CrossMark

Applied Mathematics and Nonlinear Sciences 2(2) (2017) 495–508

Applied Mathematics and Nonlinear Sciences

http://journals.up4sciences.org

On the triangular points within frame of the restricted three–body problem when both primaries are triaxial rigid bodies

Niraj Pathak[†], S. M. Elshaboury[‡]

Department of Mathematics, Faculty of Technology, Dharmsinh Desai University, Nadiad, Gujarat, India.
Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt.

Submission Info

Communicated by Elbaz I. Abouelmagd
Received Day 9th March 2017
Accepted Day 30th November 2017
Available online Day 30th November 2017

Abstract

In the framework of the restricted three–body problem when both primaries are triaxial rigid bodies, for different cases of Euler's angles, the locations of the triangular points, and the stability conditions of motion in the proximity of these points are constructed. The numerical solution is obtained by using a fourth order Runge–Kutta–Gill integrator with some graphical investigations.

Keywords: Restricted three–body problem, Triaxial rigid body, Euler angles.
AMS 2010 codes: 70F07, 70F10, 70F15.

1 Introduction

Restricted three–body problem (RTBP) plays very important role in celestial mechanics and space science. [12, 15] and [23] are very good books in celestial mechanics which explains importance of RTBP in space dynamics. Classical RTBP is explained in detailed in [15]. [2, 3, 6, 8] have studied RTBP with different perturbations like solar radiation pressure, oblateness, air drug etc. With both primaries as point masses. [7] have studied numerical integration with Lie series in the case of RTBP.

It is well known that the classical planar restricted three body problem (CRTBP) possesses five liberation or stationary points. These points are known as Lagrangian points. Out of these five points three points are collinear which are unstable where as two points are triangular which are stable in nature. [1, 4, 5, 11] studied

[†]Corresponding author.
Email address: niraj23481@yahoo.com
[‡]
Email address: Dr.Samyelshaboury@gmail.com

existence and stability of these Lagrangian points for the perturbed RTBP. Recently, [16–19] studied different family of periodic orbits and its stability using Poincaré surface section for perturbed RTBP. In recent times many perturbing forces i.e., oblateness and radiation forces of the primaries, Coriolis and centrifugal forces, variation of masses of the primaries and of the infinitesimal mass etc., have been included in the study of the restricted three body problem.

For the case, where the bigger primary is an oblate spheroid whose equatorial plane coincides with the plane of motion, [22] have studied the stability of the liberation points. A similar problem has been studied by [13]. [10] have studied existence and stability of the equilibrium points of the triaxial rigid body which is moving around another triaxial rigid body. [9] have studied the non-linear stability of triangular point L_4 in the RTBP when the bigger primary is a triaxial rigid body with its equatorial plane coincident with the plane of motion.

[14] have studied the problem when the smaller primary is a triaxial rigid body. Also [20, 21] have studied the problem when both the primaries are triaxial rigid bodies in the case of stationary rotational motion $(\theta_i = \psi_i = \phi_i = 0)$. In this paper we consider the restricted three body problem when both the primaries are triaxial rigid bodies in two cases of stationary rotational motion: $\left(\theta_i = \phi_i = \dfrac{\pi}{2}, \psi_i = 0\right)$ and $\left(\theta_i = 0, \psi_i + \phi_i = \dfrac{\pi}{2}\right)$. Also for Euler's angles $\left(\theta_i = \dfrac{\pi}{2}, \phi_i = \psi_i = 0\right)$, and $(\theta_i = 0, \psi_i + \phi_i = 0)$ we can do similar kind of analysis from Case–I and Case–II respectively.

2 Equations of motion

We shall adopt the notation and terminology of [23]. As a consequence, the distance between the primaries does not change and is taken equal to one; the sum of masses of the primaries is also taken one. The unit of time is chosen so as to make the gravitational constant unity. Besides this the principle axes of the primaries are oriented to the synodic axes by Euler's angels $(\theta_i, \psi_i, \phi_i\,(i = 1, 2))$. Since the axes are supposed to rotate with the same angular velocity as that of the rigid bodies and the bodies are moving around their center of mass without rotation, the Euler's angles remain constant throughout the motion. Using dimensionless variables, the equations of motion of the infinitesimal mass m_3 in a synodic coordinate system (x, y) are

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \frac{\partial \Omega}{\partial x}, \\ \ddot{y} + 2n\dot{x} &= \frac{\partial \Omega}{\partial y}. \end{aligned} \tag{1}$$

where,

$$\begin{aligned} \Omega = {} & \frac{n^2}{2}\left[(1-\mu) r_1^2 + \mu r_2^2\right] + \frac{(1-\mu)}{r_1} \\ & + \frac{\mu}{r_2} + \frac{(1-\mu)}{2m_1 r_1^3}\left[I_1 + I_2 + I_3 - 3I\right] \\ & + \frac{\mu}{2m_2 r_2^3}\left[I_1' + I_2' + I_3' - 3I'\right], \end{aligned} \tag{2}$$

$$\begin{aligned} r_1^2 &= (x-\mu)^2 + y^2, \\ r_2^2 &= (x+1-\mu)^2 + y^2. \end{aligned} \tag{3}$$

Here μ is the ratio of mass of the smaller primary to the total mass of primaries and $0 \le \mu \le \dfrac{1}{2}$, i.e., $\mu = \dfrac{m_2}{m_1 + m_2} \le \dfrac{1}{2}$ with $m_1 \ge m_2$ being the masses of the primaries. I_1, I_2, I_3 are the principal moments of inertia of the triaxial rigid body of mass m_1 at its center of mass, with a,b,c as its axes. I is the moment of inertia about a line joining the center of the rigid body of mass m_1 and the infinitesimal body of mass m_3 and is given by

$$I = I_1 l_1^{'2} + I_2 m_1^{'2} + I_3 n_1^{'2} \tag{4}$$

where $l_1^{'}, m_1^{'}$ and $n_1^{'}$ are the directional cosines of the line respect to its principal axes. $I_1^{'}, I_2^{'}, I_3^{'}$ are the principal moments of inertia of the triaxial rigid body of mass m_2 at its center of mass, with $a^{'}, b^{'}, c^{'}$ as its axes. $I^{'}$ is the moment of inertia about a line joining the center of the rigid body of mass m_2 and the infinitesimal body of mass m_3 and is given by

$$I^{'} = I_1^{'} l_2^{'2} + I_2^{'} m_2^{'2} + I_3^{'} n_2^{'2} \tag{5}$$

where $l_2^{'}, m_2^{'}$ and $n_2^{'}$ are the directional cosines of the line respect to its principal axes. We denote the unit vectors along the principle axes at $p_1\,(or p_2)$ by i, j, k and the unit vectors parallel to the synodic axes by I,J,K with the help of Euler's angles $(\theta_i, \psi_i, \phi_i)$, $(i = 1,2)$.

$$\begin{aligned} I &= a_{1i}\hat{i} + b_{1i}\hat{j} + c_{1i}\hat{k}, \\ J &= a_{2i}\hat{i} + b_{2i}\hat{j} + c_{2i}\hat{k}, \\ K &= a_{3i}\hat{i} + b_{3i}\hat{j} + c_{3i}\hat{k}, \end{aligned} \tag{6}$$

$(i = 1,2)$, where

$$\begin{aligned} a_{1i} &= -\sin\phi_i \sin\psi_i + \cos\theta_i \cos\phi_i \cos\psi_i, \\ a_{2i} &= \cos\phi_i \sin\psi_i + \cos\theta_i \sin\phi_i \cos\psi_i, \\ a_{3i} &= -\sin\theta_i \cos\psi_i, \\ b_{1i} &= -\sin\phi_i \cos\psi_i - \cos\theta_i \cos\phi_i \sin\psi_i, \\ b_{2i} &= \cos\phi_i \cos\psi_i - \cos\theta_i \sin\phi_i \sin\psi_i, \\ b_{3i} &= \sin\theta_i \sin\psi_i, \\ c_{1i} &= \sin\theta_i \cos\phi_i, \\ c_{2i} &= \sin\theta_i \sin\phi_i, \\ c_{3i} &= \cos\theta_i, \end{aligned} \tag{7}$$

$(i = 1,2)$.

The axes $O\,(xyz)$ have been defined by [23]. Now, Ω in equation (2) can be written as

$$\begin{aligned} I &= a_{1i}\hat{i} + b_{1i}\hat{j} + c_{1i}\hat{k}, \\ J &= a_{2i}\hat{i} + b_{2i}\hat{j} + c_{2i}\hat{k}, \\ K &= a_{3i}\hat{i} + b_{3i}\hat{j} + c_{3i}\hat{k}, \end{aligned} \tag{8}$$

$$\begin{aligned} \Omega = {} & \frac{n^2}{2}\left[(1-\mu)\,r_1^2 + \mu r_2^2\right] + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} \\ & + \frac{(1-\mu)}{2r_1^3}\left[2\,(A_1 + A_2 + A_3) - 3\frac{1}{r_1^2}\left\{\begin{array}{l} (A_2 + A_3)\,(a_{11}\,(x-\mu) + a_{21}y)^2 \\ +(A_1 + A_3)\,(b_{11}\,(x-\mu) + b_{21}y)^2 \\ +(A_2 + A_1)\,(c_{11}\,(x-\mu) + c_{21}y)^2 \end{array}\right\}\right] \\ & + \frac{\mu}{2r_2^3}\left[2\,(A_1^{\prime} + A_2^{\prime} + A_3^{\prime}) - 3\frac{1}{r_2^2}\left\{\begin{array}{l} (A_2^{\prime} + A_3^{\prime})\,(a_{12}\,(x+1-\mu) + a_{22}y)^2 \\ +(A_1^{\prime} + A_3^{\prime})\,(b_{12}\,(x+1-\mu) + b_{22}y)^2 \\ +(A_2^{\prime} + A_1^{\prime})\,(c_{12}\,(x+1-\mu) + c_{22}y)^2 \end{array}\right\}\right], \end{aligned} \tag{9}$$

where

$$A_1 = \frac{a^2}{5R^2}, \quad A_2 = \frac{b^2}{5R^2}, \quad A_3 = \frac{c^2}{5R^2}, \tag{10}$$

$$A_1^{'} = \frac{a^{'2}}{5R^2}, \quad A_2^{'} = \frac{b^{'2}}{5R^2}, \quad A_3^{'} = \frac{c^{'2}}{5R^2}, \tag{11}$$

and R is the distance between the primaries. The mean motion, n is given by,

$$\begin{aligned} n^2 = 1 &+ \frac{3}{2}\left[2\left(A_1+A_2+A_3\right) - 3a_{11}^2\left(A_2+A_3\right) - 3b_{11}^2\left(A_1+A_3\right) - 3c_{11}^2\left(A_2+A_1\right)\right] \\ &+ \frac{3}{2}\left[2\left(A_1'+A_2'+A_3'\right) - 3a_{12}^2\left(A_2'+A_3'\right) - 3b_{12}^2\left(A_1'+A_3'\right) - 3c_{12}^2\left(A_2'+A_1'\right)\right]. \end{aligned} \tag{12}$$

3 Locations of triangular points

Equation (1) permit an integral analogous to Jacobi integral

$$\begin{aligned} &\dot{x}^2 + \dot{y}^2 - 2\Omega + C = 0, \\ &f\left(x, y, \dot{x}, \dot{y}\right) = \dot{x}^2 + \dot{y}^2 - 2\Omega + C = 0. \end{aligned} \tag{13}$$

The liberation points are the singularities of the manifold Therefore, these points are the solutions of the equations $\Omega_x = 0$, $\Omega_y = 0$. We have Ω_x and Ω_y are established by [11, 12].

3.1 Case I: Euler's angles are: $\theta_i = \phi_i = \frac{\pi}{2}, \psi_i = 0$

In the case of $\left(\theta_i = \phi_i = \frac{\pi}{2}, \psi_i = 0\right)$, with the help of (11) the components of the unit vectors in the directions of synodic coordinates are $a_{3i} = -1, b_{1i} = -1, c_{2i} = 1$ and the other components are equal to zero.

$$\begin{aligned} \Omega_x = n^2 x &- \frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x+1-\mu)}{r_2^3} \\ &- \frac{3(1-\mu)(x-\mu)}{2r_1^5}\left[\left(2A_2+4A_3-A_1\right) - \frac{5}{r_1^2}\left(A_3(x-\mu)^2 + A_2y^2\right)\right] \\ &- \frac{3(\mu)(x+1-\mu)}{2r_2^5}\left[\left(2A_2'+4A_3'-A_1'\right) - \frac{5}{r_2^2}\left(A_3'(x+1-\mu)^2 + A_2'y^2\right)\right] = 0, \\ \Omega_y = n^2 y &- \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} \\ &- \frac{3(1-\mu)y}{2r_1^5}\left[\left(4A_2+2A_3-A_1\right) - \frac{5}{r_1^2}\left(A_3(x-\mu)^2 + A_2y^2\right)\right] \\ &- \frac{3(\mu)y}{2r_2^5}\left[\left(4A_2'+2A_3'-A_1'\right) - \frac{5}{r_2^2}\left(A_3'(x+1-\mu)^2 + A_2'y^2\right)\right] = 0, \end{aligned} \tag{14}$$

where

$$n^2 = 1 + \frac{3}{2}\left[\left(2A_2 - A_1 - A_3\right)\right] + \frac{3}{2}\left[\left(2A_2' - A_1' - A_3'\right)\right]. \tag{15}$$

We consider equation (14), Let the triaxial rigid body of mass m_1, be nearly a sphere of radius R_0, then

$$a \simeq R_0 + \sigma_1, \quad b \simeq R_0 + \sigma_2, \quad c \simeq R_0 + \sigma_3,$$

where$\sigma_1, \sigma_2, \sigma_3 \ll 1$. Therefore, using equation (10)

$$A_1 = \lambda_1 + \mu_1 \sigma_1, \tag{16}$$

where,

$$\lambda_1 = \frac{R_0^2}{5R^2}, \quad \mu_1 = \frac{2R_0}{5R^2}.$$

Similarly, $A_2 = \lambda_1 + \mu_1 \sigma_2, \quad A_3 = \lambda_1 + \mu_1 \sigma_3$ Again, let the triaxial rigid body of mass m_2, be nearly a sphere of radius R_0', using equation (11)

$$A_1' = \lambda_1' + \mu_1' \sigma_1 \tag{17}$$

where

$$\lambda_1' = \frac{R_0'^2}{5R^2}, \quad \mu_1' = \frac{2R_0'}{5R^2}$$

Similarly $A_2' = \lambda_1' + \mu_1' \sigma_2', \quad A_3' = \lambda_1' + \mu_1' \sigma_3'$ where $\sigma_1', \sigma_2', \sigma_3' \ll 1$. Therefore, equation (14) becomes,

$$\begin{aligned}
\Omega_x &= n^2 x - \frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x+1-\mu)}{r_2^3} \\
&\quad - \frac{3(1-\mu)\mu_1(x-\mu)}{2r_1^5}\left[(2\sigma_2 + 4\sigma_3 - \sigma_1) - \frac{5}{r_1^2}\left(\sigma_3 (x-\mu)^2 + \sigma_2 y^2\right)\right] \\
&\quad - \frac{3(\mu)\mu_1'(x+1-\mu)}{2r_2^5}\left[(2\sigma_2' + 4\sigma_3' - \sigma_1') - \frac{5}{r_2^2}\left(\sigma_3' (x+1-\mu)^2 + \sigma_2' y^2\right)\right] = 0, \\
\Omega_y &= n^2 y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} \\
&\quad - \frac{3(1-\mu)\mu_1 y}{2r_1^5}\left[(4\sigma_2 + 2\sigma_3 - \sigma_1) - \frac{5}{r_1^2}\left(\sigma_3 (x-\mu)^2 + \sigma_2 y^2\right)\right] \\
&\quad - \frac{3(\mu\mu_1')y}{2r_2^5}\left[(4\sigma_2' + 2\sigma_3' - \sigma_1') - \frac{5}{r_2^2}\left(\sigma_3' (x+1-\mu)^2 + \sigma_2' y^2\right)\right] = 0.
\end{aligned} \tag{18}$$

The mean motion n, given in equation (12), becomes

$$n^2 = 1 + \frac{3}{2}\mu_1 \left[(2\sigma_2 - \sigma_1 - \sigma_3)\right] + \frac{3}{2}\mu_1' \left[\left(2\sigma_2' - \sigma_1' - \sigma_3'\right)\right]. \tag{19}$$

The triangular points are the solutions of the equation (18) when $(y \neq 0)$. Now we suppose that the solution for equation (18) when $\sigma_i, \sigma_i' \, (i = 1, 2, 3)$ are not equal to zero as

$$\begin{aligned} r_1 &= 1 + \alpha, \\ r_2 &= 1 + \beta. \end{aligned} \tag{20}$$

where $\alpha, \beta \ll 1$. Putting the values of r_1 and r_2 from equation (20) in equation (3), we get,

$$\begin{aligned} x &= \mu - \frac{1}{2} + \beta - \alpha, \\ y &= \pm\frac{\sqrt{3}}{2}\left[1 + \frac{2}{3}(\beta + \alpha)\right]. \end{aligned} \tag{21}$$

Putting the values of r_1 and r_2 from equation (20) and x, y from equation (18), rejecting higher order terms, we get

$$\begin{aligned}\alpha &= -\frac{1}{8}\left[11\mu_1(\sigma_2-\sigma_3)+4\mu_1'(2\sigma_2'-\sigma_1'-\sigma_3')+\frac{4\mu\mu_1'}{(1-\mu)}(\sigma_3'-\sigma_2')\right],\\ \beta &= -\frac{1}{8}\left[4\mu_1(2\sigma_2-\sigma_1-\sigma_3)+11\mu_1'(\sigma_2'-\sigma_3')+\frac{4(1-\mu)\mu_1}{\mu}(\sigma_3-\sigma_2)\right].\end{aligned} \tag{22}$$

3.2 Case II:Euler's angles are: $\theta_i = 0, \psi_i + \phi_i = \frac{\pi}{2}$

In the case of $\left(\theta_i = 0, \psi_i + \phi_i = \frac{\pi}{2}\right)$, the components of directions cosines are $a_{2i} = 1, b_{1i} = -1, c_{3i} = 1$ while the other components are equal to zero.

$$\begin{aligned}\Omega_x &= n^2x-\frac{(1-\mu)(x-\mu)}{r_1^3}\\ &-\frac{\mu(x+1-\mu)}{r_2^3}-\frac{3(1-\mu)(x-\mu)}{2r_1^5}\left[(2A_2+4A_1-A_3)-\frac{5}{r_1^2}\left(A_1(x-\mu)^2+A_2y^2\right)\right]\\ &-\frac{3(\mu)(x+1-\mu)}{2r_2^5}\left[(2A_2'+4A_1'-A_3')-\frac{5}{r_2^2}\left(A_1'(x+1-\mu)^2+A_2'y^2\right)\right]=0,\\ \Omega_y &= n^2y-\frac{(1-\mu)y}{r_1^3}\\ &-\frac{\mu y}{r_2^3}-\frac{3(1-\mu)y}{2r_1^5}\left[(4A_2+2A_1-A_3)-\frac{5}{r_1^2}\left(A_1(x-\mu)^2+A_2y^2\right)\right]\\ &-\frac{3(\mu)y}{2r_2^5}\left[(4A_2'+2A_1'-A_3')-\frac{5}{r_2^2}\left(A_1'(x+1-\mu)^2+A_2'y^2\right)\right]=0.\end{aligned} \tag{23}$$

where,

$$n^2 = 1+\frac{3}{2}\left[(2A_2-A_1-A_3)\right]+\frac{3}{2}\left[(2A_2'-A_1'-A_3')\right] \tag{24}$$

Therefore, equation (23) becomes

$$\begin{aligned}\Omega_x &= n^2x-\frac{(1-\mu)(x-\mu)}{r_1^3}\\ &-\frac{\mu(x+1-\mu)}{r_2^3}-\frac{3(1-\mu)\mu_1(x-\mu)}{2r_1^5}\left[(2\sigma_2+4\sigma_1-\sigma_3)-\tfrac{5}{r_1^2}\left(\sigma_1(x-\mu)^2+\sigma_2y^2\right)\right]\\ &-\frac{3(\mu)\mu_1'(x+1-\mu)}{2r_2^5}\left[(2\sigma_2'+4\sigma_1'-\sigma_3')-\frac{5}{r_2^2}\left(\sigma_1'(x+1-\mu)^2+\sigma_2'y^2\right)\right]=0,\\ \Omega_y &= n^2y-\frac{(1-\mu)y}{r_1^3}-\frac{\mu y}{r_2^3}\\ &-\frac{3(1-\mu)\mu_1y}{2r_1^5}\left[(4\sigma_2+2\sigma_1-\sigma_3)-\frac{5}{r_1^2}\left(\sigma_1(x-\mu)^2+\sigma_2y^2\right)\right]\\ &-\frac{3(\mu\mu_1')y}{2r_2^5}\left[(4\sigma_2'+2\sigma_1'-\sigma_3')-\frac{5}{r_2^2}\left(\sigma_1'(x+1-\mu)^2+\sigma_2'y^2\right)\right]=0.\end{aligned} \tag{25}$$

The mean motion n, given in equation (12), becomes

$$n^2 = 1+\frac{3}{2}\mu_1\left[(2\sigma_2-\sigma_1-\sigma_3)\right]+\frac{3}{2}\mu_1'\left[(2\sigma_2'-\sigma_1'-\sigma_3')\right]. \tag{26}$$

Also the triangular points are the solutions of the equation (25) when $y \neq 0$. Putting the values of r_1 and r_2 from equation (20) and x, y from equation (25), rejecting higher order terms, we get

$$\begin{aligned}
\alpha &= -\frac{1}{8}\left[11\mu_1\left(\sigma_2-\sigma_1\right)+4\mu_1'\left(2\sigma_2'-\sigma_1'-\sigma_3'\right)+\frac{4\mu\mu_1'}{(1-\mu)}\left(\sigma_1'-\sigma_2'\right)\right],\\
\beta &= -\frac{1}{8}\left[4\mu_1\left(2\sigma_2-\sigma_3-\sigma_1\right)+11\mu_1'\left(\sigma_2'-\sigma_1'\right)+\frac{4\left(1-\mu\right)\mu_1}{\mu}\left(\sigma_1-\sigma_2\right)\right].
\end{aligned} \tag{27}$$

4 Small oscillation around equilibrium solutions

The particular solutions may be obtained with any desired degree of numerical accuracy. The next question is the examination of orbits in the vicinity of these particular solutions. In general, let X_o, Y_o be the coordinates correspond to any one of the particular solutions. They satisfy equations (1).

$$\begin{aligned}
\ddot{X}_0 - 2n\dot{Y}_0 &= \left(\frac{\partial \Omega}{\partial X}\right)_0,\\
\ddot{Y}_0 + 2n\dot{X} &= \left(\frac{\partial \Omega}{\partial Y}\right)_0.
\end{aligned} \tag{28}$$

In which the subscript o indicates that the particular derivatives of Ω must be evaluated for $X = X_o$, $Y = Y_o$. If now $X = X_0 + \xi$, $Y = Y_0 + \eta$ are substituted into the equations, there results,

$$\begin{aligned}
\ddot{\xi} - 2n\dot{\eta} &= \Omega^0_{xx}\xi + \Omega^0_{xy}\eta,\\
\ddot{\eta} + 2n\dot{\xi} &= \Omega^0_{yx}\xi + \Omega^0_{yy}\eta.
\end{aligned} \tag{29}$$

The super script of second partial derivatives of Ω refers to their values at $X = X_0$ and $Y = Y_0$. Specific numerical values may be obtained for any of the particular solutions. Let a solution of equations (29) be

$$\begin{aligned}
\xi &= A\exp\left(\lambda t\right),\\
\eta &= B\exp\left(\lambda t\right).
\end{aligned} \tag{30}$$

Substituting (30) into (29) gives for the equations that must be satisfied by the coefficients A and B

$$\lambda^4 + \left(4n^2 - \Omega^0_{xx} - \Omega^0_{yy}\right)\lambda^2 + \Omega^0_{xx}\Omega^0_{yy} - (\Omega^0_{xy})^2 = 0 \tag{31}$$

The character of the solution of the differential equations depends on the character of the solution for λ^2 from this quadratic equation. The solution is stable only if the quadratic has two unequal negative roots for λ^2. Now, we consider two cases of stationary rotational motion of the primaries.

4.1 The stability conditions of motion around triangular points

4.1.1 The stability conditions of case-I

From equation (30) we can easily obtain the stability conditions of L_4 and L_5. The triangular points of L_4 and L_5 are stable if the following conditions be satisfied (λ_i is pure imaginary)

$$\begin{aligned}
&\Omega^0_{xx} + \Omega^0_{yy} < 4n^2,\\
&\Omega^0_{xx}\Omega^0_{yy} > (\Omega^0_{xy})^2,\\
&\left(4n^2 - \Omega^0_{xx} - \Omega^0_{yy}\right)^2 > 4\left(\Omega^0_{xx}\Omega^0_{yy} - (\Omega^0_{xy})^2\right),
\end{aligned} \tag{32}$$

where Ω^0_{xx}, Ω^0_{yy} and Ω^0_{xy} are defined at L_4 and L_5 when the primaries are triaxial rigid bodies as,

$$\begin{aligned}
\Omega^0_{xx} &= n^2 - \frac{1}{4}(1-\mu)(1-9\alpha+12\beta) - \frac{1}{4}(\mu)(1-9\beta+12\alpha) \\
&\quad + \frac{3}{32}(1-\mu)\mu_1(41\sigma_3-37\sigma_2-4\sigma_1) + \frac{3}{32}(\mu)\mu_1'(41\sigma_3'-37\sigma_2'-4\sigma_1'), \\
\Omega^0_{yy} &= n^2 + \frac{1}{4}(1-\mu)(5-21\alpha+12\beta) + \frac{1}{4}(\mu)(5-21\beta+12\alpha) \\
&\quad + \frac{3}{32}(1-\mu)\mu_1(3\sigma_3+41\sigma_2-44\sigma_1) + \frac{3}{32}(\mu)\mu_1'(3\sigma_3'+41\sigma_2'-44\sigma_1'), \\
\Omega^0_{xy} &= -(1-\mu)\left[(1-3\alpha)\mp\frac{\sqrt{3}}{4}(7\alpha+4\beta-3)\right] - \mu\left[(1-3\beta)\mp\frac{\sqrt{3}}{4}(3-4\alpha-7\beta)\right] \\
&\quad - \frac{3}{2}(1-\mu)\mu_1\left(1\pm\frac{5\sqrt{3}}{4}\right)(2\sigma_2+4\sigma_3-\sigma_1) + \frac{15}{8}(1-\mu_1)\mu\left[\left(1\pm\frac{7\sqrt{3}}{4}\right)(3\sigma_2+\sigma_3)\mp 2\sqrt{3}\sigma_2\right] \\
&\quad - \frac{3}{2}(\mu)\mu_1'\left(1\mp\frac{5\sqrt{3}}{4}\right)(2\sigma_2'+4\sigma_3'-\sigma_1') + \frac{15}{8}(\mu_1')\mu\left[\left(1\mp\frac{7\sqrt{3}}{4}\right)(3\sigma_2'+\sigma_3')\pm 2\sqrt{3}\sigma_2'\right].
\end{aligned} \tag{33}$$

where the upper sign denotes to L_4 while the lower sign denotes to L_5. By analyzing the inequality (32), the stability conditions are

$$\begin{aligned}
&\Omega^0_{xx}\Omega^0_{yy} > \Omega^{02}_{xy}, \\
&\left(4n^2-\Omega^0_{xx}-\Omega^0_{yy}\right)^2 > 4\left(\Omega^0_{xx}\Omega^0_{yy}-\Omega^{02}_{xy}\right).
\end{aligned} \tag{34}$$

while the condition of unstable of these points is

$$4n^2 < \Omega^0_{xx} + \Omega^0_{yy} \tag{35}$$

4.1.2 The stability conditions of case-II

In this case the values of Ω^0_{xx}, Ω^0_{yy} and Ω^0_{xy} at the triangular points of L_4 and L_5 are given by

$$\begin{aligned}
\Omega^0_{xx} &= n^2 - \frac{1}{4}(1-\mu)(1-9\alpha+12\beta) - \frac{1}{4}(\mu)(1-9\beta+12\alpha) \\
&\quad + \frac{3}{32}(1-\mu)\mu_1(41\sigma_1-37\sigma_2-4\sigma_3) + \frac{3}{32}(\mu)\mu_1'(41\sigma_1'-37\sigma_2'-4\sigma_3') \\
\Omega^0_{yy} &= n^2 + \frac{1}{4}(1-\mu)(5-21\alpha+12\beta) + \frac{1}{4}(\mu)(5-21\beta+12\alpha) \\
&\quad + \frac{3}{32}(1-\mu)\mu_1(3\sigma_1+41\sigma_2-44\sigma_3) + \frac{3}{32}(\mu)\mu_1'(3\sigma_1'+41\sigma_2'-44\sigma_3'). \\
\Omega^0_{xy} &= -(1-\mu)\left[(1-3\alpha)\mp\frac{\sqrt{3}}{4}(7\alpha+4\beta-3)\right] - \mu\left[(1-3\beta)\mp\frac{\sqrt{3}}{4}(3-4\alpha-7\beta)\right] \\
&\quad - \frac{3}{2}(1-\mu)\mu_1\left(1\pm\frac{5\sqrt{3}}{4}\right)(2\sigma_2+4\sigma_1-\sigma_3) + \frac{15}{8}(1-\mu_1)\mu\left[\left(1\pm\frac{7\sqrt{3}}{4}\right)(3\sigma_2+\sigma_1)\mp 2\sqrt{3}\sigma_2\right] \\
&\quad - \frac{3}{2}(\mu)\mu_1'\left(1\mp\frac{5\sqrt{3}}{4}\right)(2\sigma_2'+4\sigma_1'-\sigma_3') + \frac{15}{8}(\mu_1')\mu\left[\left(1\mp\frac{7\sqrt{3}}{4}\right)(3\sigma_2'+\sigma_1')\pm 2\sqrt{3}\sigma_2'\right].
\end{aligned} \tag{36}$$

again the upper sign denotes to L_4 while the lower sign denotes to L_5, and the conditions of stability and instability are also given by the stated conditions in (34) and (35).

5 Numerical solutions

To obtain numerical solution of the given system for equations of motion of infinitesimal mass, we will use Runge–Kutta–Gill method with constant interval. This method is correct of order 4. Before applying method, we are shifting location of both primaries towards positive x – axis. So, new location of bigger primary is at(x, y) = ($-\mu$, 0) and that of smaller primary is at $(1-\mu, 0)$.

Since the standard Runge-Kutta–Gill fourth order method for first order differential equation $\dot{y} = f(x,y)$ is as follows.

$$y_{n+1} = y_n + \frac{1}{6}\left[k_1 + (2-\sqrt{2})k_2 + (2+\sqrt{2})k_3 + k_4\right]. \tag{37}$$

where

$$\begin{aligned}
k_1 &= hf(x_n, y_n),\\
k_2 &= hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}),\\
k_3 &= hf\left[x_n + \frac{h}{2}, y_n + \frac{k_1}{2}(-1+\sqrt{2}) + (1-\frac{1}{2}\sqrt{2})k_2\right],\\
k_4 &= hf\left[x_n + h, y_n - \frac{k_2}{2}\sqrt{2} + (1+\frac{1}{2}\sqrt{2})k_3\right].
\end{aligned} \tag{38}$$

Here, h is the step size. The system of equations (1)is second order ordinary differential equations. To convert equations (1) in to system of first order differential equations, consider, $y_1 = x$, $y_2 = \dot{x}$, $y_3 = y$ and $y_4 = \dot{y}$. Thus, equations (1) can be written as,

$$\begin{aligned}
\dot{y_1} &= y_2,\\
\dot{y_2} &= 2ny_4 + n^2 y_1 - \frac{(1-\mu)(y_1+\mu)}{r_1^3} - \mu(y_1+\mu-1)\left[\frac{1}{r_2^3}\right]\\
&\quad - \frac{1.5(1-\mu)(y_1+\mu)}{r_1^5}((2A_2+4A_3-A_1) - \frac{5}{r_1^2}(A_3((y_1+\mu)^2) + A_2 y_3^2))\\
&\quad - \frac{1.5\mu(y_1+\mu-1)}{r_2^5}((2A_2{'}+4A_3{'}-A_1{'}) - \frac{5}{r_2^2}(A_3{'}(y_1-1+\mu)^2 + A_2{'}y_3^2)),\\
\dot{y_3} &= y_4,\\
\dot{y_4} &= -2ny_2 + n^2 y_3 - \frac{(1-\mu)y_3}{r_1^3} - \mu y_3\left[\frac{1}{r_2^3}\right]\\
&\quad - \frac{1.5(1-\mu)y_3}{r_1^5}((4A_2+2A_3-A_1) - (\frac{5}{r_1^2})(A_3((y_1+\mu)^2) + A_2 y_3^2))\\
&\quad - \frac{1.5\mu y_3}{r_2^4}((4A_2{'}+2A_3{'}-A_1{'}) - (\frac{5}{r_2^2}(A_3{'}(y_1-1+\mu)^2 + A_2{'}y_3^2))).
\end{aligned} \tag{39}$$

We use Runge–Kutta–Gill fourth order method to integrate the system of first order differential equations. The algorithm for this method is as follows:

1. The system of equations are given by

$$\begin{aligned}
\dot{y_1} &= f_1(y_1, y_2, y_3, y_4),\\
\dot{y_2} &= f_2(y_1, y_2, y_3, y_4),\\
\dot{y_3} &= f_3(y_1, y_2, y_3, y_4),\\
\dot{y_4} &= f_4(y_1, y_2, y_3, y_4).
\end{aligned}$$

2. Let h be the step size.

3. Evaluate the following quantity:

$$
\begin{aligned}
k_1 &= hf_1(y_1,y_2,y_3,y_4),\\
l_1 &= hf_2(y_1,y_2,y_3,y_4),\\
m_1 &= hf_3(y_1,y_2,y_3,y_4),\\
n_1 &= hf_4(y_1,y_2,y_3,y_4).
\end{aligned}
$$

4. Now, update y_1,y_2,y_3 and y_4 as

$$
\begin{aligned}
y_1 &= y_1+0.5k_1,\\
y_2 &= y_2+0.5l_1,\\
y_3 &= y_3+0.5m_1,\\
y_4 &= y_4+0.5n_1.
\end{aligned}
$$

5. Evaluate the following quantity:

$$
\begin{aligned}
k_2 &= hf_1(y_1,y_2,y_3,y_4),\\
l_2 &= hf_2(y_1,y_2,y_3,y_4),\\
m_2 &= hf_3(y_1,y_2,y_3,y_4),\\
n_2 &= hf_4(y_1,y_2,y_3,y_4).
\end{aligned}
$$

6. Now, update y_1,y_2,y_3 and y_4 as

$$
\begin{aligned}
y_1 &= y_1+0.5k_1(-1+\sqrt{2})+k_2(1-0.5\sqrt{2}),\\
y_2 &= y_2+0.5l_1(-1+\sqrt{2})+l_2(1-0.5\sqrt{2}),\\
y_3 &= y_3+0.5m_1(-1+\sqrt{2})+m_2(1-0.5\sqrt{2}),\\
y_4 &= y_4+0.5n_1(-1+\sqrt{2})+n_2(1-0.5\sqrt{2}).
\end{aligned}
$$

7. Evaluate the following quantity:

$$
\begin{aligned}
k_3 &= hf_1(y_1,y_2,y_3,y_4),\\
l_3 &= hf_2(y_1,y_2,y_3,y_4),\\
m_3 &= hf_3(y_1,y_2,y_3,y_4),\\
n_3 &= hf_4(y_1,y_2,y_3,y_4).
\end{aligned}
$$

8. Now, update y_1,y_2,y_3 and y_4 as

$$
\begin{aligned}
y_1 &= y_1-\left[\frac{k_2}{\sqrt{2}}+(1+\frac{1}{\sqrt{2}})k_3\right],\\
y_2 &= y_2-\left[\frac{l_2}{\sqrt{2}}+(1+\frac{1}{\sqrt{2}})l_3\right],\\
y_3 &= y_3-\left[\frac{m_2}{\sqrt{2}}+(1+\frac{1}{\sqrt{2}})m_3\right],\\
y_4 &= y_4-\left[\frac{n_2}{\sqrt{2}}+(1+\frac{1}{\sqrt{2}})n_3\right].
\end{aligned}
$$

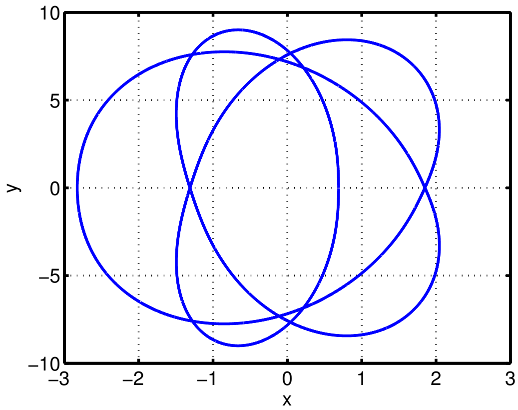

Fig. 1 Periodic orbit located at $x = 0.69$, $T = 165$ for $C = 1.92$, and $\mu = 0.01$

9. Evaluate the following quantity:

$$\begin{aligned} k_4 &= hf_1(y_1, y_2, y_3, y_4), \\ l_4 &= hf_2(y_1, y_2, y_3, y_4), \\ m_4 &= hf_3(y_1, y_2, y_3, y_4), \\ n_4 &= hf_4(y_1, y_2, y_3, y_4). \end{aligned}$$

10. Also, $y_{i,1} = x$, $y_{i,2} = \dot{x}$, $y_{i,3} = y$ and $y_{i,4} = \dot{y}$

11. Then, obtain

$$y_{i+1,1} = y_{i,1} + \frac{1}{6}\left[k_1 + (2-\sqrt{2})k_2 + (2+\sqrt{2})k_3 + k_4\right]. \tag{40}$$

12. Obtain

$$y_{i+1,2} = y_{i,2} + \frac{1}{6}\left[l_1 + (2-\sqrt{2})l_2 + (2+\sqrt{2})l_3 + l_4\right]. \tag{41}$$

13. Obtain

$$y_{i+1,3} = y_{i,3} + \frac{1}{6}\left[m_1 + (2-\sqrt{2})m_2 + (2+\sqrt{2})m_3 + m_4\right]. \tag{42}$$

14. Obtain

$$y_{i+1,4} = y_{i,4} + \frac{1}{6}\left[n_1 + (2-\sqrt{2})n_2 + (2+\sqrt{2})n_3 + n_4\right]. \tag{43}$$

15. $i = i + 1$.

16. Repeat the procedure till desire accuracy is obtained.

For obtaining periodic orbit, the values of different parameters for both cases are as follows: $A_1 = 0.01$, $A'_1 = 0.006$, $A_2 = 0.008$, $A'_2 = 0.002$, $A_3 = 0.002$ and $A'_3 = 0.001$.

5.1 Numerical solution for Case-I

Here $\theta_i = \phi_i = \dfrac{\pi}{2}, \psi_i = 0$. Figure 1 obtained using $\mu = 0.01$ and $C = 1.92$. Periodic orbit is located at $x = 0.69$ whose period is 165. Figure 2 obtained using $\mu = 0.1$ and $C = 1.1$. Periodic orbit is located at $x = 0.7124$ whose period is 173. both orbits are around both primary bodies.

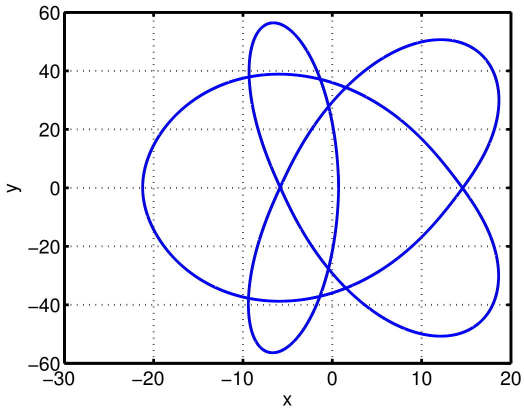

Fig. 2 Periodic orbit located at $x = 0.7124$, $T = 173$ for $C = 1.1$, and $\mu = 0.1$

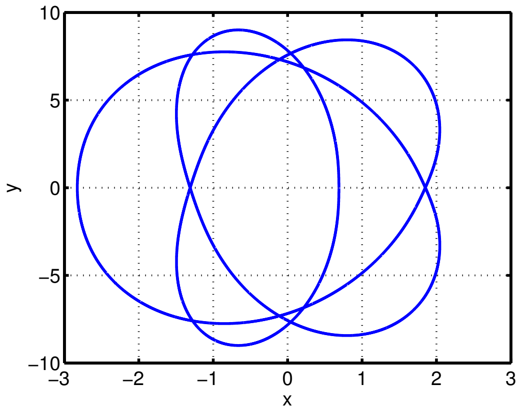

Fig. 3 Periodic orbit located at $x = 0.69$, $T = 165$ for $C = 1.92$, and $\mu = 0.01$

5.2 Numerical solution for Case-II

For obtaining periodic orbit, the values of different parameters are as following. $\theta_i = 0$, $\psi_i = \dfrac{\pi}{4}$ and $\phi_i = \dfrac{\pi}{4}$. Figure 3 obtained using $\mu = 0.01$ and $C = 1.92$. Periodic orbit is located at $x = 0.69$ whose period is 165. Figure 4 obtained using $\mu = 0.1$ and $C = 1.1$. Periodic orbit is located at $x = 0.7124$ whose period is 173. Both orbits are around both primary bodies. It can be seen that periodic orbit located at $x = 0.7124$ passing through bigger primary body very nearly whereas periodic orbit located at $x = 0.69$ passing through smaller primary body very nearly. It is observed that for both different cases, there is no change in location of periodic orbits and its period by considering value of C and μ same for both cases.

For the case of stationary rotational motion of the primaries which are triaxial rigid bodies $\left(\theta_i = \dfrac{\pi}{2}, \phi_i = \psi_i = 0\right)$. During study this case, it is worthwhile pointing out the locations of the triangular points L_4 and L_5 and the conditions of their stability can be obtained by interchanging the parameters σ_2 and σ_3 by σ_2' and σ_3' respectively in the corresponding results given in the Case-I. In addition for the case of $(\theta_i = 0, \psi_i + \phi_i = 0)$, the locations of triangular points L_4 and L_5 and the conditions of their stability can be also obtained by interchanging the parameters σ_1 and σ_2 by σ_1' and σ_2' respectively in the corresponding results given in the Case-II.

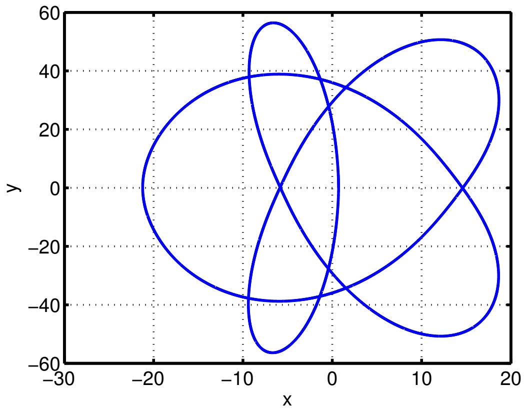

Fig. 4 Periodic orbit located at $x = 0.7124$, $T = 173$ for $C = 1.1$, and $\mu = 0.1$

6 Conclusion

In this work equations of motion of infinitesimal mass for CRTBP when both primaries are triaxial rigid bodies are analyzed. Locations of triangular points and its stability is discussed for different four cases of Euler's angles: (1) $\theta_i = \phi_i = \frac{\pi}{2}, \psi_i = 0$; (2) $\theta_i = 0, \psi_i + \phi_i = \frac{\pi}{2}$; (3) $\theta_i = \frac{\pi}{2}, \phi_i = \psi_i = 0$; (4) $\theta_i = 0, \psi_i + \phi_i = 0$. Particular the first two cases of Euler's angles are analyzed in details, the locations of the triangular points and their stability are studied. While we can do similar kind of analysis for the last two cases by using the obtained results of the first two cases. In addition the numerical solution is obtained by using a fourth order Runge–Kutta–Gill integrator with some graphical investigations.

References

[1] Abouelmagd E. I.,(2012), Existence and stability of triangular points in the restricted three–body problem with numerical applications. *Astrophysics and Space Science*, **342(1)**, 45–53. doi 10.1007/s10509-012-1162-y

[2] Abouelmagd E. I. and S. M. El-Shaboury, (2012), Periodic orbits under combined effects of oblateness and radiation in the restricted problem of three bodies. *Astrophysics and Space Science*, **341(2)**, 331–341. doi 10.1007/s10509-012-1093-7

[3] Abouelmagd E. I., (2013), The effect of photogravitational force and oblateness in the perturbed Restricted three body problem. *Astrophysics and Space Science*, **346(1)**, 51–69. doi 10.1007/s10509-013-1439-9

[4] Abouelmagd E. I. and Sharaf M. A., (2013), The motion around the libration points in the restricted three–body problem with the effect of radiation and oblateness. *Astrophysics and Space Science*, **344(2)**, 321–332. doi 10.1007/s10509-012-1335-8

[5] Abouelmagd E. I., (2013), Stability of the triangular points under combined effects of radiation and oblateness in the restricted three-body problem. *Earth, Moon, and Planets*, **110(3-4)**, 143–155. doi 10.1007/s11038-013-9415-5

[6] Abouelmagd E. I. and Guirao Juan L. G., (2016), On the perturbed restricted three-body problem. *Applied Mathematics and Nonlinear Sciences*, **1(1)**, 123–144. doi 10.21042/AMNS.2016.1.00010

[7] Abouelmagd E. I., Guirao Juan L. G. and Mostafa A., (2014), Numerical integration of the restricted thee–body problem with Lie series. *Astrophysics and Space Science*, **354(2)**, 369–378. doi 10.1007/s10509-014-2107-4

[8] Abouelmagd E. I. and Mostafa A.,(2015), Out of plane equilibrium points locations and the forbidden movement regions in the restricted three–body problem with variable mass. *Astrophysics and Space Science*, **357(1)**, 58. doi 10.1007/s10509-015-2294-7

[9] Bhatnagar K. B. and hallan P. P., (2000), The non-linear stability of L_4 in the restricted three-body problem when the bigger primary is a triaxial rigid body. *Celestial Mechanics and Dynamical Astronomy*, **77(3)**, 157–184. doi 10.1023/A:1011172032601

[10] Bhatnagar K. B. and Usha Gupta, (1986), The existence and stability of the equilibrium points of a triaxial rigid body moving around another triaxial rigid body. *Celestial Mechanics and Dynamical Astronomy*, **39**,67–83. doi 10.1007/BF01232289

[11] Choudhary R. K., (1985), On the stability of triangular libration points taking into account the light pressure for the

circular restricted problem of three bodies. *Celestial Mechanics*, **36(2)**, 165–190. doi 10.1007/BF01230650
[12] Danby J. M. A., (1962), Fundamentals of Celestial Mechanics. *Macmillan Company, New York*.
[13] Elshaboury S. M., (1991), The restricted problem of a tri-axial rigid body and two spherical bodies with variable masses. *Astrophysics and Space Science*, **186(2)**, 245–251. doi 10.1007/BF02111199
[14] Khanna M. and Bhatnagar K. B., (1999), Existence and stability of Libration points in the restricted three body problem when the smaller primary is a triaxial rigid body and the bigger one an oblate spheroid. *Indian Journal of pure and applied Mathematics*, **30(7)**, 721–734.
[15] Murray C.D. and Dermot S.F., (1999), Solar System Dynamics, *Cambridge University Press, Cambridge*.
[16] Pathak N., Sharma R. K. and Thomas V. O., (2016), Evolution of Periodic Orbits in the Sun–Saturn system. *International Journal of Astronomy and Astrophysics*, **6(2)**, 175–197. doi 10.4236/ijaa.2016.62015
[17] Pathak N. and Thomas V. O.,(2016), Evolution of the f Family Orbits in the Photo Gravitational Sun–Saturn System with Oblateness. *International Journal of Astronomy and Astrophysics*, **6(2)**, 254–271. doi 10.4236/ijaa.2016.63021
[18] Pathak N. and Thomas V. O., (2016), Analysis of Effect of Oblateness of Smaller Primary on the Evolution of Periodic Orbits. *International Journal of Astronomy and Astrophysics*, **6(3)**, 440–463. doi 10.4236/ijaa.2016.64036
[19] Pathak N. and Thomas V. O., (2016), Analysis of Effect of solar radiation pressure of bigger Primary on the Evolution of Periodic Orbits. *International Journal of Astronomy and Astrophysics*, **6(3)**, 464–493. doi 10.4236/ijaa.2016.64037
[20] Ravinder Kumar Sharma, Taqvi Z. A. and Bhatnagar K. B.,(2001), Existence and stability of libration points in the restricted three–body problem when the primaries are triaxial rigid bodies. *Celestial Mechanics And Dynamical Astronomy*, **79(2)**,119–133. doi 10.1023/A:1011168605411
[21] Ravinder Kumar Sharma, Taqvi Z. A. and Bhatnagar K. B.,(2001), Existence and stability of libration points in the restricted three body problem when the bigger primary is a triaxial rigid body and a source of radiation. *Indian Journal of pure and applied Mathematics*, **32(2)**, 255–266.
[22] Subbarao P.V. and Sharma R.K., (1975), A note on the stability of the triangular points of equilibrium in the restricted three-body problem. *Astronomy and Astrophysics*, **43**, 381–383.
[23] Szbehely V., (1967), Theory of Orbits, *Academic Press, StateplaceNew York*.