

1 Introduction and statement of the problem

We consider the following nonautonomous functional reaction-diffusion equation

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) - \Delta u(t,x) = f(u(t,x)) + b(t,u_t)(x) + g(t,x) \text{ in } (\tau,\infty)\times\Omega\,, \\ u = 0 \text{ on } (\tau,\infty)\times\partial\Omega\,, \\ u(\tau,x) = u^0(x),\ \tau\in\mathbb{R} \text{ and } x\in\Omega\,, \\ u(\tau+\theta,x) = \varphi(\theta,x),\ \theta\in[-r,0] \text{ and } x\in\Omega\,, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^{\mathbb{N}}$ is a bounded domain with smooth boundary $\partial\Omega$, $\tau\in\mathbb{R}$, $u^0\in L^2(\Omega)$ is the initial condition in τ and $\varphi\in L^2([-r,0];L^2(\Omega))$ is also the initial condition in $[\tau-r,\tau]$, $r>0$ is the length of the delay effect. For the rest we assume following assumptions conditions :

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H_1) Concerning the nonlinearity, we assume that $f \in C^1(\mathbb{R},\mathbb{R})$, there exist positive constants c,μ_0,μ_1,k and $p>2$, $N \leq \frac{2p}{p-2}$ such that

$$-c-\mu_0|u|^p \leq f(u)u \leq c-\mu_1|u|^p \;\forall u \in \mathbb{R}, \tag{1.2}$$

$$(f(u)-f(v))(u-v) \leq k(u-v)^2 \;\forall u,v \in \mathbb{R}. \tag{1.3}$$

Let us denote by

$$F(u) := \int_0^u f(s)ds.$$

From (1.2), there exist positive constants l,c',μ_0',μ_1' such that

$$|f(u)| \leq l\left(|u|^{p-1}+1\right) \;\forall u \in \mathbb{R}, \tag{1.4}$$

$$-c'-\mu_0'|u|^p \leq F(u) \leq c'-\mu_1'|u|^p \;\forall u \in \mathbb{R}. \tag{1.5}$$

H_2) The operator $b:\mathbb{R}\times L^2([-r,0];L^2(\Omega)) \to L^2(\Omega)$ is a time-dependent external force with delay, such that

(I) For all $\phi \in L^2([-r,0];L^2(\Omega))$, the function $\mathbb{R} \ni t \mapsto b(t,\phi) \in L^2(\Omega)$ is measurable;

(II) $b(t,0)=0$ for all $t \in \mathbb{R}$;

(III) $\exists L_b>0$ s.t $\forall t \in \mathbb{R}$ and $\forall \phi_1,\phi_2 \in L^2([-r,0];L^2(\Omega))$;

$$\|b(t,\phi_1)-b(t,\phi_2)\| \leq L_b\|\phi_1-\phi_2\|_{L^2([-r,0];L^2(\Omega))}; \tag{1.6}$$

(IV) $\exists C_b>0$ s.t $\forall t \geq \tau$, and $\forall u,v \in L^2([\tau-r,t];L^2(\Omega))$;

$$\int_\tau^t \|b(s,u_s)-b(s,v_s)\|^2 ds \leq C_b \int_{\tau-r}^t \|u(s)-v(s)\|^2 ds. \tag{1.7}$$

Remark 1. *From (I)-(III), for $T>\tau$ and $u \in L^2([\tau-r,T];L^2(\Omega))$ the function $\mathbb{R} \ni t \mapsto b(t,\phi) \in L^2(\Omega)$ is measurable and belongs to $L^\infty((\tau,T);L^2(\Omega))$.*

H_3) The function $g \in L^2_{loc}(\mathbb{R};L^2(\Omega))$ is an another nondelayed time-dependent external force.

For more details on differential equations with delay, we refer the reader to J. Wu [9] and J.K. Hale [5]. The purpose of this paper is to discuss the existence of pullback $\mathscr{D}$-attractor in $L^2(\Omega)\times L^2([-r,0];L^2(\Omega))$ by using a priori estimates of solutions to the problem (1.1).

This work is motivated by the work of T. Caraballo and J. Real. [1], where they proved the existence of pullback attractors for the following 2D-Navier聳Stokes model with delays :

$$\begin{cases} \frac{\partial u}{\partial t}-\nu\Delta u+\sum_{i=1}^2 u_i\frac{\partial u}{\partial x_i}=f-\nabla p+g(t,u_t) \text{ in } (\tau,\infty)\times\Omega, \\ div\, u=0 \text{ in } (\tau,\infty)\times\Omega, \\ u=0 \text{ on } (\tau,\infty)\times\partial\Omega, \\ u(\tau,x)=u_0(x),\; x\in\Omega, \\ u(t,x)=\phi(t-\tau,x),\; t\in(\tau-h,\tau) \text{ and } x\in\Omega, \end{cases} \tag{1.8}$$

where $\nu>0$ is the kinematic viscosity, u is the velocity field of the fluid, p the pressure, $\tau \in \mathbb{R}$ the initial time, u_0 the initial velocity field, f a nondelayed external force field, g another external force with delay and ϕ the initial condition in $(-h,0)$, where h is a fixed positive number.

On the other hand, the problem (1.1) without critical nonlinearity was treated by J. Li and J. Huang in [6], where they proved the existence of uniform attractor for the following non-autonomous parabolic equation with delays :

$$\begin{cases} \frac{\partial u(t,x)}{\partial t}+Au(t,x)+bu(t,x)=F(u_t)(x)+g(t,x)\; x \text{ in } \Omega, \\ u(\tau,x)=u_0(x),\; u(\tau+\theta,x)=\phi(\theta,x),\; \theta\in(-r,0). \end{cases} \tag{1.9}$$

Here Ω is a bounded domain in $\mathbb{R}^{n_0}$ with smooth boundary, $b \geq 0$, A is a densely-defined self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and with compact resolvent, F is the nonlinear term which is locally Lipschitz continuous for the initial condition, g is an external force.

In [3], J.Garcia-Luengo and P.Marin-Rubio treated the following reaction-diffusion equation with non-autonomous force in H^{-1} and delays under measurability conditions on the driving delay term :

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f(u) + g(t, u_t) + k(t) \text{ in } (\tau, \infty) \times \Omega\,, \\ u = 0 \text{ on } (\tau, \infty) \times \partial\Omega\,, \\ u(\tau + s, x) = \phi(s, x),\ s \in [-r, 0] \text{ and } x \in \Omega\,, \end{cases} \tag{1.10}$$

where $\tau \in \mathbb{R}$, $f \in C(\mathbb{R})$ the nonlinear term with critical exponent, g is an external force with delay, $k \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ a time-dependent force, ϕ the initial condition and h the lenght of the delay effect. In this work, the authors checked the existence of pullback $\mathscr{D}$-attractor in $C([-h, 0]; L^2(\Omega))$.

This paper is organized as follows. In section 2, we will prove the existence of weak solutions to the problem(1.1) by using the Faedo-Galerkin approximations, as well as the uniqueness and the continuous dependence of solution with respect to initial conditions. In section 3, we recall some definitions and abstract results on pullback $\mathscr{D}$-attractor. Then we can prove the existence of pullback $\mathscr{D}$-attractor for the nonautonomous problem with delay.

2 Existence and uniqueness of solution

First we give the concept of the solution.

Definition 1. *A weak solution of* (1.1) *is a function* $u \in L^2([\tau - r, T]; L^2(\Omega))$ *such that for all* $T > \tau$ *we have*

$$u \in L^2((\tau, T); H_0^1(\Omega)) \cap L^p((\tau, T); L^p(\Omega)) \cap C([\tau, T]; L^2(\Omega))$$

and

$$\frac{\partial u}{\partial t} \in L^2([\tau, T]; L^2(\Omega))\,,$$

with $u(t) = \varphi(t - \tau)$, *for* $t \in [\tau - r, \tau]$, *and it satisfies*

$$\int_\tau^T -\langle u, v' \rangle + \int_\tau^T \int_\Omega \nabla u \nabla v = \int_\tau^T \int_\Omega f(u) v + \int_\tau^T \langle b(t, u_t), v \rangle + \int_\tau^T \int_\Omega g v + \langle u^0, v(\tau) \rangle\,,$$

for all test functions $v \in L^2([\tau, T]; H_0^1(\Omega))$ *and* $v' \in L^2([\tau, T]; H^{-1}(\Omega))$ *such that* $v(T) = 0$.

Theorem 1. *Assume that* $g \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, *b and f satisfy (I)-(IV) and* (1.2)-(1.5) *respectively and if* $\lambda_1 > 1 + C_b/2$, *Then for all* $T > \tau$ *and all* (u^0, φ) *in* $L^2(\Omega) \times L^2([-r, 0]; L^2(\Omega))$, *there exists a unique weak solution u to the problem* (1.1).

Proof. Let us consider $\{e_k\}_{k \geq 1}$, the complete basis of $H_0^1(\Omega)$ which is given by the orthonormal eigenfunctions of Δ in $L^2(\Omega)$. We consider

$$u^m(t) = \sum_{k=1}^m \gamma_{k,m}(t) e_k\,,\ m = 1, 2, \ldots$$

which is the approximate solutions of Faedo-Galerkin of order m, that is

$$\begin{cases} \left\langle \frac{du^m}{dt}, e_k \right\rangle + \langle \Delta u^m, e_k \rangle = \langle f(u^m), e_k \rangle + \langle b(t, u_t^m), e_k \rangle + \langle g, e_k \rangle \\ \langle u^m(\tau), e_k \rangle = \langle P_m u^0, e_k \rangle = \langle u^0, e_k \rangle \text{ i.e. } P_m u^m(\tau) \to u^0 \text{ in } L^2(\Omega) \\ \langle u^m(\tau + \theta), e_k \rangle = \langle P_m \varphi(\theta), e_k \rangle = \langle \varphi(\theta), e_k \rangle\ \forall \theta \in (-r, 0) \end{cases}$$

for all $k = 1\dots m$. Where $\gamma_{k,m}(t) = \langle u^m(t), e_k\rangle$ denote the Fourier coefficients ; such that $\gamma_{m,k} \in C^1((\tau,T);\mathbb{R}) \cap L^2((\tau-r,T),\mathbb{R})$, $\gamma'_{k,m}(t)$ is absolutely continuous, and $P_m u(t) = \sum_{k=1}^m \langle u, e_k\rangle e_k$ is the orthogonal projection of $L^2(\Omega)$ and $H_0^1(\Omega)$ in $V_m = span\{e_1,\dots,e_m\}$.

It is well-known that the above finite-dimensional delayed system is well-posed (e.g. cf. [2]), at least locally. We will provide a priori estimates for the Faedo-Galerkin approximate solutions.

Claim 1. *For all $m \in \mathbb{N}^*$ and all $T > \tau$, the sequence $\{u^m\}$ is bounded in*

$$L^\infty((\tau,T);L^2(\Omega)) \cap L^2((\tau,T);H_0^1(\Omega)) \cap L^p((\tau,T);L^p(\Omega)).$$

Multiplying (1.1) by u^m and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u^m(t)\|^2 + \|\nabla u^m(t)\|^2 = \int_\Omega f(u^m)u^m + \int_\Omega b(t,u_t^m)u^m + \int_\Omega g u^m.$$

Using the hypothesis (1.2) and the Young inequality, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u^m(t)\|^2 + \|\nabla u^m(t)\|^2\\ &\le c|\Omega| - \mu_1\|u^m(t)\|^p + \frac{1}{2}\|b(t,u_t^m)\|^2 + \frac{1}{2}\|u^m(t)\|^2 + \frac{1}{2}\|g(t)\|^2 + \frac{1}{2}\|u^m(t)\|^2.\end{aligned}$$

So, one has

$$\begin{aligned}&\frac{d}{dt}\|u^m(t)\|^2 + 2\|\nabla u^m(t)\|^2 + 2\mu_1\|u^m(t)\|^p\\ &\le 2c|\Omega| + \|b(t,u_t^m)\|^2 + \|g(t)\|^2 + \|u^m(t)\|^2.\end{aligned}$$

After integrating this last estimate over $[\tau,t]$, $\tau \le t \le T$, we use (II) and (IV), so we get

$$\begin{aligned}&\|u^m(t)\|^2 + 2\int_\tau^t \|\nabla u^m(s)\|^2 ds + 2\mu_1\int_\tau^t \|u^m(s)\|^p ds\\ &\le 2c|\Omega|(t-\tau) + \|u^m(\tau)\|^2 + C_b\int_{\tau-r}^t \|u^m(s)\|^2 ds\\ &+ \int_\tau^t \|g(s)\|^2 ds + \int_\tau^t \|u^m(s)\|^2 ds,\\ &\le 2c|\Omega|(t-\tau) + \|u^m(\tau)\|^2 + C_b\int_{\tau-r}^\tau \|u^m(s)\|^2 ds + C_b\int_\tau^t \|u^m(s)\|^2 ds\\ &+ \int_\tau^t \|g(s)\|^2 ds + \int_\tau^t \|u^m(s)\|^2 ds.\end{aligned}$$

By the fact that $\lambda_1\|u\|^2 \le \|\nabla u\|^2$, one has

$$\begin{aligned}&\|u^m(t)\|^2 + 2\int_\tau^t \|\nabla u^m(s)\|^2 ds + 2\mu_1\int_\tau^t \|u^m(s)\|^p ds\\ &\le 2c|\Omega|(t-\tau) + \|u^m(\tau)\|^2 + C_b\int_{\tau-r}^\tau \|u^m(s)\|^2 ds + C_b\lambda_1^{-1}\int_\tau^t \|\nabla u^m(s)\|^2 ds\\ &+ \int_\tau^t \|g(s)\|^2 ds + \lambda_1^{-1}\int_\tau^t \|\nabla u^m(s)\|^2 ds.\end{aligned}$$

Then, we find

$$\|u^m(t)\|^2 + (2 - C_b\lambda_1^{-1} - \lambda_1^{-1}) \int_\tau^t \|\nabla u^m(s)\|^2 ds + 2\mu_1 \int_\tau^t \|u^m(s)\|^p ds$$

$$\leq 2c|\Omega|(t-\tau) + \|u^m(\tau)\|^2 + C_b \int_{\tau-r}^{\tau} \|u^m(s)\|^2 ds + \int_\tau^t \|g(s)\|^2 ds,$$

$$\leq 2c|\Omega|(T-\tau) + \|u^m(\tau)\|^2 + C_b \int_{\tau-r}^{\tau} \|u^m(s)\|^2 ds + \int_\tau^t \|g(s)\|^2 ds. \tag{2.1}$$

Since $g \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$ and for $\lambda_1 > 1 + C_b/2$, we deduce by this last estimate that for all $T > \tau$, the sequence

$$\{u^m\} \text{ is bounded in } L^\infty((\tau,T);L^2(\Omega)) \cap L^2((\tau,T);H_0^1(\Omega)) \cap L^p((\tau,T);L^p(\Omega)). \tag{2.2}$$

Also, the estimate (2.1) implies that the local solution can extended to the interval $[\tau, T]$.

Claim 2.

$$\{f(u^m)\} \textit{ is bounded in } L^q((\tau,T);L^q(\Omega)). \tag{2.3}$$

Using (1.4), we have

$$\|f(u^m(t)\|^q_{L^q(\Omega)} = \int_\Omega |f(u^m(t,x))|^q dx,$$

$$\leq l^q \int_\Omega \left(|u^m(t,x)|^{p-1} + 1\right)^q dx.$$

By the convexity of the power and the fact that $p = q(p-1)$, one has

$$\|f(u^m(t))\|^q_{L^q(\Omega)} \leq 2^{q-1} l^q \int_\Omega |u^m(t,x)|^{q(p-1)} dx + 2^{q-1} l^q |\Omega|,$$

$$\leq 2^{q-1} l^q \|u^m(t)\|^{q(p-1)}_{L^p(\Omega)} + 2^{q-1} l^q |\Omega|,$$

$$\leq 2^{q-1} l^q \|u^m(t)\|^p_{L^p(\Omega)} + 2^{q-1} l^q |\Omega|.$$

Integrating this last estimate over $[\tau, t]$, $\tau \leq t \leq T$, one obtains

$$\int_\tau^t \|f(u^m(s))\|^q_{L^q(\Omega)} ds \leq 2^{q-1} l^q \int_\tau^t \|u^m(s)\|^p_{L^p(\Omega)} ds + 2^{q-1} l^q |\Omega|(t-\tau)$$

From (2.1), we deduce that the term $\int_\tau^t \|u^m(s)\|^p_{L^p(\Omega)} ds$ is bounded, so by this last estimate we conclude that $\{f(u^m)\}$ is bounded in $L^q((\tau,T);L^q(\Omega))$, for all $T > \tau$.

Claim 3. $\left\{\frac{\partial}{\partial t} u^m\right\}$ *is bounded in* $L^2((\tau,T);L^2(\Omega))$.

Now, multiplying (1.1) by $\frac{\partial u^m}{\partial t}$ and integrating over Ω, one has

$$\left\|\frac{d}{dt} u^m(t)\right\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u^m(t)\|^2$$

$$= \int_\Omega f(u^m)\frac{\partial u^m}{\partial t} + \int_\Omega b(t, u_t^m)\frac{\partial u^m}{\partial t} + \int_\Omega g\frac{\partial u^m}{\partial t}. \tag{2.4}$$

On the other hand, we have

$$\frac{d}{dt}F(u) = \frac{dF}{du}\frac{\partial u}{\partial t},$$

$$= f(u)\frac{\partial u}{\partial t}. \tag{2.5}$$

So

$$\frac{d}{dt}\int_{\Omega} F(u) = \int_{\Omega} f(u)\frac{\partial u}{\partial t}.$$

Using this last equality in (2.4), we find

$$\left\|\frac{d}{dt}u^m(t)\right\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u^m(t)\|^2 = \frac{d}{dt}\int_{\Omega} F(u^m) + \int_{\Omega} b(t,u_t^m)\frac{\partial u^m}{\partial t} + \int_{\Omega} g\frac{\partial u^m}{\partial t}.$$

From (1.5) and Cauchy inequality, we have

$$\begin{aligned}
&\left\|\frac{d}{dt}u^m(t)\right\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u^m(t)\|^2, \\
&\leq \frac{d}{dt}\int_{\Omega}(c' - \mu_1'|u(t,x)|^p)dx + \frac{\varepsilon_1}{2}\|b(t,u_t^m)\|^2 + \frac{1}{2\varepsilon_1}\left\|\frac{d}{dt}u^m(t)\right\|^2 \\
&+ \frac{\varepsilon_2}{2}\|g(t)\|^2 + \frac{1}{2\varepsilon_2}\left\|\frac{d}{dt}u^m(t)\right\|^2.
\end{aligned}$$

After simplification, one obtains

$$\begin{aligned}
&\left(2 - \frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}\right)\left\|\frac{d}{dt}u^m(t)\right\|^2 + \frac{d}{dt}\left(\|\nabla u^m(t)\|^2 + 2\mu_1'\|u^m(t)\|^p_{L^p(\Omega)}\right) \\
&\leq \varepsilon_1\|b(t,u_t^m)\|^2 + \varepsilon_2\|g(t)\|^2.
\end{aligned}$$

We can choose $\varepsilon_1 = \varepsilon_2 = 2$ to get

$$\left\|\frac{d}{dt}u^m(t)\right\|^2 + \frac{d}{dt}\left(\|\nabla u^m(t)\|^2 + 2\mu_1'\|u^m(t)\|^p_{L^p(\Omega)}\right) \leq 2\|b(t,u_t^m)\|^2 + 2\|g(t)\|^2.$$

Integrating this last estimate over $[\tau,t]$ and using (II) and (IV), one has

$$\begin{aligned}
&\int_{\tau}^{t}\left\|\frac{d}{ds}u^m(s)\right\|^2 ds + \|\nabla u^m(t)\|^2 + 2\mu_1'\|u^m(t)\|^p_{L^p(\Omega)} \\
&\leq \|\nabla u^m(\tau)\|^2 + 2\mu_1'\|u^m(\tau)\|^p_{L^p(\Omega)} + 2C_b\int_{\tau-r}^{t}\|u^m(s)\|^2 ds + 2\int_{\tau}^{t}\|g(s)\|^2 ds, \\
&\leq \|\nabla u^m(\tau)\|^2 + 2\mu_1'\|u^m(\tau)\|^p_{L^p(\Omega)} + 2C_b\int_{\tau-r}^{\tau}\|u^m(s)\|^2 ds \\
&+ 2C_b\int_{\tau}^{t}\|u^m(s)\|^2 ds + 2\int_{\tau}^{t}\|g(s)\|^2 ds
\end{aligned}$$

Since $\lambda_1\|u\|^2 \leq \|\nabla u\|^2$, one has

$$\begin{aligned}
&\int_{\tau}^{t}\left\|\frac{d}{ds}u^m(s)\right\|^2 ds + \|\nabla u^m(t)\|^2 + 2\mu_1'\|u^m(t)\|^p_{L^p(\Omega)} \\
&\leq \|\nabla u^m(\tau)\|^2 + 2\mu_1'\|u^m(\tau)\|^p_{L^p(\Omega)} + 2C_b\int_{\tau-r}^{\tau}\|u^m(s)\|^2 ds \\
&+ 2C_b\lambda_1^{-1}\int_{\tau}^{t}\|\nabla u^m(s)\|^2 ds + 2\int_{\tau}^{t}\|g(s)\|^2 ds
\end{aligned}$$

From (2.1), we have $\int_\tau^t \|\nabla u^m(s)\|^2 ds$ is bounded and since $g \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, this last estimate gives that

$$\left\{ \frac{\partial}{\partial t} u^m \right\} \text{ is bounded in } L^2((\tau,T);L^2(\Omega)),$$

for all $T > \tau$.

From the claims (1), (2) and (3), the hypothesis (IV) and the remark (1), we can extract a subsequence (relabelled the same) such that

$$\begin{aligned}
&u^m \rightharpoonup u \text{ weakly* in } L^\infty((\tau,T);L^2(\Omega)),\\
&u^m \rightharpoonup u \text{ weakly in } L^2((\tau,T);H_0^1(\Omega)),\\
&u^m \rightharpoonup u \text{ weakly in } L^p((\tau,T);L^p(\Omega)),\\
&\frac{\partial u^m}{\partial t} \to \frac{\partial u}{\partial t} \text{ strongly in } L^2((\tau,T);L^2(\Omega)),\\
&f(u^m) \rightharpoonup \sigma' \text{ weakly in } L^q((\tau,T);L^q(\Omega)),\\
&b(.,u^m_.) \to b(.,u_.) \text{ strongly in } L^2((\tau,T);L^2(\Omega)).
\end{aligned} \tag{2.6}$$

By the Aubin-Lions lemma of compactness, we conclude that $u^m \to u$ strongly in $L^2((\tau,T);L^2(\Omega))$. Thus $u^m \to u$ a.e $[\tau,T]\times\Omega$.

Since f is continuous, we deduce that $f(u^m) \to f(u)$ a.e $[\tau,T]\times\Omega$. So from (2.3) and (lemma 1.3 in [7], p.12) we can identify σ' with $f(u)$.

To prove that $u(\tau) = u^0$, we put $v \in C^1((\tau,T);H_0^1(\Omega))$ such that $v(T) = 0$ and we note from (1.1) that

$$\begin{aligned}
\int_\tau^T -\langle u, v'\rangle + \int_\tau^T \int_\Omega \nabla u \nabla v = \int_\tau^T \int_\Omega f(u)v + \int_\tau^T \langle b(t,u_t), v\rangle\\
+ \int_\tau^T \int_\Omega gv + \langle u(\tau), v(\tau)\rangle.
\end{aligned} \tag{2.7}$$

In a similar way, from the Faedo-Galerkin approximations, we have

$$\begin{aligned}
\int_\tau^T -\langle u^m, v'\rangle \int_\tau^T \int_\Omega \nabla u^m \nabla v = \int_\tau^T \int_\Omega f(u^m)v + \int_\tau^T \langle b(t,u_t^m), v\rangle\\
+ \int_\tau^T \int_\Omega gv + \langle u^m(\tau), v(\tau)\rangle.
\end{aligned} \tag{2.8}$$

Using the fact that $u^m(\tau) \to u^0$ in $L^2(\Omega)$ and (2.6) to find

$$\begin{aligned}
\int_\tau^T -\langle u, v'\rangle + \int_\tau^T \int_\Omega \nabla u \nabla v = \int_\tau^T \int_\Omega f(u)v + \int_\tau^T \langle b(t,u_t), v\rangle\\
+ \int_\tau^T \int_\Omega gv + \langle u^0, v(\tau)\rangle.
\end{aligned} \tag{2.9}$$

Since $v(\tau)$ is given arbitrarily, comparing (2.7) and (2.9) we deduce that $u(\tau) = u^0$.

To prove that $u \in C([\tau,T];L^2(\Omega))$, we put $w^m = u^m - u$ then we have

$$\frac{\partial}{\partial t} w^m - \Delta w^m = f(u^m) - f(u) + b(t,u_t^m) - b(t,u_t).$$

Multiplying this equation by w^m and integrating over Ω, we obtain

$$\begin{aligned}
\frac{d}{dt}\|w^m(t)\|^2 + 2\|\nabla w^m(t)\|^2 = 2\int_\Omega (f(u^m) - f(u))\, w^m\\
+ 2\int_\Omega (b(t,u_t^m) - b(t,u_t))(u^m - u).
\end{aligned}$$

By (1.3), (I) and (1.6), we get

$$\frac{d}{dt}\|w^m(t)\|^2+2\|\nabla w^m(t)\|^2\leq 2k\|w^m(t)\|^2+2L_b\|w_t^m\|^2_{L^2([-r,0];L^2(\Omega))}.$$

Integrating over $[\tau,t]$, we get

$$\begin{aligned}&\|w^m(t)\|^2-\|w^m(\tau)\|^2+2\int_\tau^t\|\nabla w^m(s)\|^2ds\\ \leq\ & 2k\int_\tau^t\|w^m(s)\|^2+2L_b\int_\tau^t\int_{-r}^0\|w^m(s+\theta)\|^2d\theta ds,\\ \leq\ & 2k\int_\tau^t\|w^m(s)\|^2+2L_b\int_{-r}^0\int_{\tau-r}^t\|w^m(s)\|^2dsd\theta,\\ \leq\ & 2k\int_\tau^t\|w^m(s)\|^2+2L_br\int_{\tau-r}^\tau\|w^m(s)\|^2ds+2L_br\int_\tau^t\|w^m(s)\|^2ds.\end{aligned}$$

Therefore by by this last estimate, we can deduce that

$$\|w^m(t)\|^2\leq\|w^m(\tau)\|^2+2L_br\int_{\tau-r}^\tau\|w^m(s)\|^2ds+(2k+2L_br)\int_\tau^t\|w^m(s)\|^2ds.$$

Applying the Gronwall lemma to this estimate, we obtain

$$\|w^m(t)\|^2\leq\left(\|w^m(\tau)\|^2+2L_br\int_{\tau-r}^\tau\|w^m(s)\|^2ds\right)e^{(2k+2L_br)(t-\tau)}. \tag{2.10}$$

Since $u^m(\tau)\to u^0$ and $u^m(\tau+\theta)\to\varphi(\theta)$, the estimate (2.10) shows that $u^m\to u$ uniformly in $C([\tau,T];L^2(\Omega))$.

Finally, we prove the uniqueness and continuous dependence of the solution. Let $u^1;u^2$ be two solutions of problem (1.1) with the initial conditions $u^{0,1},u^{0,2}$ and φ^1,φ^2. Denoting that $w=u^1-u^2$ and repeating the argument as in the proof of (2.10), we find

$$\|w(t)\|^2\leq\left(\|w(\tau)\|^2+2L_br\int_{\tau-r}^\tau\|w(s)\|^2ds\right)e^{(2k+2L_br)(t-\tau)}. \tag{2.11}$$

and this completes the proof of the theorem. ■

3 existence of pullback *D*-attractors

3.1 Preliminaries of pullback *D*-attractors

First, we give some basic definitions and an abstract result on the existence of pullback attractors, which we need to obtain our results (we refer the reader to [2–4, 8]). Let (X,d) be a complete metric space, $\mathscr{P}(X)$ be the class of nonempty subsets of X, and suppose $\mathscr{D}$ is a nonempty class of parameterized sets $\widehat{D}=\{D(t):t\in\mathbb{R}\}\subset\mathscr{P}(X)$.

Definition 2. *A two parameter family of mappings $U(t,\tau):X\to X$ $t\geq\tau$, $\tau\in\mathbb{R}$, is called to be a process if*

1. $S(\tau,\tau)x=\{x\},\forall\tau\in\mathbb{R},x\in Y$;
2. $S(t,s)S(s,\tau)x=S(t,\tau)x$, $\forall t\geq s\geq\tau$, $\tau\in\mathbb{R},x\in X$.

Definition 3. *A family of bounded sets $\widehat{B}=\{B(t):t\in\mathbb{R}\}\in\mathscr{D}$ is called pullback $\mathscr{D}$-absorbing for the process $\{S(t,\tau)\}$ if for any $t\in\mathbb{R}$ and for any $\widehat{D}\in\mathscr{D}$, there exists $\tau_0(t,\widehat{D})\leq t$ such that*

$$S(t,\tau)D(\tau)\subset B(t)\quad\textit{for all}\ \ \tau\leq\tau_0(t,\widehat{D}).$$

Definition 4. *The process $S(t,\tau)$ is said to be pullback $\mathscr{D}$-asymptotically compact if for all $t\in\mathbb{R}$, all $\widehat{D}\in\mathscr{D}$, any sequence $\tau_n\to-\infty$, and any sequence $x_n\in D(\tau_n)$, the sequence $\{S(t,\tau_n)x_n\}$ is relatively compact in X.*

Definition 5. *A family $\widehat{A}=\{A(t):t\in\mathbb{R}\}\subset\mathscr{P}(X)$ is said to be a pullback 7$\mathscr{D}$-attractor for $\{S(t,\tau)\}$ if*

1. *$A(t)$ is compact for all $t\in\mathbb{R}$;*
2. *$\widehat{A}$ is invariant; i.e., $S(t,\tau)A(\tau)=A(t)$, for all $t\geq\tau$;*
3. *$\widehat{A}$ is pullback $\mathscr{D}$-attracting ; i.e.,*

$$\lim_{\tau\to-\infty} dist(S(t,\tau)D(\tau),A(t))=0,$$

for all $\widehat{D}\in\mathscr{D}$ and all $t\in\mathbb{R}$;

4. *If $\{C(t):t\in\mathbb{R}\}$ is another family of closed attracting sets then $A(t)\subset C(t)$, for all $t\in\mathbb{R}$.*

Theorem 2. *Let us suppose that the process $\{S(t,\tau)\}$ is pullback $\mathscr{D}$-asymptotically compact, and $\widehat{B}=\{B(t):t\in\mathbb{R}\}\in\mathscr{D}$ is a family of pullback $\mathscr{D}$-absorbing sets for $\{S(t,\tau)\}$. Then there exists a pullback $\mathscr{D}$-attractor $\{A(t):t\in\mathbb{R}\}$ such that*

$$A(t)=\bigcap_{s\leq t}\overline{\bigcup_{\tau\leq s}S(t,\tau)B(\tau)}.$$

3.2 Construction of the associated process

Now, we will apply the above results in the phase space $H:=L^2(\Omega)\times L^2([-r,0];L^2(\Omega))$, which is a Hilbert space with the norm

$$\|(u^0,\varphi)\|_H^2=\|\nabla u^0\|^2+\int_{-r}^{0}\|\varphi(\theta)\|^2d\theta,$$

with $(u^0,\varphi)\in H$. To this aim, We consider $g\in L^2_{loc}(\mathbb{R};L^2(\Omega))$, $b:\mathbb{R}\times L^2([-r,0];L^2(\Omega))\to L^2(\Omega)$ with the hypotheses (I)-(IV) and $f\in C^1(\mathbb{R};\mathbb{R})$ verifying (1.2)-(1.5). Then the family of mappings

$$\begin{aligned}S(t,\tau):H&\to H\\ (u^0,\varphi)&\longmapsto S(t,\tau)(u^0,\varphi)=(u(t),u_t),\end{aligned}\tag{3.1}$$

with $t\geq\tau$, $\tau\in\mathbb{R}$ and u is the weak solution to (1.1), defines a process.

On the other hand, we construct the family of mappings

$$\begin{aligned}U(t,\tau):H&\to C([-r,0];L^2(\Omega))\\ (u^0,\varphi)&\longmapsto U(t,\tau)(u^0,\varphi)=u_t,\ \forall t\geq\tau+r,\end{aligned}\tag{3.2}$$

which we will use in our analysis. Of course, it is sensible to expect that the both operators should be related. Let us consider the linear mapping

$$\begin{aligned}j:C([-r,0];L^2(\Omega))&\to l^2(\Omega)\times C([-r,0];L^2(\Omega))\\ \varphi&\longmapsto j(\varphi)=(\varphi(0),\varphi).\end{aligned}$$

This map is obviously continuous from $C([-r,0];L^2(\Omega))$ into H. We note that for all $(u^0,\varphi)\in H$ provided that $t\geq\tau+r$, so we write

$$S(t,\tau)(u^0,\varphi)=j(U(t,\tau)(u^0,\varphi)),\ \forall(u^0,\varphi)\in H,\ \forall t\geq\tau+r.$$

To check the continuity of the process, we need the following lemma.

Lemma 1. *Let (u^0,φ), $(v^0,\phi)\in H$ be two couples of initial conditions for the problem (1.1) and u, v be the corresponding solutions to (1.1). Then there exists a positive constant $\nu := 2(\frac{1}{2}+k+\frac{C_b}{2}-\lambda_1)>0$, such that*

$$\|u(t)-v(t)\|^2 \le \left(\|u^0-v^0\|^2 + C_b\|\varphi-\phi\|^2\right)e^{\nu(t-\tau)}, \ \forall t\ge\tau. \tag{3.3}$$

It also holds

$$\|u_t-v_t\|^2_{C([-r,0];L^2(\Omega))} \le \left(\|u^0-v^0\|^2 + C_b\|\varphi-\phi\|^2\right)e^{\nu(t-r-\tau)}, \ \forall t\ge\tau+r. \tag{3.4}$$

Proof. From (1.1), one has

$$\frac{\partial}{\partial t}(u-v)-\Delta(u-v) = f(u)-f(v)+b(t,u_t)-b(t,v_t).$$

We put $w=u-v$, we obtain

$$\frac{\partial w}{\partial t}-\Delta w = f(u)-f(v)+b(t,u_t)-b(t,v_t).$$

Multiplying this equation by w and integrating it over Ω, one gets

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2+\|\nabla w(t)\|^2 = \int_\Omega (f(u)-f(v))\,w + \int_\Omega (b(t,u_t)-b(t,v_t))\,w.$$

Using (1.3) and Cauchy-Schwarz inequality, one has

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2+\|\nabla w(t)\|^2 \le k\|w(t)\|^2 + \|b(t,u_t)-b(t,v_t)\|\|w(t)\|.$$

Since $\lambda_1\|w(t)\|^2\le\|\nabla w(t)\|^2$ and by the Young inequality, one finds

$$\begin{aligned}\frac{d}{dt}\|w(t)\|^2+2\lambda_1\|w(t)\|^2 &\le \frac{d}{dt}\|w(t)\|^2+2\|\nabla w(t)\|^2,\\ &\le 2k\|w(t)\|^2+\|b(t,u_t)-b(t,v_t)\|^2+\|w(t)\|^2.\end{aligned}$$

Therefore, one has

$$\frac{d}{dt}\|w(t)\|^2 \le 2\left(\frac{1}{2}+k-\lambda_1\right)\|w(t)\|^2+\|b(t,u_t)-b(t,v_t)\|^2.$$

Integrating this last estimate from τ to t and using (1.7), one obtains

$$\begin{aligned}\|w(t)\|^2 \le{}& \|w(\tau)\|^2+2\left(\frac{1}{2}+k-\lambda_1\right)\int_\tau^t\|w(s)\|^2ds\\ &+\int_\tau^t\|b(s,u_s)-b(s,v_s)\|^2ds,\\ \le{}& \|w(\tau)\|^2+2\left(\frac{1}{2}+k-\lambda_1\right)\int_\tau^t\|w(s)\|^2ds+C_b\int_{\tau-r}^t\|w(s)\|^2ds,\\ \le{}& \|w(\tau)\|^2+2\left(\frac{1}{2}+k-\lambda_1\right)\int_\tau^t\|w(s)\|^2ds+C_b\int_{\tau-r}^\tau\|w(s)\|^2ds\\ &+C_b\int_\tau^t\|w(s)\|^2ds,\\ \le{}& \|w(\tau)\|^2+C_b\int_{\tau-r}^\tau\|w(s)\|^2ds+2\left(\frac{1}{2}+k+\frac{C_b}{2}-\lambda_1\right)\int_\tau^t\|w(s)\|^2ds.\end{aligned}$$

By the Gronwall lemma, for all $t \geq \tau$, one deduces

$$\begin{aligned}\|w(t)\|^2 &\leq \left(\|w(\tau)\|^2 + C_b \int_{\tau-r}^{\tau} \|w(s)\|^2 ds\right) e^{\nu(t-\tau)}, \\ &\leq \left(\|u^0 - v^0\|^2 + C_b \|\varphi - \phi\|^2_{L^2([-r,0];L^2(\Omega))}\right) e^{\nu(t-\tau)},\end{aligned}$$

and by this last estimate, we proved (3.3). Now, assume that $t \geq \tau + r$, so $t + \theta \geq \tau$ for all $\theta \in [-r, 0]$ and one has

$$\begin{aligned}\|w(t+\theta)\|^2 &\leq \left(\|u^0 - v^0\|^2 + C_b \|\varphi - \phi\|^2_{L^2([-r,0];L^2(\Omega))}\right) e^{\nu(t+\theta-\tau)}, \\ &\leq \left(\|u^0 - v^0\|^2 + C_b \|\varphi - \phi\|^2_{L^2([-r,0];L^2(\Omega))}\right) e^{\nu(t-r-\tau)}.\end{aligned}$$

Hence, we conclude

$$\|w_t\|_{C([-r,0];L^2(\Omega))} \leq \left(\|u^0 - v^0\|^2 + C_b \|\varphi - \phi\|^2_{L^2([-r,0];L^2(\Omega))}\right) e^{\nu(t-r-\tau)}.$$

By this last estimate we finished the proof of this lemma. ■

Theorem 3. *Under the previous assumptions, the mapping $S(.,.)$ defined in (3.1), is a continuous process for all $\tau \leq t$.*

Proof. The proof of this theorem is as the proof of Theorem 9 in [1]. The uniqueness of the solutions implies that $S(.,.)$ is a process. For the continuity of $S(.,.)$, we use the previous lemma. We consider (u^0, φ), $(v^0, \phi) \in H$ and u, v are their corresponding solutions. Firstly, if we take $t \geq \tau + r$, it follows from (3.4)

$$\begin{aligned}\|u_t - v_t\|^2_{L^2([-r,0];L^2(\Omega))} &= \int_{-r}^{0} \|u(t+\theta) - v(t+\theta)\|^2 d\theta, \\ &\leq \int_{-r}^{0} \sup_{s\in[-r,0]} \|u(t+s) - v(t+s)\|^2 d\theta, \\ &\leq r\left(\|u^0 - v^0\|^2 + C_b \|\varphi - \phi\|^2\right) e^{\nu(t-r-\tau)}.\end{aligned}$$

Now, for $t \in [\tau, \tau + r]$, we deduce

$$\begin{aligned}\|u_t - v_t\|^2_{L^2([-r,0];L^2(\Omega))} &= \int_{-r}^{0} \|u(t+\theta) - v(t+\theta)\|^2 d\theta, \\ &\leq \left(r\|u^0 - v^0\|^2 + (C_b r + 1)\|\varphi - \phi\|^2\right) e^{\nu(t-r-\tau)}.\end{aligned}$$

So, for all $t \geq \tau$, we have

$$\|u_t - v_t\|^2_{L^2([-r,0];L^2(\Omega))} \leq \left(r\|u^0 - v^0\|^2 + (C_b r + 1)\|\varphi - \phi\|^2\right) e^{\nu(t-r-\tau)}.$$

Hence, by this last estimate and (3.3) we deduce the continuity of $S(t, \tau)$. ■

3.3 Existence of pullback D-absorbing set in $C([-r,0];L^2(\Omega))$ and H

Firstly, we need to the following lemma, it relates the absorption properties for the mappings with those of process S in the fact that, proving those for U yields to similar properties for S.

Lemma 2. *Assume that the family of bounded sets $\{B(t) : t \in \mathbb{R}\}$ in the space $C([-r,0];L^2(\Omega))$ is pullback $\mathscr{D}$-absorbing for the mapping $U(.,.)$. Then the family of bounded sets $\{j(B(t)) : t \in \mathbb{R}\}$ in $L^2(\Omega) \times C([-r,0];L^2(\Omega))$ is pullback $\mathscr{D}$-absorbing for the process $S(.,.)$.*

Proof. Let $\{D(t) : t \in \mathbb{R}\}$ be a family bounded sets in H, so there exists $T > r$ such that

$$U(t,\tau)D(\tau) \subset B(t)\,,\ \forall t-\tau \geq T\,.$$

On the other hand, we have

$$S(t,\tau)(u^0,\varphi) = j(U(t,\tau)(u^0,\varphi))\,,$$

it follows that

$$S(t,\tau)(u^0,\varphi) = j(U(t,\tau)(u^0,\varphi)) \subset j(B(t))\,,\ \forall t-\tau \geq T\,.$$

■

Remark 2. *Noticing that the word absorbing used in this papier should be interpreted in a generalized sense, since U is not a process.*

Now, we need the following estimations.

Lemma 3. *Assume that $g \in L^2_{loc}(\mathbb{R};L^2(\Omega))$, there exists a small enough $\alpha < 2\lambda_1 - 2 - C_b$ such that*

$$\int_{-\infty}^{t} e^{\alpha t}\|g(s)\|^2 ds < \infty\,, \tag{3.5}$$

the function f satisfies (1.2)-(1.5) and b fulfills conditions (I)-(IV) and

$$\int_{\tau}^{t} e^{\sigma s}\|b(s,u_s)-b(s,v_s)\|^2 ds \leq C_b \int_{\tau-r}^{t} e^{\sigma s}\|u(s)-v(s)\|^2 ds\,. \tag{3.6}$$

Then we have

$$\begin{aligned}\|u(t)\|^2 \leq\ & e^{-\alpha(t-\tau)}\|u(\tau)\|^2 + C_b e^{-\alpha(t-\tau)}\int_{\tau-r}^{\tau}\|u(s)\|^2 ds \\ & + 2c|\Omega|\alpha^{-1}\left(1-e^{-\alpha(t-\tau)}\right) + e^{-\alpha t}\int_{-\infty}^{t} e^{\alpha s}\|g(s)\|^2 ds\,,\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\eta e^{-\alpha t}\int_{\tau}^{t} e^{\alpha s}\|u(s)\|^2 ds + 2\mu_1 e^{-\alpha t}\int_{\tau}^{t} e^{\alpha s}\|u(s)\|^p_{L^p(\Omega)} ds \\ &\leq e^{-\alpha(t-\tau)}\|u(\tau)\|^2 + C_b e^{-\alpha(t-\tau)}\int_{\tau-r}^{\tau}\|u(s)\|^2 ds + 2c|\Omega|\alpha^{-1}\left(1-e^{-\alpha(t-\tau)}\right) \\ &+ e^{-\alpha t}\int_{-\infty}^{t} e^{\alpha s}\|g(s)\|^2 ds\,,\end{aligned} \tag{3.8}$$

where $\eta := 2\lambda_1 - 2 - \alpha - C_b$.

Proof. Multiplying (1.1) by u and integrating over Ω, one has

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|\nabla u(t)\|^2 = \int_\Omega f(u)u + \int_\Omega b(t,u_t)u + \int_\Omega gu\,.$$

By (1.2), Cauchy-Shwarz and Young inequalities, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|\nabla u(t)\|^2 + \mu_1\|u(t)\|^p_{L^p(\Omega)} \leq c|\Omega| + \frac{1}{2}\|b(t,u_t)\|^2 + \frac{1}{2}\|g(t)\|^2 + \|u(t)\|^2\,.$$

Since $\lambda_1\|u\|^2 \leq \|\nabla u\|^2$ and after calculation, one has

$$\frac{d}{dt}\|u(t)\|^2 + 2(\lambda_1 - 1)\|u(t)\|^2 + 2\mu_1\|u(t)\|^p_{L^p(\Omega)} \leq 2c|\Omega| + \|b(t,u_t)\|^2 + \|g(t)\|^2 .$$

Now, we multiply this last estimate by $e^{\alpha t}$ such that $0 < \alpha < 2\lambda_1 - 2 - C_b$, so one gets

$$\begin{aligned} & e^{\alpha t}\frac{d}{dt}\|u(t)\|^2 + 2(\lambda_1 - 1)e^{\alpha t}\|u(t)\|^2 + 2\mu_1 e^{\alpha t}\|u(t)\|^p_{L^p(\Omega)} \\ & \leq 2c|\Omega|e^{\alpha t} + e^{\alpha t}\|b(t,u_t)\|^2 + e^{\alpha t}\|g(t)\|^2 . \end{aligned} \tag{3.9}$$

On the other hand, we have

$$\frac{d}{dt}\left(e^{\alpha t}\|u(t)\|^2\right) = \alpha e^{\alpha t}\|u(t)\|^2 + e^{\alpha t}\frac{d}{dt}\|u(t)\|^2$$

We substitute (3.9) in this equality, we find

$$\begin{aligned} \frac{d}{dt}\left(e^{\alpha t}\|u(t)\|^2\right) \leq \alpha e^{\alpha t}\|u(t)\|^2 - 2(\lambda_1 - 1)e^{\alpha t}\|u(t)\|^2 - 2\mu_1 e^{\alpha t}\|u(t)\|^p_{L^p(\Omega)} \\ + 2c|\Omega|e^{\alpha t} + e^{\alpha t}\|b(t,u_t)\|^2 + e^{\alpha t}\|g(t)\|^2 . \end{aligned}$$

Integrating this last estimate over $[\tau, t]$, one obtains

$$\begin{aligned} e^{\alpha t}\|u(t)\|^2 \leq {} & e^{\alpha\tau}\|u(\tau)\|^2 + 2c|\Omega|\alpha^{-1}(e^{\alpha t} - e^{\alpha\tau}) \\ & + (\alpha + 2 - 2\lambda_1)\int_\tau^t e^{\alpha s}\|u(s)\|^2 ds - 2\mu_1\int_\tau^t e^{\alpha s}\|u(s)\|^p_{L^p(\Omega)} ds \\ & + \int_\tau^t e^{\alpha s}\|b(s,u_s)\|^2 ds + \int_\tau^t e^{\alpha s}\|g(s)\|^2 ds . \end{aligned}$$

Using (3.6) and (II), one has

$$\begin{aligned} e^{\alpha t}\|u(t)\|^2 \leq {} & e^{\alpha\tau}\|u(\tau)\|^2 + 2c|\Omega|\alpha^{-1}(e^{\alpha t} - e^{\alpha\tau}) \\ & + (\alpha + 2 - 2\lambda_1)\int_\tau^t e^{\alpha s}\|u(s)\|^2 ds - 2\mu_1\int_\tau^t e^{\alpha s}\|u(s)\|^p_{L^p(\Omega)} ds \\ & + C_b\int_{\tau-r}^t e^{\alpha s}\|u(s)\|^2 ds + \int_\tau^t e^{\alpha s}\|g(s)\|^2 ds . \end{aligned} \tag{3.10}$$

On the other hand, we have

$$\begin{aligned} \int_{\tau-r}^t e^{\alpha s}\|u(s)\|^2 ds &= \int_{\tau-r}^\tau e^{\alpha s}\|u(s)\|^2 ds + \int_\tau^t e^{\alpha s}\|u(s)\|^2 ds , \\ &\leq e^{\alpha\tau}\int_{\tau-r}^\tau \|u(s)\|^2 ds + \int_\tau^t e^{\alpha s}\|u(s)\|^2 ds . \end{aligned} \tag{3.11}$$

So by (3.10) and (3.11), one finds

$$\begin{aligned} e^{\alpha t}\|u(t)\|^2 \leq {} & e^{\alpha\tau}\|u(\tau)\|^2 + 2c|\Omega|\alpha^{-1}(e^{\alpha t} - e^{\alpha\tau}) \\ & + (\alpha + 2 - 2\lambda_1 + C_b)\int_\tau^t e^{\alpha s}\|u(s)\|^2 ds - 2\mu_1\int_\tau^t e^{\alpha s}\|u(s)\|^p_{L^p(\Omega)} ds \\ & + C_b e^{\alpha\tau}\int_{\tau-r}^\tau \|u(s)\|^2 ds + \int_\tau^t e^{\alpha s}\|g(s)\|^2 ds . \end{aligned}$$

Hence, by (3.5) we obtain

$$\|u(t)\|^2 + (2\lambda_1 - \alpha - 2 - C_b)e^{-\alpha t}\int_\tau^t e^{\alpha s}\|u(s)\|^2 ds$$
$$+ 2\mu_1 e^{-\alpha t}\int_\tau^t e^{\alpha s}\|u(s)\|^p_{L^p(\Omega)} ds$$
$$\leq e^{-\alpha(t-\tau)}\|u(\tau)\|^2 + 2c|\Omega|\alpha^{-1}\left(1 - e^{-\alpha(t-\tau)}\right) + C_b e^{-\alpha(t-\tau)}\int_{\tau-r}^{\tau}\|u(s)\|^2 ds$$
$$+ e^{-\alpha t}\int_{-\infty}^t e^{\alpha s}\|g(s)\|^2 ds.$$

Thus, for $\eta := 2\lambda_1 - \alpha - 2 - C_b > 0$, by this last estimate we get

$$\|u(t)\|^2 \leq e^{-\alpha(t-\tau)}\|u(\tau)\|^2 + C_b e^{-\alpha(t-\tau)}\int_{\tau-r}^{\tau}\|u(s)\|^2 ds$$
$$+ 2c|\Omega|\alpha^{-1}\left(1 - e^{-\alpha(t-\tau)}\right) + e^{-\alpha t}\int_{-\infty}^t e^{\alpha s}\|g(s)\|^2 ds,$$

and

$$\eta e^{-\alpha t}\int_\tau^t e^{\alpha s}\|u(s)\|^2 ds + 2\mu_1 e^{-\alpha t}\int_\tau^t e^{\alpha s}\|u(s)\|^p_{L^p(\Omega)} ds$$
$$\leq e^{-\alpha(t-\tau)}\|u(\tau)\|^2 + C_b e^{-\alpha(t-\tau)}\int_{\tau-r}^{\tau}\|u(s)\|^2 ds + 2c|\Omega|\alpha^{-1}\left(1 - e^{-\alpha(t-\tau)}\right)$$
$$+ e^{-\alpha t}\int_{-\infty}^t e^{\alpha s}\|g(s)\|^2 ds,$$

for all $t \geq \tau$. So by these two estimations the proof of the lemma is finished. ■

Proposition 1. *Under the assumptions in lemma (3). Then the family $\{B_1(t) : t \in \mathbb{R}\}$ given by*

$$B_1(t) = \overline{B}_{C([-r,0];L^2(\Omega))}(0, R_1(t)),$$

with

$$R_1^2(t) = e^{\alpha r}\left(2c|\Omega|\alpha^{-1} + e^{-\alpha t}\int_{-\infty}^t e^{\alpha t}\|g(s)\|^2 ds\right), \ \forall t \in \mathbb{R};$$

is pullback $\mathscr{D}$-absorbing for the mapping $U(t,\tau)$. Moreover, the family $\{B_0(t) : t \in \mathbb{R}\}$ given by

$$B_0(t) = \overline{B}_{L^2(\Omega))}(0, R_1(t)) \times \overline{B}_{L^2([-r,0];L^2(\Omega))}\left(0, \sqrt{r}R_1(t)\right) \subset H, \ \forall t \in \mathbb{R},$$

is pullback $\mathscr{D}$-absorbing for the process S defined by (3.1).

Proof. The first part may be proved as follows.
By definition, we have

$$\|U(t,\tau)(u^0,\varphi)\|^2_{C([-r,0];L^2(\Omega))} = \sup_{s\in[-r,0]}\|u(t+s)\|^2.$$

From (3.7), if we take $t \geq \tau + r$, so $t + \theta \geq \tau$. Then one has

$$\|u(t+\theta)\|^2 \leq e^{-\alpha(t+\theta-\tau)}\|u(\tau)\|^2 + C_b e^{-\alpha(t+\theta-\tau)}\|\varphi\|^2_{L^2([-r,0];L^2(\Omega))}$$
$$+ 2c|\Omega|\alpha^{-1}\left(1 - e^{-\alpha(t+\theta-\tau)}\right) + e^{-\alpha(t+\theta)}\int_{-\infty}^{t+\theta} e^{\alpha s}\|g(s)\|^2 ds,$$

which implies that

$$\begin{aligned}\sup_{s\in[-r,0]}\|u(t+s)\|^2 &\le e^{-\alpha(t-r-\tau)}\|u(\tau)\|^2+C_b e^{-\alpha(t-r-\tau)}\|\varphi\|^2_{L^2([-r,0];L^2(\Omega))}\\ &+2c|\Omega|\alpha^{-1}e^{\alpha r}\left(e^{-\alpha r}-e^{-\alpha(t-\tau)}\right)\\ &+e^{-\alpha(t-r)}\int_{-\infty}^{t}e^{\alpha s}\|g(s)\|^2ds,\end{aligned}\tag{3.12}$$

On the one hand, we have

$$\begin{aligned}\|\varphi\|^2_{L^2([-r,0];L^2(\Omega))} &= \int_{-r}^{0}\|\varphi(\theta)\|^2d\theta,\\ &\le \int_{-r}^{0}\sup_{s\in[-r,0]}\|\varphi(s)\|^2d\theta,\\ &\le r\|\varphi\|^2_{C([-r,0];L^2(\Omega))}.\end{aligned}\tag{3.13}$$

Therefore by (3.12), (3.13) and the fact that $u(\tau)=\varphi(0)$, we obtain

$$\begin{aligned}&\sup_{s\in[-r,0]}\|u(t+s)\|^2 \le e^{-\alpha(t-r-\tau)}\|\varphi(0)\|^2+C_b r e^{-\alpha(t-r-\tau)}\|\varphi\|^2_{C([-r,0];L^2(\Omega))}\\ &+2c|\Omega|\alpha^{-1}e^{\alpha r}\left(e^{-\alpha r}-e^{-\alpha(t-\tau)}\right)+e^{-\alpha(t-r)}\int_{-\infty}^{t}e^{\alpha s}\|g(s)\|^2ds,\\ &\le (1+C_b r)e^{-\alpha(t-r-\tau)}\|\varphi\|^2_{C([-r,0];L^2(\Omega))}+2c|\Omega|\alpha^{-1}e^{\alpha r}\\ &+e^{-\alpha(t-r)}\int_{-\infty}^{t}e^{\alpha s}\|g(s)\|^2ds.\end{aligned}$$

Then, we find

$$\begin{aligned}&\|U(t,\tau)(u^0,\varphi)\|^2_{C([-r,0];L^2(\Omega))} = \sup_{s\in[-r,0]}\|u(t+s)\|^2,\\ &\le (1+C_b r)e^{-\alpha(t-r-\tau)}\|\varphi\|^2_{C([-r,0];L^2(\Omega))}\\ &+e^{\alpha r}\left(2c|\Omega|\alpha^{-1}+e^{-\alpha t}\int_{-\infty}^{t}e^{\alpha s}\|g(s)\|^2ds\right),\end{aligned}\tag{3.14}$$

for all $(u^0,\varphi)\in H$ and all $t\ge\tau+r$.

Let $\mathscr{R}$ be the set of all functions $\rho:\mathbb{R}\longrightarrow(0,+\infty)$ such that

$$\lim_{t\to-\infty}e^{\alpha t}\rho^2(t)=0.$$

By $\mathscr{D}$ we denote the class of all families $\widehat{\mathbf{D}}=\{D(t):t\in\mathbb{R}\}\subset\mathscr{P}(C([-r,0];L^2(\Omega)))$ such that $D(t)\subset\overline{\mathbf{B}}_{C([-r,0];L^2(\Omega))}(0,\rho(t))$, for some $\rho\in\mathscr{R}$, where we denote by $\overline{\mathbf{B}}_{C([-r,0];L^2(\Omega))}(0,\rho(t))$ the closed ball in $C([-r,0];L^2(\Omega))$ centered at 0 with radius $\rho(t)$. Let

$$R_1^2(t)=e^{\alpha r}\left(2c|\Omega|\alpha^{-1}+e^{-\alpha t}\int_{-\infty}^{t}e^{\alpha s}\|g(s)\|^2ds\right).$$

Thus, for all $\widehat{D}\in\mathscr{D}$ and all $t\in\mathbb{R}$, by (3.14) there exists $\tau_0(\widehat{D},t)\le t$ such that

$$\|U(t,\tau)(u^0,\varphi)\|^2_{C([-r,0];L^2(\Omega))}\le R_1^2(t),\tag{3.15}$$

for all $\tau \leq \tau_0(\widehat{D},t)$; i.e., $B_1(t) = \overline{B}_{C([-r,0];L^2(\Omega))}(0,R_1(t))$ is pullback $\mathscr{D}$-absorbing for the mapping $U(t,\tau)$.

Concerning the second part, we observe that $\{j(B(t)), t \in \mathbb{R}\}$ is a family of pullback $\mathscr{D}$-absorbing sets for the process S. On the other hand, since

$$\|\varphi\|^2_{L^2([-r,0];L^2(\Omega))} \leq r\|\varphi\|^2_{C([-r,0];L^2(\Omega))},$$

and

$$j(B(t)) = \left\{(\varphi(0),\varphi) \,:\, \varphi \in \overline{B}_{C([-r,0];L^2(\Omega))}(0,R_1(t))\right\},$$

we deduce that

$$j(B(t)) \subset \overline{B}_{L^2(\Omega))}(0,R_1(t)) \times \overline{B}_{L^2([-r,0];L^2(\Omega))}\left(0,\sqrt{r}R_1(t)\right) = B_0(t),$$

which implies that the family $\{B_0(t) \,:\, t \in \mathbb{R}\}$ is pullback $\mathscr{D}$-absorbing sets for the process S. ■

3.4 Existence of pullback D-absorbing set in $C([-r,0];H^1_0(\Omega))$

Proposition 2. *Suppose that conditions of lemma (3) are satisfied, if there exists a sufficiently small α^* such that*

$$\alpha < \alpha^* < \min\left\{2\frac{\lambda_1 - 1}{\lambda_1}, 2\mu_1\right\}.$$

Then the family $\{B_2(t) : t \in \mathbb{R}\}$ given by

$$B_2(t) = \overline{B}_{C([-r,0];H^1_0(\Omega))}(0,R_2(t)),$$

where

$$R_2^2(t) = 2c|\Omega|\left(\alpha^{*-1}e^{\alpha^* r} + 2C_b\alpha^{-1}\eta^{-1}e^{\alpha r}\right) + 2C_b\eta^{-1}e^{-\alpha(t-r)}\int_{-\infty}^t e^{\alpha s}\|g(s)\|^2 ds + 2e^{-\alpha^*(t-r)}\int_{-\infty}^t e^{\alpha^* s}\|g(s)\|^2 ds, \ \forall t \in \mathbb{R},$$

is pullback $\mathscr{D}$-absorbing for the mapping $U(t,\tau)$.

Proof.
Multipying (1.1) by $u + \frac{\partial u}{\partial t}$ and integrating over Ω, we obtain

$$\left\|\frac{d}{dt}u(t)\right\|^2 + \frac{1}{2}\frac{d}{dt}\left(\|u(t)\|^2 + \|\nabla u(t)\|^2\right) = \int_\Omega f(u)\left(u + \frac{\partial u}{\partial t}\right) + \int_\Omega b(t,u_t)\left(u + \frac{\partial u}{\partial t}\right) + \int_\Omega g\left(u + \frac{\partial u}{\partial t}\right).$$

Using (1.2), (1.5), (2.5) and Young inequality, one finds

$$2\left\|\frac{d}{dt}u(t)\right\|^2 + \frac{d}{dt}\left(\|u(t)\|^2 + \|\nabla u(t)\|^2 + 2\mu_1'\|u(t)\|^p_{L^p(\Omega)}\right) + 2\|\nabla u(t)\|^2 + 2\mu_1\|u(t)\|^p_{L^p(\Omega)} \leq 2c|\Omega| + 2\|b(t,u_t)\|^2 + 2\|g(t)\|^2 + 2\left\|\frac{d}{dt}u(t)\right\|^2 + 2\|u(t)\|^2.$$

By the fact that $\lambda_1\|u\|^2 \leq \|\nabla u\|^2$, after simplification one has

$$\frac{d}{dt}\left(\|u(t)\|^2 + \|\nabla u(t)\|^2 + 2\mu_1'\|u(t)\|^p_{L^p(\Omega)}\right) + 2(1-\lambda_1^{-1})\|\nabla u(t)\|^2 + 2\mu_1\|u(t)\|^p_{L^p(\Omega)} \leq 2c|\Omega| + 2\|b(t,u_t)\|^2 + 2\|g(t)\|^2.$$

Since α in lemma (3) is small enough, we can choose a positive constant α^* sufficiently small with $\alpha < \alpha^* < \min\left\{2\frac{\lambda_1-1}{\lambda_1}, 2\mu_1\right\}$, such that

$$2(1-\lambda_1^{-1})\|\nabla u(t)\|^2 \geq \alpha^*(\|u(t)\|^2+\|\nabla u(t)\|^2).$$

So, we can write

$$\frac{d}{dt}\gamma_1(t)+\alpha^*\gamma_1(t) \leq 2c|\Omega|+2\|b(t,u_t)\|^2+2\|g(t)\|^2, \tag{3.16}$$

where

$$\gamma_1(t) = \|u(t)\|^2+\|\nabla u(t)\|^2+2\mu_1'\|u(t)\|^p_{L^p(\Omega)}. \tag{3.17}$$

Multiplying (3.16) by $e^{\alpha^* t}$, one has

$$e^{\alpha^* t}\frac{d}{dt}\gamma_1(t)+\alpha^* e^{\alpha^* t}\gamma_1(t) \leq 2c|\Omega|e^{\alpha^* t}+2e^{\alpha^* t}\|b(t,u_t)\|^2+2e^{\alpha^* t}\|g(t)\|^2. \tag{3.18}$$

On the other hand, we have

$$\frac{d}{dt}\left(e^{\alpha^* t}\gamma_1(t)\right) = \alpha^* e^{\alpha^* t}\gamma_1(t)+e^{\alpha^* t}\frac{d}{dt}\gamma_1(t) \tag{3.19}$$

Then, by (3.18) and (3.19), we obtain

$$\begin{aligned}\frac{d}{dt}\left(e^{\alpha^* t}\gamma_1(t)\right) &\leq \alpha^* e^{\alpha^* t}\gamma_1(t)-\alpha^* e^{\alpha^* t}\gamma_1(t)+2c|\Omega|e^{\alpha^* t}\\ &+2e^{\alpha^* t}\|b(t,u_t)\|^2+2e^{\alpha^* t}\|g(t)\|^2,\\ &\leq 2c|\Omega|e^{\alpha^* t}+2e^{\alpha^* t}\|b(t,u_t)\|^2+2e^{\alpha^* t}\|g(t)\|^2.\end{aligned}$$

Integrating this last one from τ to t, one gets

$$\begin{aligned}e^{\alpha^* t}\gamma_1(t) &\leq e^{\alpha^*\tau}\gamma_1(\tau)+2c|\Omega|\int_\tau^t e^{\alpha^* s}ds+2\int_\tau^t e^{\alpha^* s}\|b(s,u_s)\|^2 ds\\ &+2\int_\tau^t e^{\alpha^* s}\|g(s)\|^2 ds,\\ &\leq e^{\alpha^*\tau}\gamma_1(\tau)+2c|\Omega|\alpha^{*-1}\left(e^{\alpha^* t}-e^{\alpha^*\tau}\right)+2\int_\tau^t e^{\alpha^* s}\|b(s,u_s)\|^2 ds\\ &+2\int_\tau^t e^{\alpha^* s}\|g(s)\|^2 ds.\end{aligned}$$

From (3.5) and (3.6), one finds

$$\begin{aligned}e^{\alpha^* t}\gamma_1(t) &\leq e^{\alpha^*\tau}\gamma_1(\tau)+2c|\Omega|\alpha^{*-1}\left(e^{\alpha^* t}-e^{\alpha^*\tau}\right)+2C_b\int_{\tau-r}^t e^{\alpha^* s}\|u(s)\|^2 ds\\ &+2\int_{-\infty}^t e^{\alpha^* s}\|g(s)\|^2 ds,\\ &\leq e^{\alpha^*\tau}\gamma_1(\tau)+2c|\Omega|\alpha^{*-1}\left(e^{\alpha^* t}-e^{\alpha^*\tau}\right)+2C_b e^{\alpha^*\tau}\int_{\tau-r}^\tau \|u(s)\|^2 ds\\ &+2C_b\int_\tau^t e^{\alpha^* s}\|u(s)\|^2 ds+2\int_{-\infty}^t e^{\alpha^* s}\|g(s)\|^2 ds.\end{aligned}$$

We multiply this estimate by $e^{-\alpha^* t}$, we obtain

$$\begin{aligned}\gamma_1(t) &\le e^{-\alpha^*(t-\tau)}\gamma_1(\tau) + 2C_b e^{-\alpha^*(t-\tau)}\int_{\tau-r}^{\tau}\|u(s)\|^2 ds \\ &+ 2c|\Omega|\alpha^{*-1}\left(1-e^{-\alpha^*(t-\tau)}\right) + 2C_b e^{-\alpha^* t}\int_{\tau}^{t} e^{\alpha^* s}\|u(s)\|^2 ds \\ &+ 2e^{-\alpha^* t}\int_{-\infty}^{t} e^{\alpha^* s}\|g(s)\|^2 ds. \end{aligned}\tag{3.20}$$

On the one hand, since $H_0^1(\Omega)\subset L^2(\Omega)$ and $H_0^1(\Omega)\subset L^p(\Omega)$, we have

$$\begin{aligned}\gamma_1(\tau) &= \|u(\tau)\|^2 + \|\nabla u(\tau)\|^2 + 2\mu_1'\|u(\tau)\|^p_{L^p(\Omega)}, \\ &\le (1+\lambda_1^{-1})\|\nabla u(\tau)\|^2 + 2\mu_1'\|u(\tau)\|^p_{L^p(\Omega)}, \\ &\le (1+\lambda_1^{-1})\|\nabla u(\tau)\|^2 + k_1\|\nabla u(\tau)\|^p, \\ &\le k_2(1+\lambda_1^{-1})\|\nabla u(\tau)\|^p + k_1\|\nabla u(\tau)\|^p, \\ &\le k_3\|\nabla u(\tau)\|^p. \end{aligned}\tag{3.21}$$

So, by (3.17), (3.20) and (3.21), one finds

$$\begin{aligned}\|u(t)\|^2 + \|\nabla u(t)\|^2 + 2\mu_1'\|u(t)\|^p_{L^p(\Omega)} &\le k_3 e^{-\alpha^*(t-\tau)}\|\nabla u(\tau)\|^p \\ + 2C_b e^{-\alpha^*(t-\tau)}\int_{\tau-r}^{\tau}\|u(s)\|^2 ds &+ 2c|\Omega|\alpha^{*-1}\left(1-e^{-\alpha^*(t-\tau)}\right) \\ + 2C_b e^{-\alpha^* t}\int_{\tau}^{t} e^{\alpha^* s}\|u(s)\|^2 ds &+ 2e^{-\alpha^* t}\int_{-\infty}^{t} e^{\alpha^* s}\|g(s)\|^2 ds. \end{aligned}$$

From this last estimate and (3.8), we have

$$\begin{aligned}\|\nabla u(t)\|^2 &\le k_3 e^{-\alpha^*(t-\tau)}\|\nabla u(\tau)\|^p + 2C_b e^{-\alpha^*(t-\tau)}\int_{\tau-r}^{\tau}\|u(s)\|^2 ds \\ &+ 2c|\Omega|\alpha^{*-1}\left(1-e^{-\alpha^*(t-\tau)}\right) + 2e^{-\alpha^* t}\int_{-\infty}^{t} e^{\alpha^* s}\|g(s)\|^2 ds \\ &+ 2C_b e^{-\alpha^* t}\int_{\tau}^{t} e^{\alpha^* s}\|u(s)\|^2 ds, \\ &\le k_3 e^{-\alpha^*(t-\tau)}\|\nabla u(\tau)\|^p + 2C_b e^{-\alpha^*(t-\tau)}\int_{\tau-r}^{\tau}\|u(s)\|^2 ds \\ &+ 2c|\Omega|\alpha^{*-1}\left(1-e^{-\alpha^*(t-\tau)}\right) + 2e^{-\alpha^* t}\int_{-\infty}^{t} e^{\alpha^* s}\|g(s)\|^2 ds \\ &+ 2C_b\eta^{-1}e^{-\alpha(t-\tau)}\|u(\tau)\|^2 + 2C_b^2\eta^{-1}e^{-\alpha(t-\tau)}\int_{\tau-r}^{\tau}\|u(s)\|^2 ds \\ &+ 4C_b c|\Omega|\alpha^{-1}\eta^{-1}\left(1-e^{-\alpha(t-\tau)}\right) + 2C_b\eta^{-1}e^{-\alpha t}\int_{-\infty}^{t} e^{\alpha s}\|g(s)\|^2 ds, \\ &\le k_3 e^{-\alpha^*(t-\tau)}\|\nabla u(\tau)\|^p + 2C_b\eta^{-1}e^{-\alpha(t-\tau)}\|u(\tau)\|^2 \\ &+ 2C_b\left(e^{-\alpha^*(t-\tau)} + C_b\eta^{-1}e^{-\alpha(t-\tau)}\right)\int_{\tau-r}^{\tau}\|u(s)\|^2 ds \\ &+ 2c|\Omega|\alpha^{*-1}\left(1-e^{-\alpha^*(t-\tau)}\right) + 4C_b c|\Omega|\alpha^{-1}\eta^{-1}\left(1-e^{-\alpha(t-\tau)}\right) \\ &+ 2e^{-\alpha^* t}\int_{-\infty}^{t} e^{\alpha^* s}\|g(s)\|^2 ds + 2C_b\eta^{-1}e^{-\alpha t}\int_{-\infty}^{t} e^{\alpha s}\|g(s)\|^2 ds. \end{aligned}$$

In the fact that

$$\|\varphi\|^2_{L^2([-r,0];L^2(\Omega))} \le r\|\varphi\|^2_{C([-r,0];L^2(\Omega))},$$

one has

$$\begin{aligned}
&\|\nabla u(t)\|^2 \le k_3 e^{-\alpha^*(t-\tau)}\|\nabla u(\tau)\|^p + 2C_b\eta^{-1}e^{-\alpha(t-\tau)}\|u(\tau)\|^2\\
&+ 2C_b r\left(e^{-\alpha^*(t-\tau)} + C_b\eta^{-1}e^{-\alpha(t-\tau)}\right)\|\varphi\|^2_{C([-r,0];L^2(\Omega))}\\
&+ 2c|\Omega|\alpha^{*-1}\left(1-e^{-\alpha^*(t-\tau)}\right) + 4C_b c|\Omega|\alpha^{-1}\eta^{-1}\left(1-e^{-\alpha(t-\tau)}\right)\\
&+ 2e^{-\alpha^* t}\int_{-\infty}^{t} e^{\alpha^* s}\|g(s)\|^2 ds + 2C_b\eta^{-1}e^{-\alpha t}\int_{-\infty}^{t} e^{\alpha s}\|g(s)\|^2 ds\\
&\le k_3 e^{-\alpha^*(t-\tau)}\|\nabla u(\tau)\|^p + 2C_b\eta^{-1}e^{-\alpha(t-\tau)}\|u(\tau)\|^2\\
&+ 2C_b r\left(e^{-\alpha^*(t-\tau)} + C_b\eta^{-1}e^{-\alpha(t-\tau)}\right)\|\varphi\|^2_{C([-r,0];L^2(\Omega))}\\
&+ 2c|\Omega|\alpha^{*-1} + 4C_b c|\Omega|\alpha^{-1}\eta^{-1}\\
&+ 2e^{-\alpha^* t}\int_{-\infty}^{t} e^{\alpha^* s}\|g(s)\|^2 ds + 2C_b\eta^{-1}e^{-\alpha t}\int_{-\infty}^{t} e^{\alpha s}\|g(s)\|^2 ds.
\end{aligned} \tag{3.22}$$

If we take $t \ge \tau + r$ *i.e.* $t+\theta \ge \tau$, it follows

$$\begin{aligned}
&\|\nabla u(t+\theta)\|^2 \le k_3 e^{-\alpha^*(t+\theta-\tau)}\|\nabla u(\tau)\|^p + 2C_b\eta^{-1}e^{-\alpha(t+\theta-\tau)}\|u(\tau)\|^2\\
&+ 2C_b r\left(e^{-\alpha^*(t+\theta-\tau)} + C_b\eta^{-1}e^{-\alpha(t+\theta-\tau)}\right)\|\varphi\|^2_{C([-r,0];L^2(\Omega))}\\
&+ 2c|\Omega|\alpha^{*-1}\left(1-e^{-\alpha^*(t+\theta-\tau)}\right) + 4C_b c|\Omega|\alpha^{-1}\eta^{-1}\left(1-e^{-\alpha(t+\theta-\tau)}\right)\\
&+ 2e^{-\alpha^*(t+\theta)}\int_{-\infty}^{t+\theta} e^{\alpha^* s}\|g(s)\|^2 ds + 2C_b\eta^{-1}e^{-\alpha(t+\theta)}\int_{-\infty}^{t+\theta} e^{\alpha s}\|g(s)\|^2 ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|U(t,\tau)(u^0,\varphi)\|^2_{C([-r,0];H_0^1(\Omega))} = \sup_{\theta\in[-r,0]}\|\nabla u(t+\theta)\|^2,\\
&\le k_3 e^{-\alpha^*(t-r-\tau)}\|\nabla u(\tau)\|^p + 2C_b\eta^{-1}e^{-\alpha(t-r-\tau)}\|u(\tau)\|^2\\
&+ 2C_b r\left(e^{-\alpha^*(t-r-\tau)} + C_b\eta^{-1}e^{-\alpha(t-r-\tau)}\right)\|\varphi\|^2_{C([-r,0];L^2(\Omega))}\\
&+ 2c|\Omega|\alpha^{*-1}\left(1-e^{-\alpha^*(t-r-\tau)}\right) + 4C_b c|\Omega|\alpha^{-1}\eta^{-1}\left(1-e^{-\alpha(t-r-\tau)}\right)\\
&+ 2e^{-\alpha^*(t-r)}\int_{-\infty}^{t} e^{\alpha^* s}\|g(s)\|^2 ds + 2C_b\eta^{-1}e^{-\alpha(t-r)}\int_{-\infty}^{t} e^{\alpha s}\|g(s)\|^2 ds.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&\|U(t,\tau)(u^0,\varphi)\|^2_{C([-r,0];H_0^1(\Omega))}\\
&\le k_3 e^{-\alpha^*(t-r-\tau)}\|\nabla u(\tau)\|^p + 2C_b\eta^{-1}e^{-\alpha(t-r-\tau)}\|u(\tau)\|^2\\
&+ 2C_b r\left(e^{-\alpha^*(t-r-\tau)} + C_b\eta^{-1}e^{-\alpha(t-r-\tau)}\right)\|\varphi\|^2_{C([-r,0];L^2(\Omega))}\\
&+ 2c|\Omega|\left(\alpha^{*-1}e^{\alpha^* r} + 2C_b\alpha^{-1}\eta^{-1}e^{\alpha r}\right)\\
&+ 2C_b\eta^{-1}e^{-\alpha(t-r)}\int_{-\infty}^{t} e^{\alpha s}\|g(s)\|^2 ds\\
&+ 2e^{-\alpha^*(t-r)}\int_{-\infty}^{t} e^{\alpha^* s}\|g(s)\|^2 ds.
\end{aligned} \tag{3.23}$$

Similarly to the Lemma 3, let $\mathscr{R}$ be the set of all functions $\rho : \mathbb{R} \longrightarrow (0,+\infty)$ such that

$$\lim_{t\to-\infty} e^{\alpha^* t}\rho^2(t) = 0,$$

by $\mathscr{D}$ we denote the class of all families $\widehat{\mathbf{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathscr{P}(C([-r,0];H_0^1(\Omega)))$ such that $D(t) \subset \overline{\mathbf{B}}_{C([-r,0];H_0^1(\Omega))}(0,\rho(t))$, for some $\rho \in \mathscr{R}$, where we denote by $\overline{\mathbf{B}}_{C([-r,0];H_0^1(\Omega))}(0,\rho(t))$ the closed ball in $C([-r,0];H_0^1(\Omega))$ centered at 0 with radius $\rho(t)$. Let

$$\begin{aligned} R_2^2(t) &= 2c|\Omega|\left(\alpha^{*-1}e^{\alpha^* r} + 2C_b\alpha^{-1}\eta^{-1}e^{\alpha r}\right) \\ &\quad + 2C_b\eta^{-1}e^{-\alpha(t-r)}\int_{-\infty}^{t} e^{\alpha s}\|g(s)\|^2 ds + 2e^{-\alpha^*(t-r)}\int_{-\infty}^{t} e^{\alpha^* s}\|g(s)\|^2 ds. \end{aligned}$$

Thus, for all $\widehat{D} \in \mathscr{D}$ and all $t \in \mathbb{R}$, by (3.23) there exists $\tau_0(\widehat{D},t) \le t$ such that

$$\|U(t,\tau)(u^0,\varphi)\|^2_{C([-r,0];H_0^1(\Omega))} \le R_2^2(t), \tag{3.24}$$

for all $\tau \le \tau_0(\widehat{D},t)$, this means that $B_2(t) = \overline{B}_{C([-r,0];H_0^1(\Omega))}(0,R_2(t))$ is pullback $\mathscr{D}$-absorbing for the mapping $U(t,\tau)$.

The proof of the proposition is completed. ■

3.5 Existence of pullback *D*-attractor

To prove the existence of pullback $\mathscr{D}$-attractor, we need to prove the following lemma.

Lemma 4. *Assume that conditions of lemma (3) are satisfied. Then the process $\{S(t,\tau)\}$ corresponding to (1.1) is pullback $\mathscr{D}$-asymptotically compact.*

Proof. Let $t \in \mathbb{R}$, $\widehat{D} \in \mathscr{D}$, a sequences $\tau_n \to_{n\to+\infty} -\infty$ and $(u^{0,n},\varphi^n) \in D(\tau_n)$, be fixed. We have to check that the sequence

$$\{S(t,\tau_n)(u^{0,n},\varphi^n)\} = \{(u(t,\tau_n,(u^{0,n},\varphi^n)), u_t(.,\tau_n,(u^{0,n},\varphi^n)))\},$$

is relatively compact in H. In order to show this, we need to prove that the sequence

$$\{U(t,\tau_n)(u^{0,n},\varphi^n)\} = \{u_t(.,\tau_n,(u^{0,n},\varphi^n))\}$$

is relatively compact in $C([-r,0];L^2(\Omega))$. To this end, we use the Ascoli-Arzela theorem. In other words, we check

(a) the equicontinuity property for the sequence $\{u_t(.,\tau_n,(u^{0,n},\varphi^n))\} := \{u_t^n(.)\}$, i.e. $\forall\varepsilon > 0, \exists\delta > 0$ such that if $|\theta_1 - \theta_2| \le \delta$, then $\|u_t^n(\theta_1) - u_t^n(\theta_2)\| \le \varepsilon$, for all $\theta_1 > \theta_2 \in [-r,0]$;

(b) the uniform boundedness of $\{u_t^n(\theta)\}$, for all $\theta \in [-r,0]$.

In order to prove (b), we consider u^n, u the corresponding solutions to (1.1), so by Lemma 1 we can deduce that $\{u_t^n\}$ and $\{u_t\}$ are uniformly bounded in $C([-r,0];L^2(\Omega))$.

To prove (a), we proceed as follows :

$$\begin{aligned} \|u_t^n(\theta_1) - u_t^n(\theta_2)\| &= \|u(t+\theta_1) - u(t+\theta_2)\|, \\ &= \left\|\int_{t+\theta_2}^{t+\theta_1} u'(s)ds\right\|, \\ &\le \int_{t+\theta_2}^{t+\theta_1} \|u'(s)\|ds, \\ &\le \int_{t+\theta_2}^{t+\theta_1}\Big(\|\Delta u(s)\| + \|f(u(s))\| + \|b(s,u_s)\| \\ &\quad + \|g(s)\|\Big)ds. \end{aligned} \tag{3.25}$$

Now, we estimate the terms on the right hand side of this inequality

1). From the Holder inequality, we have

$$\int_{t+\theta_2}^{t+\theta_1} \|\Delta u(s)\| ds \leq \left(\int_{t+\theta_2}^{t+\theta_1} ds \right)^{1/2} \left(\int_{t+\theta_2}^{t+\theta_1} \|\Delta u(s)\|^2 ds \right)^{1/2},$$
$$\leq |\theta_1 - \theta_2|^{1/2} \left(\int_{t+\theta_2}^{t+\theta_1} \|\Delta u(s)\|^2 ds \right)^{1/2}.$$

On the one hand, we have

$$\|\Delta u\|^2 \leq \lambda_m \|\nabla u\|^2.$$

So, using this inequality in (3.22) and integrating it over $[t+\theta_2, t+\theta_1]$, one obtain

$$\begin{aligned}
&\int_{t+\theta_2}^{t+\theta_1} \|\Delta u(s)\|^2 ds \leq \lambda_m \int_{t+\theta_2}^{t+\theta_1} \|\nabla u(s)\|^2 ds \\
&\leq k_3 \lambda_m \int_{t+\theta_2}^{t+\theta_1} e^{-\alpha^*(s-\tau)} \|\nabla u(\tau)\|^p ds + 2C_b \eta^{-1} \lambda_m \int_{t+\theta_2}^{t+\theta_1} e^{-\alpha(s-\tau)} \|u(\tau)\|^2 ds \\
&+ 2C_b r \lambda_m \int_{t+\theta_2}^{t+\theta_1} \left(e^{-\alpha^*(s-\tau)} + C_b \eta^{-1} e^{-\alpha(s-\tau)} \right) \|\varphi\|^2_{C([-r,0];L^2(\Omega))} ds \\
&+ 2c|\Omega| \lambda_m \int_{t+\theta_2}^{t+\theta_1} \left(\alpha^{*-1} \left(1 - e^{-\alpha^*(s-\tau)} \right) + C_b \alpha^{-1} \eta^{-1} \left(1 - e^{-\alpha(s-\tau)} \right) \right) ds \\
&+ 2\lambda_m \int_{t+\theta_2}^{t+\theta_1} e^{-\alpha^* s} \int_{-\infty}^{s} e^{\alpha^* s'} \|g(s')\|^2 ds' ds \\
&+ 2C_b \eta^{-1} \lambda_m \int_{t+\theta_2}^{t+\theta_1} e^{-\alpha s} \int_{-\infty}^{s} e^{\alpha s'} \|g(s')\|^2 ds' ds.
\end{aligned}$$

Therefore, one obtains

$$\begin{aligned}
&\int_{t+\theta_2}^{t+\theta_1} \|\Delta u(s)\|^2 ds \\
&\leq k_3 \lambda_m \|\nabla u(\tau)\|^p \alpha^{*-1} e^{-\alpha^*(t-\tau)} \left(e^{-\alpha^* \theta_2} - e^{-\alpha^* \theta_1} \right) \\
&+ 2C_b \eta^{-1} \lambda_m \|u(\tau)\|^2 \alpha^{-1} e^{-\alpha(t-\tau)} \left(e^{-\alpha \theta_2} - e^{-\alpha \theta_1} \right) \\
&+ 2C_b r \lambda_m \|\varphi\|^2_{C([-r,0];L^2(\Omega))} \alpha^{*-1} e^{-\alpha^*(t-\tau)} \left(e^{-\alpha^* \theta_2} - e^{-\alpha^* \theta_1} \right) \\
&+ 2C_b^2 r \lambda_m \|\varphi\|^2_{C([-r,0];L^2(\Omega))} \eta^{-1} \alpha^{-1} e^{-\alpha(t-\tau)} \left(e^{-\alpha \theta_2} - e^{-\alpha \theta_1} \right) \\
&+ 2c|\Omega| \lambda_m \alpha^{*-1} \left(1 - \alpha^{*-1} e^{-\alpha^*(t-\tau)} \left(e^{-\alpha^* \theta_2} - e^{-\alpha^* \theta_1} \right) \right) \\
&+ 2c|\Omega| C_b \eta^{-1} \lambda_m \alpha^{-1} \left(1 - \alpha^{-1} e^{-\alpha(t-\tau)} \left(e^{-\alpha \theta_2} - e^{-\alpha \theta_1} \right) \right) \\
&+ 2\lambda_m \alpha^{*-1} \left(e^{-\alpha^* \theta_2} - e^{-\alpha^* \theta_1} \right) e^{-\alpha^* t} \int_{-\infty}^{t} e^{\alpha^* s'} \|g(s')\|^2 ds' \\
&+ 2C_b \eta^{-1} \lambda_m \alpha^{-1} \left(e^{-\alpha \theta_2} - e^{-\alpha \theta_1} \right) e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha s'} \|g(s')\|^2 ds' \qquad (3.26) \\
&\to 0 \text{ when } \theta_1 \to \theta_2.
\end{aligned}$$

Hence, it follows that

$$\int_{t+\theta_2}^{t+\theta_1} \|\Delta u(s)\| ds \leq |\theta_1 - \theta_2|^{1/2} \left(\int_{t+\theta_2}^{t+\theta_1} \|\Delta u(s)\|^2 ds \right)^{1/2} \to 0 \text{ when } \theta_1 \to \theta_2 \,.$$

2). From the Holder inequality, we have

$$\int_{t+\theta_2}^{t+\theta_1} \|f(u(s))\| ds \leq |\theta_1 - \theta_2|^{1/2} \cdot \left(\int_{t+\theta_2}^{t+\theta_1} \|f(u(s))\|^2 ds \right)^{1/2} . \tag{3.27}$$

Using (1.4) and the convexity of the power, one gets

$$\begin{aligned} \|f(u(t))\|^2 &= \int_\Omega |f(u(t,x))|^2 dx, \\ &\leq 2l^2 \|u(t)\|^{2(p-1)} + 2l^2 |\Omega| \,. \end{aligned}$$

Integrating this estimate over $[t+\theta_2, t+\theta_1]$, one finds

$$\int_{t+\theta_2}^{t+\theta_1} \|f(u(s))\|^2 ds \leq 2l^2 \int_{t+\theta_2}^{t+\theta_1} \|u(s)\|^{2(p-1)} ds + 2l^2 |\Omega| \cdot |\theta_1 - \theta_2| \,.$$

Since $\lambda_1 \|u\|^2 \leq \|\nabla u\|^2$, we have

$$\int_{t+\theta_2}^{t+\theta_1} \|f(u(s))\|^2 ds \leq 2l^2 \lambda_1^{(p-1)} \int_{t+\theta_2}^{t+\theta_1} \|\nabla u(s)\|^{2(p-1)} ds + 2l^2 |\Omega| \cdot |\theta_1 - \theta_2| \,. \tag{3.28}$$

From (3.22), one has

$$\begin{aligned} \|\nabla u(t)\|^{2(p-1)} \leq \Big\{ & k_3 e^{-\alpha^*(t-\tau)} \|\nabla u(\tau)\|^p + 2C_b \eta^{-1} e^{-\alpha(t-\tau)} \|u(\tau)\|^2 \\ & + 2C_b r \left(e^{-\alpha^*(t-\tau)} + C_b \eta^{-1} e^{-\alpha(t-\tau)} \right) \|\varphi\|^2_{C([-r,0];L^2(\Omega))} \\ & + 2c|\Omega| \alpha^{*-1} + 4C_b c |\Omega| \alpha^{-1} \eta^{-1} \\ & + 2e^{-\alpha^* t} \int_{-\infty}^t e^{\alpha^* s} \|g(s)\|^2 ds + 2C_b \eta^{-1} e^{-\alpha t} \int_{-\infty}^t e^{\alpha s} \|g(s)\|^2 ds \Big\}^{(p-1)} . \end{aligned}$$

By applying the convexity of power three times, one gets

$$\begin{aligned} & \|\nabla u(t)\|^{2(p-1)} \\ & \leq 2^{2(p-2)} \left(k_3 \|\nabla u(\tau)\|^p + 2C_b r \|\varphi\|^2_{C([-r,0];L^2(\Omega))} \right)^{(p-1)} e^{-(p-1)\alpha^*(t-\tau)} \\ & + 2^{2(p-2)} \left(2C_b \eta^{-1} \|u(\tau)\|^2 + 2C_b^2 r \eta^{-1} \|\varphi\|^2_{C([-r,0];L^2(\Omega))} \right)^{(p-1)} e^{-(p-1)\alpha(t-\tau)} \\ & + 2^{2(p-2)} \left(2c|\Omega| \alpha^{*-1} + 4C_b c |\Omega| \alpha^{-1} \eta^{-1} \right)^{(p-1)} \\ & + 2^{3(p-2)} 2^{(p-1)} \left(\int_{-\infty}^t e^{\alpha^* s} \|g(s)\|^2 ds \right)^{(p-1)} e^{-(p-1)\alpha^* t} \\ & + 2^{3(p-2)} (2C_b \eta^{-1})^{(p-1)} \left(\int_{-\infty}^t e^{\alpha s} \|g(s)\|^2 ds \right)^{(p-1)} e^{-(p-1)\alpha t} . \end{aligned}$$

Integrating it over $[t+\theta_2, t+\theta_1]$, one obtains

$$\begin{aligned}
&\int_{t+\theta_2}^{t+\theta_1} \|\nabla u(s)\|^{2(p-1)} ds \\
&\leq 2^{2(p-2)} \left(k_3 \|\nabla u(\tau)\|^p + 2C_b r \|\varphi\|^2_{C([-r,0];L^2(\Omega))}\right)^{(p-1)} \int_{t+\theta_2}^{t+\theta_1} e^{-(p-1)\alpha^*(s-\tau)} ds \\
&+ 2^{2(p-2)} \left(2C_b \eta^{-1} \|u(\tau)\|^2 + 2C_b^2 r \eta^{-1} \|\varphi\|^2_{C([-r,0];L^2(\Omega))}\right)^{(p-1)} \int_{t+\theta_2}^{t+\theta_1} e^{-(p-1)\alpha(s-\tau)} ds \\
&+ 2^{2(p-2)} \left(2c|\Omega|\alpha^{*-1} + 4C_b c|\Omega|\alpha^{-1}\eta^{-1}\right)^{(p-1)} |\theta_1 - \theta_2| \\
&+ 2^{3(p-2)} 2^{(p-1)} \left(\int_{-\infty}^t e^{\alpha^* s} \|g(s)\|^2 ds\right)^{(p-1)} \int_{t+\theta_2}^{t+\theta_1} e^{-(p-1)\alpha^* s} ds \\
&+ 2^{3(p-2)} (2C_b \eta^{-1})^{(p-1)} \left(\int_{-\infty}^t e^{\alpha s} \|g(s)\|^2 ds\right)^{(p-1)} \int_{t+\theta_2}^{t+\theta_1} e^{-(p-1)\alpha s} ds.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\int_{t+\theta_2}^{t+\theta_1} \|\nabla u(s)\|^{2(p-1)} ds \leq\; & C_1' e^{-(p-1)\alpha^*(t-\tau)} \left(e^{-(p-1)\alpha^*\theta_2} - e^{-(p-1)\alpha^*\theta_1}\right) \\
&+ C_2' e^{-(p-1)\alpha(t-\tau)} \left(e^{-(p-1)\alpha\theta_2} - e^{-(p-1)\alpha\theta_1}\right) \\
&+ C_3' |\theta_1 - \theta_2| + C_4' e^{-(p-1)\alpha t} \left(e^{-(p-1)\alpha\theta_2} - e^{-(p-1)\alpha\theta_1}\right) \\
&+ C_5' e^{-(p-1)\alpha^* t} \left(e^{-(p-1)\alpha^*\theta_2} - e^{-(p-1)\alpha^*\theta_1}\right) \\
&\to 0 \text{ as } \theta_1 \to \theta_2.
\end{aligned}$$

Hence by (3.27), (3.28) and this last estimate we deduce that

$$\int_{t+\theta_2}^{t+\theta_1} \|f(u(s))\| ds \to 0 \text{ as } \theta_1 \to \theta_2.$$

3). Similarly, by the Holder inequality, we have

$$\int_{t+\theta_2}^{t+\theta_1} \|b(s,u_s)\| ds \leq |\theta_1 - \theta_2|^{1/2} \cdot \left(\int_{t+\theta_2}^{t+\theta_1} \|b(s,u_s)\|^2 ds\right)^{1/2}. \tag{3.29}$$

On the other hand, by (II), (1.7) and since $\lambda_1 \|u\|^2 \leq \|\nabla u\|^2$, one has

$$\begin{aligned}
\int_{t+\theta_2}^{t+\theta_1} \|b(s,u_s)\|^2 ds &\leq C_b \int_{t+\theta_2-r}^{t+\theta_1} \|u(s)\|^2 ds \\
&\leq \int_{t+\theta_2-r}^{t+\theta_2} \|u(s)\|^2 ds + \int_{t+\theta_2}^{t+\theta_1} \|u(s)\|^2 ds \\
&\leq \|\varphi\|^2_{L^2([-r,0];L^2(\Omega))} + \lambda_1^{-1} \int_{t+\theta_2}^{t+\theta_1} \|\nabla u(s)\|^2 ds.
\end{aligned} \tag{3.30}$$

By (3.26), it follows that

$$\int_{t+\theta_2}^{t+\theta_1} \|\nabla u(s)\|^2 ds \to 0 \text{ as } \theta_1 \to \theta_2.$$

Then, from (3.29), (3.30) and this last estimate, we deduce that

$$\int_{t+\theta_2}^{t+\theta_1} \|b(s,u_s)\| ds \ \to 0 \text{ when } \theta_1 \to \theta_2 .$$

4). Finally, we use the Holder inequality to obtain

$$\int_{t+\theta_2}^{t+\theta_1} \|g(s)\| ds \leq |\theta_1-\theta_2|^{1/2} \cdot \left(\int_{t+\theta_2}^{t+\theta_1} \|g(s)\|^2 ds \right)^{1/2} .$$

Since $g \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, one gets

$$\int_{t+\theta_2}^{t+\theta_1} \|g(s)\| ds \leq |\theta_1-\theta_2|^{1/2} \cdot \|g\|_{L^2([t+\theta_2,t+\theta_1];L^2(\Omega))} \to 0 \text{ when } \theta_1 \to \theta_2 .$$

Consequently, by 1), 2), 3), 4) and (3.25), we deduce that

$$\|u(t+\theta_1)-u(t+\theta_2)\| \ \to 0 \text{ when } \theta_1 \to \theta_2 ,$$

and this ensures the equicontinuity property in $C([-r,0];L^2(\Omega))$; i.e. the sequence $\{U(t,\tau_n)(u^{0,n},\varphi^n)\}$ is relatively compact in $C([-r,0];L^2(\Omega))$.

Since we have $S(t,\tau_n)(u^{0,n},\varphi^n) = j(U(t,\tau_n)(u^{0,n},\varphi^n))$, so $\{S(t,\tau_n)(u^{0,n},\varphi^n)\}$ is relatively compact in the space $L^2(\Omega)\times C([-r,0];L^2(\Omega))$ and by the continuous injection of $L^2(\Omega)\times C([-r,0];L^2(\Omega))$ in H, we deduce that $\{S(t,\tau_n)(u^{0,n},\varphi^n)\}$ is relatively compact in H. The proof of this lemma is completed. ■

By Proposition 1 and Lemma 4, we proved that the process $S(t,\tau)$ has a pullback $\mathscr{D}$-absorbing set and it is pullback $\mathscr{D}$-asymptotically compact, then by Theorem 2 we can deduce the following result.

Theorem 4. *The process $\{S(t,\tau)\}$ corresponding to (1.1) has a pullback $\mathscr{D}$-attractor $\widehat{A} = \{A(t) : t \in \mathbb{R}\}$ in H. Furetheremore, $\widehat{A} \subset L^2(\Omega)\times C([-r,0];L^2(\Omega))$.*

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