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Dirichlet series and analytical solutions of MHD viscous flow with suction / blowing

Vishwanath B. Awati [†]

Department of Mathematics, Rani Channamma University, Belagavi-591 156, INDIA

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Abstract

This paper presents Dirichlet series and approximate analytical solutions of magnetohydrodynamic (MHD) flow due to a suction / blowing caused by boundary layer of an incompressible viscous flow. The governing nonlinear partial differential equations of momentum equations are reduced into a set of nonlinear ordinary differential equations (ODE) by using a classical similarity transformation along with appropriate boundary conditions. Both nonlinearity and infinite interval demand novel mathematical tools for their analysis. We use elegant fast converging Dirichlet series and approximate analytical solutions (method of stretching of variables) of these nonlinear differential equations. These methods have advantages over pure numerical methods for obtaining derived quantities accurately for various values of the parameters involved at a stretch and also they are valid in much larger parameter domains as compared with DTM-Padé and classical numerical schemes.

Keywords: Magnetohydrodynamics (MHD); boundary layer flow; Dirichlet series; Powell method; method of stretching variables.
AMS 2010 codes: 76Dxx, 76Wxx.

1 Introduction

The study of MHD boundary layer incompressible viscous fluid flow has many important applications in engineering and science, viz. power generator, the cooling reactors, design of heat exchangers and MHD accelerator. Na [1] discussed the hydrodynamic problem of Hiemenz flow and illustrates the solution of boundary value problem using finite difference method. The stagnation-point flows of electrically conducting fluids in the presence of large transverse magnetic field strengths have been discussed by Ariel [2]. The study of MHD flow in a porous and non-porous media due to the effect of magnetic fields on the performance of many systems using electrically conducting fluids has an important concept in science and engineering. Raptis et al. [3] discussed hydrodynamic free convection flow through a porous medium between two parallel plates. Takhar and Ram [4]

[†]Corresponding author.

Email address: awati_vb@yahoo.com

have studied the MHD forced and free convection flow of water at 4°C through a porous medium in the presence of uniform transverse magnetic field for local similarity equations. Wazwaz [5] analyzed an efficient analytical and numerical procedure for solving boundary value problems for higher order integro-differential equations using Adomian decomposition method. Wazwaz [6] studied the modified decomposition method for analytic treatment of nonlinear differential equations appears in boundary layers in fluid mechanics. Wazwaz [7] discussed a treatment of two forms of third order nonlinear Blasius equation which arises from boundary layer equations by solving variational iteration method. Kuo [8] studied the thermal boundary layer problems in a semi-infinite plate using differential transformation method. Many researchers have studied Blasius equation among Arikoglu and Ozkol [9], Fang et al. [10] and Fang and Lee [11] have discussed importance for all boundary layer equations. The Blasius equation describes as non-dimensional velocity distribution in the laminar boundary layer over a semi-infinite flat plate.

The present investigation is to analyse MHD flow due to a suction / blowing caused by boundary layer of an incompressible viscous flow. The solution of resulting third order nonlinear boundary value problem with an infinite interval is obtained using Dirichlet series method and method of stretching of variables. For a specific type of boundary conditions, i.e., $f'(\infty) = 0$, Dirichlet series solution is particularly useful for obtaining solution and derived quantities exactly. The necessary conditions for existence and uniqueness of these solutions may also be found in [12, 13]. A general discussion of convergence of Dirichlet series may also be found in Riesz [14]. The accuracy as well as uniqueness of solution can be confirmed using other powerful semi-numerical schemes. Recently, Dirichlet series method has been successively employed to solve many types of boundary value problems with infinite intervals. Sachdev et al. [15] have discussed various problems arises from fluid dynamics of stretching sheet using this approach and found more accurate solution compared with earlier numerical findings. Recently, Awati and Bujurke [16], Awati et al. [17, 18] and Kudenatti et al. [19] have analysed the problems from MHD boundary layer flow with nonlinear stretching sheet using these methods and found more accurate results compared with the classical numerical methods. In this article, we present Dirichlet series solution and an approximate analytical method i.e method of stretching of variables. This method is quite easy to use especially for nonlinear ordinary differential equations and requires less computer time compared with pure numerical methods. All above successful applications which verifies validity, effectiveness and flexibility of the Dirichlet series method.

The present paper is organized as follows. In Section 2 mathematical formulation of the proposed problem with relevant boundary conditions is given. Section 3 is devoted to solution of the problem using Dirichlet series. Section 4 presents the solution by means of method of stretching of variables. In Section 5 detailed results obtained are compared with the corresponding numerical schemes and Section 6 is about the conclusion.

2 Formulation of the problem

Consider a steady two-dimensional MHD viscous flow due to suction / blowing which consist of continuity and momentum equations of viscous incompressible fluid is electrically conducting under the influence of an applied magnetic field normal to stretching sheet. The velocity components u, v along the x, y directions, respectively, and assume that U_∞ is the free stream velocity which is uniform and constant. The governing two-dimensional boundary layer equations are ([20])

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho} u, \quad (2)$$

where $\nu = \mu/\rho$ is the kinematic viscosity, (i.e., μ is the dynamic viscosity and ρ is the fluid density) and σ is the electrical conductivity of fluid; the extremal electric field and polarization effects are negligible. The magnetic

field with strength B_0 is applied in y -direction and induced magnetic field is neglected. The relevant boundary conditions for the present flow becomes

$$u = 0; \quad v = v_w \quad \text{at} \quad y = 0 \quad (3)$$

$$u = U_\infty \quad \text{at} \quad x = 0, \quad (4)$$

where v_w is the velocity across stretching sheet when $v_w < 0$ and it is blowing velocity when $v_w > 0$. Introducing the stream function $\psi(x, y)$ as

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \quad (5)$$

The mathematical significance is that equation of continuity i.e Eq. (1) is satisfied automatically and momentum boundary layer equation becomes

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3} - \frac{\sigma B_0^2}{\rho} \frac{\partial \psi}{\partial y} \quad (6)$$

Eq. (1) to (2) along with boundary conditions in both Eqs. (3) and (4) admit similarity solution. We use following similarity variables (Kuo [8]):

$$f(\eta) = \frac{\psi}{\sqrt{U_\infty \nu x}} \quad \text{where} \quad \eta = y \sqrt{\frac{U_\infty}{\nu x}}. \quad (7)$$

Substituting Eq. (7) with (5) into Eq. (6) reduce to the following nonlinear ordinary differential equation [20]

$$f''' + \frac{1}{2} f f'' - M f' = 0, \quad ' = \frac{d}{d\eta}. \quad (8)$$

The appropriate boundary conditions become (Wazwaz [5] and Wazwaz [6])

$$f(0) = f_w, \quad f'(0) = 1, \quad f'(\infty) = 0, \quad (9)$$

where $M = \frac{\sigma B_0^2}{\rho U_\infty}$ is the magnetic parameter and f_w is suction / injection parameter. In the present work we treat $f(0) = f_w$ and $f_w > 0$ correspond to suction, $f_w < 0$ correspond to blowing and $f_w = 0$ is the case when surface is impermeable.

3 Dirichlet series solution

We seek Dirichlet series solution of Eq. (8) satisfying derivative boundary condition at infinity of Eq. (9) in the form [12, 13]

$$f = \gamma_1 + 12\gamma \sum_{i=1}^{\infty} b_i a^i \exp(-i\gamma\eta), \quad (10)$$

where γ and a are parameters which are to be determined. Substituting Eq. (10) into Eq. (8), we get

$$\sum_{i=1}^{\infty} (-\gamma^2 i^3 + \frac{1}{2} \gamma \gamma_1 i^2 + M i) b_i a^i \exp(-i\gamma\eta) + 12\gamma^2 \sum_{i=2}^{\infty} \sum_{k=1}^{i-1} \left\{ \frac{1}{2} k^2 \right\} b_k b_{i-k} a^i \exp(-i\gamma\eta) = 0. \quad (11)$$

For $i = 1$, we have

$$\gamma_1 = 2(\gamma^2 - M). \quad (12)$$

Substituting Eq. (12) into Eq. (11) we get the recurrence relation in the form

$$b_i = \frac{12\gamma^2}{i(i-1)[\gamma^2 i + M]} \sum_{k=1}^{i-1} \left\{ \frac{1}{2} k^2 \right\} b_k b_{i-k},$$

where $i = 2, 3, 4, \dots$. If the Eq. (10) converges absolutely when $\gamma > 0$ for some η_0 , this series converges absolutely and uniformly in the half plane $\text{Re } \eta \geq \text{Re } \eta_0$ and represents an analytic $\frac{2\pi}{\gamma}$ periodic function $f = f(\eta_0)$ such that $f'(\infty) = 0$. The Eq. (10) contains two unknown free parameters, namely, a and γ to be determined from remaining boundary conditions of Eq. (9) at $\eta = 0$:

$$f(0) = 2 \left(\frac{\gamma^2 - M}{\gamma} \right) + 12\gamma \sum_{i=1}^{\infty} b_i a^i = f_w \quad (13)$$

and

$$f'(0) = 12\gamma^2 \sum_{i=1}^{\infty} (-i) b_i a^i = 1. \quad (14)$$

The solution of above nonlinear equations Eq. (13) and Eq. (14) yield unknown parameters a and γ . The solution of the above equations is equivalent to the unconstrained minimization of the functional

$$\left[2 \left(\frac{\gamma^2 - M}{\gamma} \right) + 12\gamma \sum_{i=1}^{\infty} b_i a^i - f_w \right]^2 + \left[12\gamma^2 \sum_{i=1}^{\infty} (-i) b_i a^i - 1 \right]^2.$$

We use Powell's method of conjugate directions (Press et al. [21]) which is one of the most efficient techniques for solving unconstrained optimization problems. This helps in finding the unknown parameters a and γ uniquely for different values of M and f_w . Alternatively, Newton's method is also used to determine the unknown parameters a and γ accurately. The shear stress at the surface of the problem is given by

$$f''(0) = 12\gamma \sum_{i=1}^{\infty} b_i a^i (i\gamma)^2.$$

The velocity profiles of the problem is given by

$$f'(\eta) = 12\gamma^2 \sum_{i=1}^{\infty} (-i) b_i a^i \exp(-i\gamma\eta).$$

4 Approximate analytical method

Most of the nonlinear ODE arising in MHD boundary layer equations which are not easy for obtaining analytical solutions. In this regards, attempts have been made to develop approximate analytical methods for the solution of these problems. Method of stretching of variables is used here for the solution of nonlinear ODE. In this method, we choose suitable derivative function H' such that derivative boundary conditions are satisfied automatically and integration of H' will satisfy remaining boundary condition. Substitutions of this resulting function into given equation gives residual of the form $R(\xi, \alpha)$ which is called "defect function". Using least squares method, the residual of the defect function can be minimized (Ariel [22]). Using the transformation $f = f_w + F$ into Eq. (8), we get

$$F''' + \frac{1}{2}(f_w + F)F'' - MF' = 0, \quad ' = \frac{d}{d\eta} \quad (15)$$

and the boundary conditions in Eq. (9) become

$$F(0) = 0, \quad F'(0) = 1, \quad F'(\infty) = 0. \quad (16)$$

f_w	Dirichlet series method			Numerical	Method of stretching of variables
	a	γ	$f''(0)$	$f''(0)$	$f''(0)$
-1	-0.0965256	1.0189385	-0.8456998	-0.849425600	-0.858678
0	-0.0621294	1.2387362	-1.0819500	-1.082828844	-1.080120
1	-0.0406485	1.5109842	-1.3537160	-1.371011260	-1.358680

Table 1 Various values of suction or blowing parameters f_w obtained by Dirichlet series and method of stretching of variables with their numerical values for $M = 1$.

We introduce two variables ξ and G in the form

$$G(\xi) = \alpha F(\eta) \quad \text{and} \quad \xi = \alpha \eta, \quad (17)$$

where $\alpha > 0$ is an amplification factor. In view of Eq. (17), the system in Eqs. (15-16) is transformed to the form

$$\alpha^2 G''' + \frac{1}{2}(f_w \alpha + G)G'' - MG' = 0, \quad ' = \frac{d}{d\xi} \quad (18)$$

and the boundary conditions in Eq. (16) become

$$G(0) = 0, \quad G'(0) = 1, \quad G'(\infty) = 0. \quad (19)$$

We choose a suitable derivative function

$$G' = \exp(-\xi) \quad (20)$$

which satisfies the derivative boundary conditions in Eq. (19). Integrating Eq. (20) with respect to ξ from 0 to ξ using conditions (19), we get

$$G = 1 - \exp(-\xi). \quad (21)$$

Substituting Eq. (21) into Eq. (18), we get the residual of defect function

$$R(\xi, \alpha) = \left(\alpha^2 - \frac{1}{2}f_w \alpha - \frac{1}{2} - M \right) \exp(-\xi) + \frac{1}{2} \exp(-2\xi). \quad (22)$$

Using the least squares method as discussed in Ariel [22], Eq. (22) can be minimized for which

$$\frac{\partial}{\partial \alpha} \int_0^\infty R^2(\xi, \alpha) d\xi = 0. \quad (23)$$

Substituting Eq. (22) into Eq. (23) and solving cubic equation for a positive root, we get

$$\alpha = \frac{f_w}{4} \quad \text{and} \quad \alpha = \frac{1}{6} \left(\frac{3}{2}f_w \pm \sqrt{3} \sqrt{2 + 12M + \frac{3}{4}f_w^2} \right). \quad (24)$$

Once the amplification factor is calculated, then using Eq. (15), the original function f can be written as

$$f = f_w + \frac{1}{\alpha} (1 - \exp(-\alpha \eta)) \quad (25)$$

with α defined in Eq. (24). Thus Eq. (25) gives the solution of Eq. (8) for different values M and f_w .

M	Dirichlet series method			Numerical	Method of stretching of variables
	a	γ	$f''(0)$	$f''(0)$	$f''(0)$
1	-0.0621294	1.2387362	-1.0819500	-1.082828844	-1.080120
2	-0.0356405	1.5870945	-1.4727840	-1.473039150	-1.471960
5	-0.0156185	2.3469129	-2.2732890	-2.273317103	-2.273030

Table 2 Various values of magnetic parameters M obtained by Dirichlet series and method of stretching of variables with their numerical values for $f_w = 0$.

Fig. 1 Velocity profiles $f'(\eta)$ for various values of porosity parameters f_w with $M = 0$.

Fig. 2 Velocity profiles $f'(\eta)$ for various values of porosity parameters f_w for $M = 1.0$.

5 Results and discussion

The computations, that have been performed for third order nonlinear boundary value problems with infinite domain, were solved semi-numerically using powerful techniques such as Dirichlet series method and approximate analytical method, viz. method of stretching of variables. In the Dirichlet series method it is important

Fig. 3 Velocity profiles $f'(\eta)$ for various values of magnetic parameters M for $f_w = 0$.

Fig. 4 Velocity profiles $f'(\eta)$ for various values of magnetic parameters M for $f_w = 1.0$.

that the edge boundary condition, $\eta \rightarrow \infty$, is automatically satisfied. The numerical computations were performed for various values of physical parameters involved in the equation, viz. the magnetic parameter M and suction/blowing parameter $s = f_w$. The present solution is also validated by comparing it with the previously published work of Thiagarajan and Senthikumar [20], as shown in both Tables 1 and 2.

The graphs for function $f'(\eta)$, i.e., velocity profiles which corresponds to velocity components u, v are plotted against η for different values of parameters f_w and M . Both Figs. 1 and 2 display the plot of dimensionless velocity field $f'(\eta)$ for suction/blowing parameter at $f_w = 1, 0, -1$ for $M = 0, M = 1$. Fig. 3 shows velocity profiles $f'(\eta)$ for different values of magnetic parameter M with suction/blowing parameter f_w . It is evident that magnetic parameter M increases while velocity profile $f'(\eta)$ decreases. This shows that the effect of magnetic field is to decelerate velocity. Fig. 4 shows the effect of magnetic parameter M with $f_w > 0$ over dimensionless velocity $f'(\eta)$. It predicts that effect of magnetic parameter is to reduce the dimensionless velocity. In Fig. 5, the dimensionless velocity for different values of magnetic parameters M and $f_w < 0$ is presented. It is inferred that for increasing values of magnetic parameter velocity profiles $f'(\eta)$ decreases. In all the figures it conveys

Fig. 5 Velocity profiles $f'(\eta)$ for various values of magnetic parameters M for $f_w = -1.0$.

Fig. 6 Variation of $f''(0)$ with f_w for various values of magnetic parameters M .

that the effect of magnetic field is to reduce velocity. Fig. 6 shows the effect of suction/ blowing parameter f_w on skin friction $f'(0)$ for different values of magnetic parameter M . It is observed that effects of magnetic field and suction/blowing parameters have similar effects over skin friction. Fig. 7 shows skin friction $f'(0)$ against magnetic parameter M for different values of suction/blowing parameters f_w . It is evident that effect of magnetic field is to decrease skin friction and also it shows that the skin friction decreases while $f_w > 0$ increases for $f_w < 0$.

6 Conclusion

In this article, we describe the analysis of boundary value problem for third order nonlinear ordinary differential equation over an infinite interval arising in MHD boundary layer equations. The Dirichlet series and approximate analytical methods, viz. method of stretching of variables are used to solve two point boundary value problem. The semi-numerical schemes described here offer advantages over solutions obtained by using

Fig. 7 Variation of $f''(0)$ with M for various values of porosity parameters f_w .

DTM-Padé and pure numerical methods, etc. The convergence of Dirichlet series method is given. The results are displayed in tables and graphically, the effects of various emerging parameters are discussed in detail. These methods are of wide applicability especially for nonlinear differential equations.

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