Applied Mathematics and Nonlinear Sciences

<http://journals.up4sciences.org>

Symmetry Reductions for a Generalized Fifth Order KdV Equation

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Submission Info

Communicated by Juan L.G. Guirao

Received 7th June 2017

Accepted 30th November 2017

Available online 30th November 2017

Abstract

In this work, Lie symmetry analysis is performed on a generalized fifth-order KdV equation. This equation describes many nonlinear problems with great physical interest in mathematical physics, nonlinear dynamics and plasma physics, among them it is a useful model for the description of wave phenomena in plasma and solid state and internal solitary waves in shallow waters. Group invariant solutions are obtained which allow us to transform the equation into ordinary differential equations. Furthermore, taking into account the conservation laws that the ordinary differential equation admits we reduce the order of the equations. Finally, we obtain some exact solutions.

Keywords: Generalized fifth-order KdV, Lie Symmetries, Optimal system, Exact solutions.

AMS 2010 codes: 35C07, 35Q35, 76M60.

1 Introduction

Nonlinear equations have a great interest in Mathematics and Physics because almost all of the physics problems are nonlinear due to its nature. Furthermore, a big amount of physics models are described by time evolution equations that in fact, also are nonlinear partial differential equations. That is why, nonlinear evolution equations are widely used as models to describe many important complex physical phenomena in various fields of science, such as the Einstein field equations that describe gravitational fields [16], Navier-Stokes equations of fluid dynamics [8, 24] or the Koterweg de Vries equations (KdV) which model waves on shallow water surfaces [17, 25].

In the present paper we are going to focus our attention in Koterweg de Vries equations. The KdV equation

$$u_t + \lambda uu_x + \mu u_{xxx} = 0 \quad (1)$$

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was arised to model shallow water waves with weak nonlinearities and it is probably the most studied nonlinear evolution equation due to its wide applicability.

The KdV equation (1) was generalized to a standard fifth order equation of the form

$$u_t + au^2u_x + bu_xu_{xx} + cuu_{xxx} + du_{xxxxx} = 0 \quad (2)$$

where a , b , c and d are non zero parameters and $u(x, t)$ is a sufficiently smooth function.

As the original KdV, this equation (2) models shallow water waves, however its applications are including shallow-water waves near critical value of surface tension and waves in nonlinear LC circuit with mutual inductance between neighbouring inductors [7].

In the present paper we take the first step to generalized the standard fifth order KdV (2) and extend the obtained results to a more general case in which appears an unknown function of u . Particularly, we have studied the following form of KdV of fifth order

$$u_t + u_xu_{xx} + f(u)u_x + buu_{xxx} + u_{xxxxx} = 0. \quad (3)$$

where b is a constant and f a function.

First of all, Lie point symmetries have been obtained applying the Lie classical method [11, 13, 20]. Several well-know researchers obtain symmetries because they can be use to obtain systematically exact solutions of the equations. Remark that these solutions are usually with soliton or compacton structures, which are attracting lots of interest nowadays [18, 19, 21]. Lie symmetries also play an important role simplifying models and helping to understand bifurcations of nonlinear systems.

Furthermore, once the Lie symmetries and their corresponding group invariant solutions have been obtained we have transformed the equation (3) into an ordinary differential equation and we have applied the double reduction method given by Sjöberg [23]. That way, we have solved the ordinary differential equation (ODE), whose solutions provide solutions of the original partial differential equation (3). Examples about the method are in [6, 12]

The double reduction method emphasizes and uses the relation between symmetries and conservation laws. In fact, its first step is to obtained the conservation laws for the ODE, which have been obtained applying the direct method of the multipliers [3, 4] proposed by Anco and Bluman. This method have extensively applied to partial differential equation because allow to obtain all conservation laws for the corresponding equation. Some examples can be found in [9, 10, 14].

The structure of the work is as follows. In Section 2, we have applied the Lie group method of infinitesimals transformations to the generalized fifth order KdV (3) and we have reported its reductions obtained from the optimal system of subalgebras. Moreover, in Section 3 the double reduction method have been applied to the ODE already obtained in the previous section and, of course, their conservation laws have been obtained. Finally, exact solutions for the ODE and, consequently, for the studied generalized fifth order KdV have been showed.

2 Lie Symmetries

To apply the classical method to (3) we consider the one-parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$x^* = x + \varepsilon \xi(x, t, u) + O(\varepsilon^2), \quad (4)$$

$$t^* = t + \varepsilon \tau(x, t, u) + O(\varepsilon^2),$$

$$u^* = u + \varepsilon \eta(x, t, u) + O(\varepsilon^2),$$

where ε is the group parameter. Then one requires that this transformation leaves invariant the set of solutions of (3). This yields to an overdetermined, linear system of equations for the infinitesimals $\xi(x, t, u)$, $\tau(x, t, u)$ and $\eta(x, t, u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{X} = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \eta(x, t, u) \frac{\partial}{\partial u}. \quad (5)$$

Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface condition

$$\Phi \equiv \xi \frac{\partial u}{\partial x} + \tau \frac{\partial u}{\partial t} - \eta = 0. \quad (6)$$

We consider the classical Lie group symmetry analysis of equation (3). Invariance of equation (3) under a Lie group of point transformations with infinitesimal generator (5) leads to a set of 63 determining equations for the infinitesimals $\xi(x, t, u)$, $\tau(x, t, u)$ and $\eta(x, t, u)$. The solutions of this system depends on b and $f(u)$.

1. For b arbitrary constant and f arbitrary function, the only symmetries admitted by (3) are the group of space and time translations, which are defined by the infinitesimal generators

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = \frac{\partial}{\partial t}.$$

In this case, we obtain travelling wave reductions

$$z = x - \lambda t, \quad u = h(z),$$

where $h(z)$ satisfies

$$h'''' + bhh''' + h'h'' + f(h)h' - \lambda h' = 0. \quad (7)$$

2. For b arbitrary constant and $f(u) = au^2 + c$, we obtain one extra symmetry

$$\mathbf{X}_3^1 = (x + 4ct) \frac{\partial}{\partial x} + 5t \frac{\partial}{\partial t} - 2u \frac{\partial}{\partial u}.$$

3. For $b = 0$, $f(u) = au^2 + du + c$ and $a \neq 0$, we obtain one extra symmetry, and this symmetry is defined by the following infinitesimal generator:

$$\mathbf{X}_3^2 = (x + 4ct - \frac{d^2}{a}t) \frac{\partial}{\partial x} + 5t \frac{\partial}{\partial t} - (2u + \frac{d}{a}) \frac{\partial}{\partial u}.$$

4. For $b = 0$, $f(u) = au + c$, we obtain one extra symmetry

$$\mathbf{X}_3^3 = at \frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

5. For $b = 0$, $f(u) = c$, we obtain $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3^1$ and $\mathbf{X}_3^3|_{a=0}$.

In order to determine solutions of PDE (3) that are not equivalent by the action of the group, we must calculate the one-dimensional optimal system [22]. Next we construct a table showing the separate adjoint actions of each

element in X_i^j , $i = 1, \dots, 3$ and $j = 1, 2$. This construction is done by summing the Lie series. The commutator table, and its adjoint table are as follows.

Table 1: Commutator table for the Lie algebra $\{X_1, X_2, X_3^1\}$.

	X_1	X_2	X_3^1
X_1	0	0	X_1
X_2	0	0	$4cX_1 + 5X_2$
X_3^1	$-X_1$	$-4cX_1 - 5X_2$	0

Table 2: Adjoint table for the Lie algebra $\{X_1, X_2, X_3^1\}$.

Ad	X_1	X_2	X_3^1
X_1	X_1	X_2	$X_3^1 - \varepsilon X_1$
X_2	X_1	X_2	$X_3^1 - \varepsilon(4cX_1 + 5X_2)$
X_3^1	$e^\varepsilon X_1$	Y_1	X_3^1

where

$$Y_1 = e^{5\varepsilon} X_2 + 4c\varepsilon X_1 \left(1 + \sum_{p=1}^{\infty} \left(\frac{\varepsilon^p}{(p+1)!} \sum_{q=0}^p 5^q \right) \right). \quad (8)$$

For b arbitrary constant and $f(u) = au^2 + c$, the one-dimensional optimal system of subalgebras is given by the set $\{X_3^1, \lambda X_1 + X_2\}$. Taking into account X_3^1 we obtain,

$$z = (x - ct)t^{-1/5}, \quad u = h(z)t^{-2/5},$$

where $h(z)$ satisfies

$$5h'''' + 5bh'h''' + 5h_z h'' + 5ah^2 h' - zh' - 2h = 0. \quad (9)$$

Table 3: Commutator table for the Lie algebra $\{X_1, X_2, X_3^2\}$.

	X_1	X_2	X_3^2
X_1	0	0	X_1
X_2	0	0	$\left(4c - \frac{d^2}{a}\right) X_1 + 5X_2$
X_3^2	$-X_1$	$-\left(4c - \frac{d^2}{a}\right) X_1 - 5X_2$	0

Table 4: Adjoint table for the Lie algebra $\{X_1, X_2, X_3^2\}$.

Ad	X_1	X_2	X_3^2
X_1	X_1	X_2	$X_3^2 - \varepsilon X_1$
X_2	X_1	X_2	$X_3^2 - \varepsilon \left(\left(4c - \frac{d^2}{a}\right) X_1 + 5X_2 \right)$
X_3^2	$e^\varepsilon X_1$	Y_2	X_3^2

where $Y_2 = e^{5\varepsilon} X_2 + \left(4c - \frac{d^2}{a}\right) \varepsilon X_1 \left(1 + \sum_{p=1}^{\infty} \left(\frac{\varepsilon^p}{(p+1)!} \sum_{q=0}^p 5^q \right) \right)$.

For $b = 0$, $f(u) = au^2 + du + c$ and $a \neq 0$, the one-dimensional optimal system of subalgebras is given by the set $\{X_3^2, \lambda X_1 + X_2\}$. Taking into account X_3^2 we obtain,

$$z = \left(x + \left(\frac{d^2}{4a} - c \right) t \right) t^{-1/5}, \quad u = h(z)t^{-2/5} - \frac{d}{2a},$$

where $h(z)$ satisfies

$$5h'''' + 5h'h'' + 5ah^2 h' - zh' - 2h = 0. \quad (10)$$

Table 5: Commutator table for the Lie algebra $\{X_1, X_2, X_3^3\}$.

	X_1	X_2	X_3^3
X_1	0	0	0
X_2	0	0	aX_1
X_3^3	0	$-aX_1$	0

Table 6: Adjoint table for the Lie algebra $\{X_1, X_2, X_3^3\}$.

Ad	X_1	X_2	X_3^3
X_1	X_1	X_2	X_3^3
X_2	X_1	X_2	$X_3^3 - a\varepsilon X_1$
X_3^3	X_1	$X_2 + a\varepsilon X_1$	X_3^3

For $b = 0$, $f(u) = au + c$ and $a \neq 0$, the one-dimensional optimal system of subalgebras is given by the set

$\{\lambda X_1 + X_2, X_2 + \mu X_3^3\}$. We obtain the reduction for $X_2 + \mu X_3^3$

$$z = x - \frac{a\mu t^2}{2}, \quad u = h(z) + \mu t,$$

where $h(z)$ satisfies

$$h'''' + h'h'' + ah'h' + ch' + \mu = 0. \quad (11)$$

Table 7: Commutator table for the Lie algebra $\{X_1, X_2, X_3^1, X_3^3|_{a=0}\}$.

	X_1	X_2	X_3^1	$X_3^3 _{a=0}$
X_1	0	0	X_1	0
X_2	0	0	$4cX_1 + 5X_2$	0
X_3^1	$-X_1$	$-4cX_1 - 5X_2$	0	$2X_3^3 _{a=0}$
$X_3^3 _{a=0}$	0	0	$-2X_3^3 _{a=0}$	0

Table 8: Adjoint table for the Lie algebra $\{X_1, X_2, X_3^1, X_3^3|_{a=0}\}$.

Ad	X_1	X_2	X_3^1	X_4
X_1	X_1	X_2	$X_3^1 - \varepsilon X_1$	$X_3^3 _{a=0}$
X_2	X_1	X_2	$X_3^1 - \varepsilon(4cX_1 + 5X_2)$	$X_3^3 _{a=0}$
X_3^1	$e^\varepsilon X_1$	Y_1	X_3^1	$e^{-2\varepsilon} X_3^3 _{a=0}$
$X_3^3 _{a=0}$	X_1	X_2	$X_3^1 + 2\varepsilon X_3^3 _{a=0}$	$X_3^3 _{a=0}$

where Y_1 is given by (3). The one-dimensional optimal system of subalgebras is given by the set $\{X_3^1, \lambda X_1 + X_2 + \mu X_3^3|_{a=0}\}$. For $\lambda X_1 + X_2 + \mu X_3^3|_{a=0}$

$$z = x - \lambda t, \quad u = h(z) + \mu t,$$

where $h(z)$ satisfies

$$h'''' + h'h'' + (c - \lambda)h' + \mu = 0. \quad (12)$$

3 Exact solutions by double reduction method

Another powerful application of conservation laws taking into account the relationship between Lie symmetries and conservation laws is the so called double reduction method given by Sjöberg [23]. Sjöberg introduced this method in order to get solutions of a q th partial differential equation from the solutions of an ordinary differential equation of order $q-1$.

In [5] by using the general method of conservation law multipliers Anco developed symmetry properties of conservation laws of partial differential equations. The author proved that conservation laws that are symmetry invariant or symmetry homogeneous have at least one important application: any symmetry-invariant conservation law will reduce to a first integral for the ODE obtained by symmetry reduction of the given PDE when symmetry-invariant solutions $u(t, x)$ are sought. This provides a direct reduction of order of the ODE.

In Ref. [3, 4, 22] the authors show that all non-trivial conservation laws arise from multipliers. Specifically, when we move off of the set of solutions of equation (12), every non-trivial local conservation law is equivalent to one that can be expressed in the characteristic form

$$D_z \tilde{X} = (h'''' + bh'h'' + h'h' + f(h)h' - \lambda h') Q, \quad (13)$$

where $Q(z, h, h', h'', \dots)$ is the multiplier. In general, a function $Q(z, h, h', \dots)$ is a multiplier if it is non-singular on the set of solutions $h(z)$ of equation (12), and if its product with equation (3) is a divergence expression with respect to z . There is a one-to-one correspondence between non-trivial multipliers and non-trivial conservation laws in characteristic form.

The determining equation to obtain all multipliers is

$$\frac{\delta}{\delta h} \left((h'''' + bhh''' + h'h'' + f(h)h' - \lambda h')Q \right) = 0. \quad (14)$$

This equation must hold off of the set of solutions of equation (12). Once the multipliers are found, the corresponding non-trivial conservation laws are obtained either by using a homotopy formula Ref. [2–4].

We will now find all multipliers Q , and we will obtain corresponding conservation laws. The determining equation (14) splits with respect to the variables h''', h''', h''', h'''' . This yields a linear determining system for Q which can be solved by the same algorithmic method used to solve the determining equation for infinitesimal symmetries.

For the multiplier $Q(z, h, h', h'')$ we construct the following conservation laws

- For b arbitrary constant and f arbitrary function, for the multiplier

$$Q = 1,$$

we obtain the following conservation law:

$$D_z[\int (f(h) - \lambda) dh + 1/2 (-b + 1) h'^2 + bhh'' + h'''] = 0. \quad (15)$$

- For $b = \frac{1}{2}$ and f arbitrary function, besides of the multiplier $Q = 1$ we obtain

$$Q = h.$$

We get the corresponding conservation law for $Q = u$:

$$D_z[hh'''' - h'h''' + 1/2 h^2 h'' + 1/2 h''^2 - \int h(\lambda - f(h)) dh] = 0. \quad (16)$$

- For $b = \frac{1}{2}$ and $f = \frac{3}{40} h^2 + c_1$, besides of the multiplier $Q = 1$ we obtain

$$Q = h'' + \frac{3h^2}{20}.$$

This multiplier yields to the following conservation law:

$$D_z \left[\frac{9h^5}{4000} + \frac{(200c_1 - 200\lambda + 300h'')h^3}{4000} + \frac{3h^2 h''''}{20} + \frac{(-1200h'h'''' + 1600h''^2)h}{4000} + \frac{(1200h'^2 + 4000h''')h''}{4000} \right. \\ \left. + \frac{(2000c_1 - 2000\lambda)h'^2}{4000} - 1/2 h''^2 \right] = 0. \quad (17)$$

For $b = 1/2$, from (15) and (16) we obtain the system

$$\int [f(h) - \lambda] dh + 1/4 h'^2 + 1/2 h h'' + h'''' + k_1 = 0, \quad (18)$$

$$hh'''' - h'h''' + 1/2 h^2 h'' + 1/2 h''^2 - \int h(\lambda - f(h)) dh - k_2 = 0. \quad (19)$$

From equation (18) we deduce h'''' and substituting into equation (19) we derive the equation

$$-h \int [f(h)] dh + \lambda h^2 - 1/4 h h'^2 - h k_1 - h' h''' + 1/2 h''^2 + \int (h f(h) - h \lambda) dh + k_2 = 0. \quad (20)$$

For $f(h) = \lambda$, $k_1 = k_2 = 0$ we deduce the follows solution of equation (20)

$$h(z) = e^{\int g(\zeta) d\zeta + k_3},$$

where $\frac{d}{d\zeta} g(\zeta) = 1/8 \frac{(15\zeta^2+2)(g(\zeta))^3}{\zeta} + 7/2 (g(\zeta))^2 + 1/2 \frac{g(\zeta)}{\zeta}$.

For $b = 1/2$ and $f = \frac{3}{40} h^2 + c_1$, from (15) and (17) we obtain the system

$$1/40 h^3 + h c_1 - h \lambda + 1/4 (h')^2 + 1/2 h'' h + h'''' + k_1 = 0, \quad (21)$$

$$\begin{aligned} \frac{9h^5}{4000} + \frac{(200c_1 - 200\lambda + 300h'')h^3}{4000} + \frac{3h^2 h''''}{20} + \frac{(-1200h' h''' + 1600h''^2)h}{4000} \\ + \frac{(1200h'^2 + 4000h''''')h''}{4000} + \frac{(2000c_1 - 2000\lambda)h'^2}{4000} - 1/2 h'''^2 + k_3 = 0. \end{aligned} \quad (22)$$

From equation (21) we deduce h'''' and substituting into equation (22) we derive the equation

$$\begin{aligned} -\frac{3h^5}{2000} - 1/10 c_1 h^3 + 1/10 \lambda h^3 - 1/40 h^3 h'' - \frac{3h^2 (h')^2}{80} - \frac{3h^2 k_1}{20} - 3/10 h h' h''' - 1/10 h (h'')^2 - c_1 h h'' \\ + \lambda h h'' + 1/20 h'' (h')^2 - k_1 h'' + 1/2 (h')^2 c_1 - 1/2 \lambda (h')^2 - 1/2 (h''')^2 + k_3 = 0. \end{aligned} \quad (23)$$

If $c_1 = k_1 = k_3 = 0$, we obtain the following solution for the equation (23)

$$h(z) = -40 \text{WeierstrassP}(z + z_1, (1/2)\lambda, z_2)$$

where WeierstrassP is the Weierstrass elliptic function, which is defined by $\text{WeierstrassP}(z, g_2, g_3) = 1/z^2 + \sum (1/(z - \omega)^2 - 1/\omega^2, \omega)$ where sums and products range over $\omega = 2m_1\omega_1 + 2m_2\omega_2$ such that $m_1, m_2 \in (Z \times Z) - (0, 0)$. Quantities g_2 and g_3 are known as the invariants and are related to ω_1 and ω_2 by $g_2 = 60 \sum_{\omega} \left(\frac{1}{\omega^4} \right)$, $g_3 = 140 \sum_{\omega} \left(\frac{1}{\omega^6} \right)$, [1].

4 Conclusions

In this paper, we have studied a generalized fifth-order KdV equation from Lie symmetries view point. We have established a symmetry classification of equation (3) in terms of the arbitrary constants b and the arbitrary function $f(u)$. By using the adjoint representation of the symmetry group on its Lie algebra, we have constructed an optimal system of one-dimensional subalgebras. We have obtained the similarity-reduced equations for each element of optimal system as well as some group invariant solutions. Moreover, double reduction method is used to obtain some exact solutions.

5 Acknowledgements

The authors are grateful to the Universidad de Cádiz and Junta de Andalucía (FQM-201 group) for their support. The authors also expresses their sincere gratitude to the Plan Propio de Investigación 2013 de la Universidad de Cádiz.

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