CrossMark

Applied Mathematics and Nonlinear Sciences 2(1) (2017) 271–284

Applied Mathematics and Nonlinear Sciences

http://journals.up4sciences.org

Solving Poisson's Equations with fractional order using Haarwavelet

I. K. Youssef[1]
M. H. El Dewaik[2] †

1. Department of Mathematics, Ain Shams University, Cairo, Email address: kaoud22@hotmail.com
2. Department of Basic Science, The British University, Cairo, Egypt

Submission Info

Communicated by E.I. Abouelmagd
Received 5th April 2017
Accepted 28th June 2017
Available online 28th June 2017

Abstract

The algebraic structure of the linear system appears in solving fractional order Poisson's equation by Haar wavelet collocation approach is considered. The fractional derivative is described in the Caputo sense. Comparison with the classical integer case as a limiting process is illustrated. Numerical comparison is made between the solution using the Haar wavelet method and the finite difference method. The results confirms the accuracy for the Haar wavelet method.

Keywords: Fractional Poisson's equation, Finite difference, Wavelet, Haar Wavelet
AMS 2010 codes: 26A33, 46F12, 65R10.

1 Introduction

Wavelet is a wave like oscillation with a magnitude that begins at zero, increases, and then decreases back to zero. It can typically be visualized as a brief oscillation like one recorded by a seismograph or heat monitor. Generally, wavelets are purposefully crafted to have specific properties that make them useful for signal processing.

The Fourier transform is a useful tool to analyze the frequency components of the signal. However, if we take the Fourier transform over the whole time axis, we cannot tell at what instant a particular frequency rises. Short time Fourier transform uses a sliding window to find spectrogram, which gives the information of both time and frequency. But still another problem exists: The length of window limits the resolution in frequency. Wavelet transform seems to be a solution to the problem above. Wavelet transforms are based on small wavelets

†Corresponding author.
Email address: marwa.eldewaik@bue.edu.eg

with limited duration. The translated-version wavelets locate where we concern. Whereas the scaled-version wavelets allow us to analyze the signal in different scale, [1] and [2]. In the last few decades many authors pointed out that derivatives and integrals of non-integer order are very suitable for the description of properties of various real material, e.g. polymers. It has been shown that new fractional order models are more adequate than previously used integer models. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer order models [1].

2 History

The first literature that relates to the wavelet transform is Haar wavelet. It was proposed by the mathematician AlfrdHaar in 1909. However, the concept of the wavelet did not exist at that time. Until 1981, the concept was proposed by the geophysicist Jean Morlet. Afterward, Morlet and the physicist Alex Grossman invented the term wavelet in 1984. Before 1985, Haar wavelet was the only orthogonal wavelet people know. A lot of researchers even thought that there was no orthogonal wavelet except Haar wavelet. Fortunately, the mathematician Yves Meyer constructed the second orthogonal wavelet called Meyer wavelet in 1985. As more and more scholars joined in this field, the 1^{st} international conference was held in France in 1987. In 1988, StephaneMallat and Meyer proposed the concept of multi resolution. In the same year, Ingrid Daubechies found a systematical method to construct the compact support orthogonal wavelet. In 1989, Mallat proposed the fast wavelet transform. With the appearance of this fast algorithm, the wavelet transform had numerous applications in the signal processing field, [1]. In 1910, Haar showed that certain square wave functions could be translated and scaled to create a basis set that span the space L^2. Years later, it was seen that the system of Haar is a particular wavelet system. In comparison with other techniques, which use the same structure of building bases functions and introduce the solution as a linear combination of those base. The Haar wavelet is simple, can implement standard algorithms with high accuracy for a small number of grid points. The simplicity in building the wavelet bases from any function which use only two operations translation and dilation [3], this can be easily seen in Haar wavelet.The simple form of the mother function in Haar wavelet as we see below makes the processes of dilation and translation an easy work and the introduced wavelet family is orthogonal not only linearly independent. Although, the wavelet function appeared in 1910, their use in the solution of differential equations does not appear until recently [4–6], last twenty years.In 2017 Kaoud and El Dewaik, [7] have used Haar wavelet technique to solve Poisson's equation on a unit square domain with collocation points $j/16\,, j = 1,3,\ldots,15$. The results obtained here can be seen as a generalization to those we have obtained in [7]. The classical integer case can be seen as limiting process as the order of the fractional derivative appears the integer case.

3 Fractional Derivatives

There are many definitions for fractional order differentiation in fractional calculus e.g: Riemann- Lioville, Caputo fractional and Grünwald-Letnikov fractional. They are given as follows, [8]:

3.1 Riemann-Liouville derivative

Let $f(x)\,\varepsilon L^1,\ \alpha \in R^+$. Then the fractional order integral of function $f(x)$ of order α is defined as

$$ {}_a^R J_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt. \tag{1}$$

The Riemann-Liouville derivative of order α , for $x \in [a,b]$,is defined by

$$ D_{RL}^{\alpha} u(x) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \int_a^x u(\xi)\,(x-\xi)^{m-\alpha-1}\,d\xi. \tag{2}$$

where$\Gamma(.)$is the Gamma function, m-1$<\alpha<$m and m=$[\alpha]+1$, with $[\alpha]$ denoting the integer part of α.

3.2 Caputo Fractional Derivatives

A different representation of the fractional derivative was proposed by Caputo,

$$D_C^{\alpha}u(x)=\frac{1}{\Gamma(m-\alpha)}\int_a^x\frac{d^m u}{d\,\xi^m}(\xi)(x-\xi)^{m-\alpha-1}d\xi. \tag{3}$$

Where , m-1$<\alpha<$m and m=$[\alpha]+1$. The Caputo representation has some advantages over the Riemann-Liouville representation. The most advantage is that the Caputo-derivatives of a constant is zero, whereas for the Riemann-Liouville is not.

3.3 Grünwald-Letnikov fractional

Another way to represent the fractional derivatives is by the Grunwald-Letnikov formula, that is, for $\alpha>0$

$$D_{GL}^{\alpha}u(x)=\lim_{\Delta x\to 0}\frac{1}{\Delta x^{\alpha}}\sum_{k=0}^{\lfloor\frac{x-\alpha}{\Delta x}\rfloor}(-1)^k\frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)}u(x-k\Delta x) \tag{4}$$

4 Haar Functions

In 1910 Haar showed that certain square wave functions could be translated and scaled to create a basis set that span $L^2([0,1])$, [9].

The scaling function should have a compact support over $0\leq x\leq 1$, therefore

$$h_0(x)=\begin{cases}1, 0<x\leq 1,\\ 0, otherwise\end{cases} \tag{5}$$

And the mother wavelets function $h_1(t)$ as:

$$h_1(x)=\begin{cases}1, & 0\leq x<\frac{1}{2},\\ -1, & \frac{1}{2}\leq x<1,\\ 0, & otherwise\end{cases} \tag{6}$$

All the other subsequent functions are generated from $h_1(x)$ with two operations: translation and dilation That is

$$h_n(x)=h_1(2^j x-k); n\ \geq 1 \tag{7}$$

where n=2^j+k, $0\leq j$, $0\leq k<2^j$.

$h_0(t)$ is also included to make this set complete.

The Haar wavelets are orthogonal in the sense,

$$\int_0^1 h_i(t)h_l(t)\,dt=2^{-j}\delta_{il}$$

$$=\begin{cases}2^{-j} & i=l=2^j+k\\ 0 & i\neq l\end{cases}$$

Therefore, they form a set of basis functions.

4.1 Function approximation

It is accepted that any square integrable functionin the interval [0,1], $y(t)\,\varepsilon\, L^2[0,1]$ can be expanded in a Haar series in the form

$$y(t)=\sum_{n=0}^{\infty} c_n h_n(t)$$

Where the coefficients c_n are determined by$c_n = 2^j \int_0^1 y(t)\, h_n(t)\, dt$
with,$n = 2^j + k, \quad j \geq 0,\ \ 0 \leq k < j$

The series expansion of $y(t)$ contains infinite terms. If $y(t)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then y(t) will be terminated at finite terms, [10] that is

$$y(t)=\sum_{n=0}^{m-1} c_n h_n(t) \ = \ \mathrm{C}_m^{\mathrm{T}} \mathrm{h}_m(t)$$

Where the coefficients vector $\mathrm{C}_m^{\mathrm{T}}$and the Haar function vector $\mathrm{h}_m(t)$ are defined as

$$\mathrm{C}_{(m)}^{\mathrm{T}} = [c_0,\ c_1, \ldots,\ c_{m-1}]$$

And

$$\mathrm{h}_m(t) = [\,h_0(t),\ h_1(t),\ \ldots,\ h_{m-1}(t)]^{\mathrm{T}}$$

where T is denotes the transpose.

To facilitate the comparison with the structured systems appears in the finite difference treatment we use eight collocation points at the points $\frac{j}{16}$, $j = 1,\ 3,\ \cdots,\ 15$and the first eight Haar wavelet can be expressed as

$$h_0(t) = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1],$$

$$h_1(t) = [1 \quad 1 \quad 1 \quad 1 \quad -1 \quad -1 \quad -1 \quad -1],$$

$$h_2(t) = [1 \quad 1 \quad -1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0],$$

$$h_3(t) = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad -1 \quad -1],$$

$$h_4(t) = [1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],$$

$$h_5(t) = [0 \quad 0 \quad 1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0],$$

$$h_6(t) = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -1 \quad 0 \quad 0],$$

$$h_7(t) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -1].$$

5 Fractional Integration of Haar wavelets

The fractional integrals of the first eight Haar wavelets can be expressed as

$$q_0 = {}_0^R J_t^{\alpha} h_0(t) = \frac{1}{\alpha\Gamma(\alpha)} t^{\alpha}, \qquad 0 \leq t < 1,$$

$$q_1 = {}_0^R J_t^{\alpha} h_1(t) = \frac{1}{\alpha\Gamma(\alpha)} \begin{cases} t^{\alpha}, & 0 \leq t < \frac{1}{2} \\ t^{\alpha} - 2(t-\frac{1}{2})^{\alpha}, & \frac{1}{2} \leq t < 1 \end{cases}$$

$$q_2 = {}_0^R J_t^{\alpha} h_2(t) = \frac{1}{\alpha\,\Gamma(\alpha)} \begin{cases} t^{\alpha}, & 0 \leq t < \frac{1}{4} \\ t^{\alpha} - 2(t-\frac{1}{4})^{\alpha}, & \frac{1}{4} \leq t < \frac{1}{2} \\ t^{\alpha} - 2(t-\frac{1}{4})^{\alpha} + (t-\frac{1}{2})^{\alpha}, & \frac{1}{2} \leq t < 1 \end{cases}$$

$$q_3 = {}_0^R J_t^\alpha h_3(t) = \frac{1}{\alpha\,\Gamma(\alpha)}\begin{cases}(t-\frac{1}{2})^\alpha, & \frac{1}{2}\le t<\frac{3}{4}\\ (t-\frac{1}{2})^\alpha-2(t-\frac{3}{4})^\alpha, & \frac{3}{4}\le t<1\end{cases}$$

$$q_4 = {}_0^R J_t^\alpha h_4(t) = \frac{1}{\alpha\,\Gamma(\alpha)}\begin{cases}t^\alpha, & 0\le t<\frac{1}{8}\\ t^\alpha-2(t-\frac{1}{8})^\alpha, & \frac{1}{8}\le t<\frac{1}{4}\\ t^\alpha-2(t-\frac{1}{8})^\alpha+(t-\frac{1}{4})^\alpha, & \frac{1}{4}\le t<1\end{cases}$$

$$q_5 = {}_0^R J_t^\alpha h_5(t) = \frac{1}{\alpha\,\Gamma(\alpha)}\begin{cases}(t-\frac{1}{4})^\alpha, & \frac{1}{4}\le t<\frac{3}{8}\\ (t-\frac{1}{4})^\alpha-2(t-\frac{3}{8})^\alpha, & \frac{3}{8}\le t<\frac{1}{2}\\ (t-\frac{1}{4})^\alpha-2(t-\frac{3}{8})^\alpha+(t-\frac{1}{2})^\alpha, & \frac{1}{2}\le t<1\end{cases}$$

$$q_6 = {}_0^R J_t^\alpha h_6(t) = \frac{1}{\alpha\,\Gamma(\alpha)}\begin{cases}(t-\frac{1}{2})^\alpha, & \frac{1}{2}\le t<\frac{5}{8}\\ (t-\frac{1}{2})^\alpha-2(t-\frac{5}{8})^\alpha, & \frac{5}{8}\le t<\frac{3}{4}\\ (t-\frac{1}{2})^\alpha-2(t-\frac{5}{8})^\alpha+(t-\frac{3}{4})^\alpha, & \frac{3}{4}\le t<1\end{cases}$$

$$q_7 = {}_0^R J_t^\alpha h_7(t) = \frac{1}{\alpha\,\Gamma(\alpha)}\begin{cases}(t-\frac{3}{4})^\alpha, & \frac{3}{4}\le t<\frac{7}{8}\\ (t-\frac{3}{4})^\alpha-2(t-\frac{7}{8})^\alpha, & \frac{7}{8}\le t<1\end{cases}$$

6 The solution of Fractional Poisson's equation using Haar wavelet method

Fractional Poisson's equation has the form

$$\frac{\partial^\alpha u}{\partial x^\alpha}+\frac{\partial^\alpha u}{\partial y^\alpha}=F(x,y),1<\alpha\le 2 \tag{8}$$

$$0\le x\le 1\ ,\ 0\le y\le 1,$$

With boundary conditions

$$\left.\begin{aligned}u(x,0)&=f_1(x)\\ u(x,1)&=f_2(x)\end{aligned}\right\}\qquad 0\le x\le 1 \tag{9}$$

$$\left.\begin{aligned}u(0,y)&=g_1(y)\\ u(1,y)&=g_2(y)\end{aligned}\right\}\qquad 0\le y\le 1. \tag{10}$$

According to the two-dimensional multi-resolution analysis, [11], any function u(x,y) which is square integrable on [0,1]×[0,1] can be expressed in terms of two dimensional Haar series as follows

$$u(x,y)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{i,j}h_i(x)h_j(y) \tag{11}$$

This series can be taken as an approximation for the solution of Poisson's equation. Moreover, the expansion of u(x,y) can be terminated.

$$u(x,y)=\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2}a_{i,j}h_i(x)h_j(y) \tag{12}$$

where the wavelet coefficients $a_{i,j}$ i=1,2,...,$2M_1$, j=1,2,...,$2M_2$are to be determined.

The approach of Haar wavelet depends on writing the dominant derivative term in the form

$$u_{x^\alpha y^\alpha}=\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2}a_{i,j}h_i(x)h_j(y) \tag{13}$$

Integrating (13) with respect to y in the limits [0,y]

$$u_{x^\alpha y^{\alpha-1}}=\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2}a_{i,j}h_i(x)P_j(y)+C_1(x) \tag{14}$$

Integrating (14) with respect to y (fractional of order $\alpha-1$), we get

$$u_{x^\alpha} = \sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_{i,j} h_i(x) q_j(y) + y^{\alpha-1}\frac{C_1(x)}{(\alpha-1)\Gamma(\alpha-1)} + C_2(x) \tag{15}$$

Using the boundary and the initial conditions we can get $C_1(x)$ *and* $C_2(x)$ .And accordingly one can obtain

$$u_{x^\alpha}(x,y) = \sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_{i,j} h_i(x)\left[q_j(y) - y^{\alpha-1}q_j(1)\right] + y^{\alpha-1}\frac{\partial^\alpha f_2(x)}{\partial x^\alpha} + (1-y^{\alpha-1})\frac{\partial^\alpha f_1(x)}{\partial x^\alpha} \tag{16}$$

Similarly, integrating (13) with respect to x in the limits $[0,x]$

$$u_{x^{\alpha-1}y^\alpha} = \sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_{i,j} h_j(y) P_i(x) + C_3(y) \tag{17}$$

Integrating (17) with respect to x (fractional of order $\alpha-1$), we get

$$u_{y^\alpha} = \sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_{i,j} h_j(y) q_i(x) + x^{\alpha-1}\frac{C_3(y)}{(\alpha-1)\Gamma(\alpha-1)} + C_4(y) \tag{18}$$

Using the boundary and the initial conditions we can get $C_3(y)$ *and* $C_4(y)$. And accordingly one can obtain

$$u_{x^\alpha}(x,y) = \sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_{i,j} h_j(y)\left[q_i(x) - x^{\alpha-1}q_i(1)\right] + x^{\alpha-1}\frac{\partial^\alpha g_2(x)}{\partial y^\alpha} + (1-x^{\alpha-1})\frac{\partial^\alpha g_1(x)}{\partial y^\alpha} \tag{19}$$

Then we Integrate equation (16) two times with respect to x and using equation (10), we obtain

$$\begin{aligned} u(x,y) &= \textstyle\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_{i,j}\left[q_i(x) - x^{\alpha-1}q_i(1)\right]\left[q_j(y) - y^{\alpha-1}q_j(1)\right] \\ &+ x^{\alpha-1}g_2(y) + \left(1-x^{\alpha-1}\right)g_1(y) + y^{\alpha-1}f_2(x) \\ &+ \left(1-y^{\alpha-1}\right)f_1(x) - x^{\alpha-1}y^{\alpha-1}f_2(1) + x^{\alpha-1}\left(1-y^{\alpha-1}\right)f_1(1) \\ &- \left(1-x^{\alpha-1}\right)y^{\alpha-1}f_2(0) - (1-x^{\alpha-1})(1-y^{\alpha-1})f_1(0) \end{aligned} \tag{20}$$

The wavelet collocation points are defined by

$$x_l = \frac{l-0.5}{2M_1}, \qquad l = 1,2,\ldots,\ 2M_1 \tag{21}$$

$$y_n = \frac{n-0.5}{2M_2}, \qquad n = 1,2,\ldots,\ 2M_2 \tag{22}$$

Substituting equations (17) and (18) in equation (8), and replacing x *by* x_l and y *by* y_n in the obtained equations and equation (16), we arrive at

$$\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_{i,j} A(i,j,l,n) = \emptyset(x_l,y_n) \tag{23}$$

Where

$$A(i,j,l,n) = h_i(x_l)\left[q_j(y_n) - y_n^{\alpha-1}q_j(1)\right] + \left[q_i(x_l) - x_l^{\alpha-1}q_i(1)\right]h_j(y_n) \tag{24}$$

$$\emptyset(x_l,y_n) = \left(y_n^{\alpha-1}-1\right)f_1''(x_l) - y_n^{\alpha-1}\frac{\partial^\alpha f_2(x_l)}{\partial x^\alpha} + \left(x_l^{\alpha-1}-1\right)\frac{\partial^\alpha g_1(y_n)}{\partial y^\alpha} - x_l^{\alpha-1}\frac{\partial^\alpha g_2(y_n)}{\partial y^\alpha} + F(x_l,y_n) \tag{25}$$

$$\begin{aligned} u(x_l,y_n) &= \textstyle\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_{i,j}\left[q_i(x_l) - x_l^{\alpha-1}q_i(1)\right]\left[q_j(y_n) - y_n^{\alpha-1}q_j(1)\right] \\ &+ x_l^{\alpha-1}g_2(y_n) + \left(1-x_l^{\alpha-1}\right)g_1(y_n) + y_n^{\alpha-1}f_2(x_l) \\ &+ \left(1-y_n^{\alpha-1}\right)f_1(x_l) - y_n^{\alpha-1}x_l^{\alpha-1}f_2(1) - x_l^{\alpha-1}\left(1-y_n^{\alpha-1}\right)f_1(1) \\ &- \left(1-x_l^{\alpha-1}\right)y_n^{\alpha-1}f_2(0) - (1-x_l^{\alpha-1})(1-y_n^{\alpha-1})f_1(0) \end{aligned} \tag{26}$$

The coefficients $a_{i,j}$, i=1,2,...,$2M_1$, j=1,2,...,$2M_2$ are found from equation (19). Then we substitute in equation (22) to obtain the Haar solution at the collocation points x_l, $l = 1,2,\ldots 2M_1$, j $= 1,2,\ldots,2M_2$.

7 Comparison between the resulting coefficients matrix in case of finite difference and Haar wavelet methods

This section is a generalization of a previous work done for the integer case [7]. The properties of the resulting linear system using Haar wavelet method are investigated.

Theorem 1 The coefficient matrix is symmetric matrix as shown in the following

$$\frac{1}{\alpha\Gamma(\alpha)}\begin{bmatrix} D_1^{\alpha} & A_1^{\alpha} & A_2^{\alpha} & A_3^{\alpha} \\ (A_1^{\alpha})^T & D_2^{\alpha} & B_1^{\alpha} & B_2^{\alpha} \\ (A_2^{\alpha})^T & (B_1^{\alpha})^T & D_3^{\alpha} & C_1^{\alpha} \\ (A_3^{\alpha})^T & (B_2^{\alpha})^T & (C_1^{\alpha})^T & D_4^{\alpha} \end{bmatrix} \tag{27}$$

Where D_1^{α} ,D_2^{α} ,D_3^{α} ,D_4^{α} ,A_1^{α} ,A_2^{α} ,A_3^{α} ,B_1^{α} ,B_2^{α} ,and C_1^{α} are illustrated in the appendix. For the integer case (at $\alpha = 2$) we have

$$D_1^2 = \frac{1}{64}\begin{bmatrix} -7 & -11 & -11 & -7 \\ -11 & -15 & -15 & -11 \\ -11 & -15 & -15 & -11 \\ -7 & -11 & -11 & -7 \end{bmatrix}, D_2^2 = \frac{1}{64}\begin{bmatrix} -3 & -3 & 3 & 3 \\ -3 & -3 & 3 & 3 \\ 3 & 3 & -3 & -3 \\ 3 & 3 & -3 & -3 \end{bmatrix}, D_3^2 = \frac{1}{128}\begin{bmatrix} 0 & 4 & 3 & 1 \\ 4 & -8 & -3 & -1 \\ 3 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix},$$

$$D_4^2 = \frac{1}{128}\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 3 \\ -1 & -3 & -8 & 4 \\ 1 & 3 & 4 & 0 \end{bmatrix}, A_1^2 = \frac{1}{64}\begin{bmatrix} -5 & -5 & 5 & 5 \\ -9 & -9 & 9 & 9 \\ -9 & -9 & 9 & 9 \\ -5 & -5 & 5 & 5 \end{bmatrix},$$

$$A_2^2 = \frac{1}{128}\begin{bmatrix} -7 & 11 & 3 & 1 \\ -15 & 19 & 3 & 1 \\ -15 & 19 & 3 & 1 \\ -7 & 11 & 3 & 1 \end{bmatrix}, A_3^2 = \frac{1}{128}\begin{bmatrix} -1 & -3 & -11 & 7 \\ -1 & -3 & -19 & 15 \\ -1 & -3 & -19 & 15 \\ -1 & -3 & -11 & 7 \end{bmatrix},$$

$$B_1^2 = \frac{1}{128}\begin{bmatrix} -3 & 7 & 3 & 1 \\ -3 & 7 & 3 & 1 \\ 3 & -7 & -3 & -1 \\ 3 & -7 & -3 & -1 \end{bmatrix}, B_2^2 = \frac{1}{128}\begin{bmatrix} -1 & -3 & -7 & 3 \\ -1 & -3 & -7 & 3 \\ 1 & 3 & 7 & -3 \\ 1 & 3 & 7 & -3 \end{bmatrix},$$

$$C_1^2 = \frac{1}{128}\begin{bmatrix} -1 & -3 & -4 & 0 \\ 1 & 3 & 8 & -4 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Theorem 2

As $\alpha \to 2$

$\frac{1}{\alpha\Gamma(\alpha)}D_1^{\alpha} \to D_1^2$, $\frac{1}{\alpha\Gamma(\alpha)}D_2^{\alpha} \to D_2^2$,
$\frac{1}{\alpha\Gamma(\alpha)}D_3^{\alpha} \to D_3^2$, $\frac{1}{\alpha\Gamma(\alpha)}D_4^{\alpha} \to D_4^2$,
$\frac{1}{\alpha\Gamma(\alpha)}A_1^{\alpha} \to A_1^2$, $\frac{1}{\alpha\Gamma(\alpha)}A_2^{\alpha} \to A_2^2$,
$\frac{1}{\alpha\Gamma(\alpha)}A_3^{\alpha} \to A_3^2 \frac{1}{\alpha\Gamma(\alpha)}B_1^{\alpha} \to B_1^2$,
$\frac{1}{\alpha\Gamma(\alpha)}B_2^{\alpha} \to B_2^2$ and $\frac{1}{\alpha\Gamma(\alpha)}C_1^{\alpha} \to C_1^2$.

Finite difference approximations of fractional derivatives

In (2014) I.K.Youssef and A.M.Shoukr, [12], represented the structure of the coefficient matrix of fractional Poisson's equation using finite difference method.

In this method the integral in Caputo's formula is replaced by a finite sum of integrals at the discretization points, and approximate the second order derivative by using the standard finite difference formula, then the finite difference formula of fractional Poisson's equation takes the form:

$$\sum_{k=0}^{i-1} b_k(U_{i-k+1,j} - 2U_{i-k,j} + U_{i-k-1,j} \quad + b_s^*(U_{i,j-s+1} - 2U_{i,j-s} + U_{i,j-s-1}) = f_{i,j}$$

The structure of coefficient matrix

$$A = \begin{bmatrix} A''1 & (b_0^*)*I & 0 & 0 \\ (b_0^* - 2b_1^*)*I & A1 & (b_0^*)*I & 0 \\ (b_1^* - 2b_2^*)*I & (b_0^* - 2b_1^* - 2b_2^*)*I & A1 & (b_0^*)*I \\ (b_2^* - 2b_3^*)*I & (b_1^* - 2b_2^* - 2b_3^*)*I & (b_0^* - 2b_1^* - 2b_2^*)*I & A1 \end{bmatrix},$$

$$A1 = \begin{bmatrix} (b_1 - 2b_0 - 2b_0^* + b_1^*) & b_0 & 0 & 0 \\ b_0 - 2b_1 & (b_1 - 2b_0 - 2b_0^* + b_1^*) & b_0 & 0 \\ b_1 - 2b_2 & (b_0^* - 2b_1^* - 2b_2^*)*I & (b_1 - 2b_0 - 2b_0^* + b_1^*) & b_0 \\ b_2 - 2b_3 & (b_1^* - 2b_2^* - 2b_3^*)*I & (b_0^* - 2b_1^* - 2b_2^*)*I & (b_1 - 2b_0 - 2b_0^* + b_1^*) \end{bmatrix}$$

$$(A''1)_{ij} = \begin{cases} -2b_0 - 2b_0^\bullet & if\, i = j \\ b_1 - 2b_0 - 2b_0^* + b_1^\bullet & if\, i = j = 2,3,\ldots,N-1 \\ b_0 & if\, j = i+1, i = 1,2,\ldots,N-2 \\ b_{i-2} - 2b_{i-1} & if\, i = 2,3,\ldots,N-1,\ j = 1 \\ b_{i-j-1} - 2b_{i-j} - 2b_{i-j+1} & if\, i \succ j,\ j = 2,3,\ldots,N-1,\ j = 1 \\ 0 & othereise \end{cases}$$

While in case of the finite difference method the resulting coefficient matrix is block tri diagonal matrix with the natural ordering is considered [13], [14].

8 Numerical Results and Discussion

The following fractional Poisson's equation:

$$\frac{\partial^\alpha U(x,y)}{\partial x^\alpha} + \frac{\partial^\alpha U(x,y)}{\partial y^\alpha} = f(x,y)$$

Was considered on a finite domain $0 \le x \le 1$and $0 \le y \le 1$ with the non-homogeneous function $f(x,y) = \Gamma(\alpha+1)(x^\alpha + y^\alpha)$ and the boundary conditions:

$$U(x,0) = U(0,y) = 0, U(x,1) = x^\alpha,\ U(1,\ y) = y^\alpha.$$

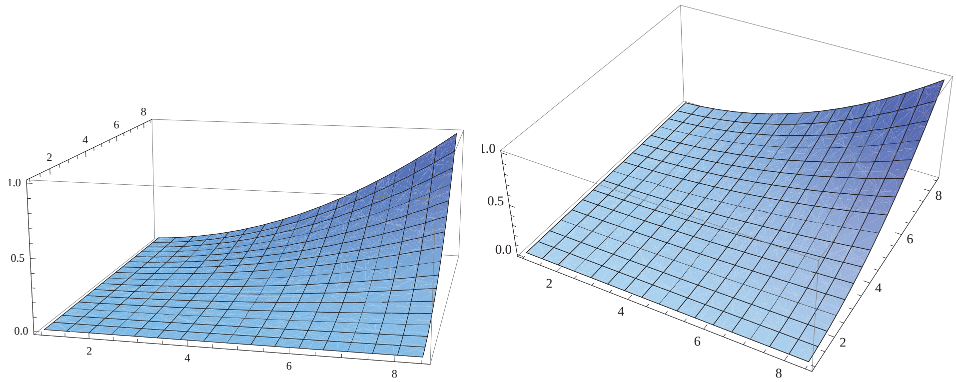
This fractional Poisson's equation has the exact solution $U(x,y) = (xy)^\alpha$. The fractional Poisson absolute error is defined by:

Error = $\frac{1}{(m-1)^2}\sqrt{\sum_{i,j=1}^{m-1}(U_{i,j} - u_{i,j})^2}$, in which $U_{i,j}$ and $u_{i,j}$ are the exact and numerical solutions respectively, [15].

This problem is solved using Haar wavelet method. The results show higher accuracy compared with the finite difference method, [15].

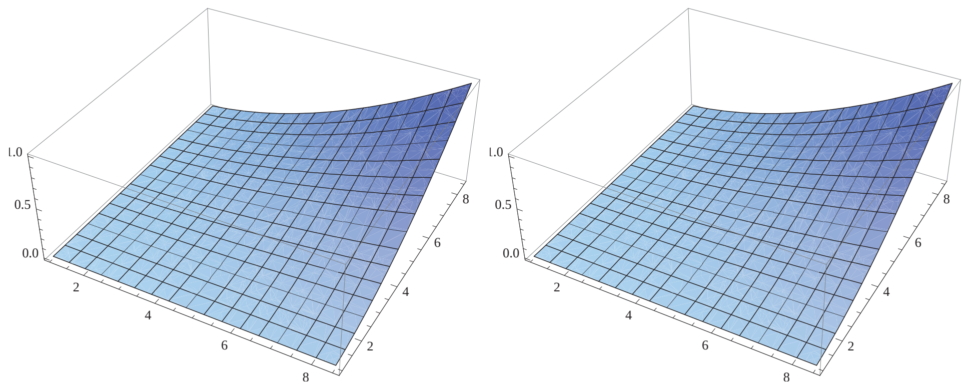
The approximate solution at $\alpha = 2$ error of order 10^{-19}

The approximate solution at $\alpha = 1.75$ error of order 10^{-18}



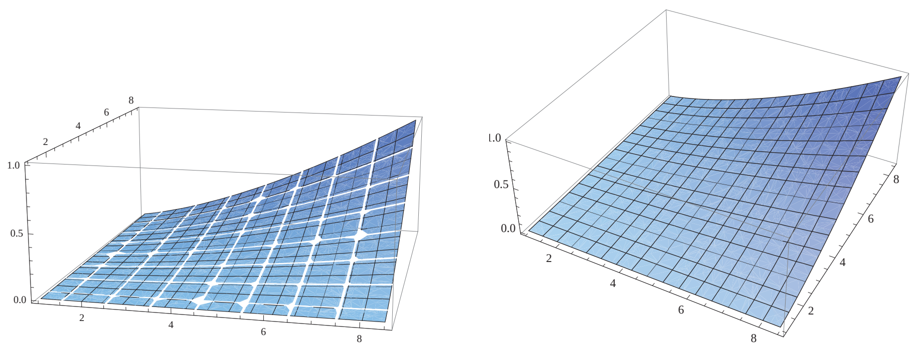
(a) *The approximate solution at* $\alpha = 2$. (b) *The exact solution at* $\alpha = 2$

Fig. 1 Comparison between the approximate and exact solutions when $\alpha = 2$



(a) *The approximate solution at* $\alpha = 1.75$. (b) *The exact solution at* $\alpha = 1.75$

Fig. 2 Comparison between the approximate and exact solutions when $\alpha = 1.75$

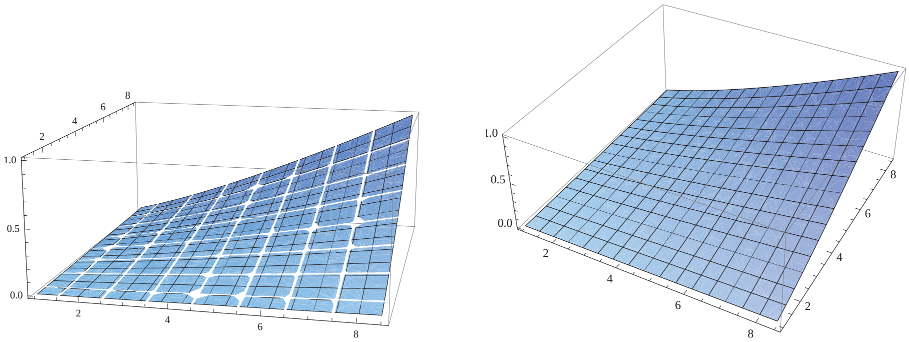


(a) *The approximate solution at* $\alpha = 1.5$. (b) *The exact solution at* $\alpha = 1.5$

Fig. 3 Comparison between the approximate and exact solutions when $\alpha = 1.5$

9 Conclusion

The wavelet solution gives reliable results for the fractional order Poisson's equation as in the integer case. The numerical results obtained generalize the results of the classical integer case. Moreover, the matrix structure



(a) The approximate solution at $\alpha = 1.25$.

(b) The exact solution at $\alpha = 1.25$

Fig. 4 Comparison between the approximate and exact solutions when $\alpha = 1.25$

of the linear system (27) has the symmetric block structure. Comparison with the corresponding matrix appears in the finite difference treatment help in building the block structure [12]. The memory and hereditary behaviors of the fractional order derivatives appears with the coefficients through the Gamma function factors.

10 Appendix

$$D_1^{\alpha} = [d1_{ij}]$$

$$d1_{11} = \frac{-1}{4} + 2^{1-3\alpha}, d1_{22} = \frac{-3}{4} + 2^{1-3\alpha}(3)^{\alpha}, d1_{33} = \frac{-5}{4} + 2^{1-3\alpha}(5)^{\alpha}, d1_{44} = \frac{-7}{4} + 2^{1-3\alpha}(7)^{\alpha}$$

$$d1_{12} = d1_{21} = \frac{-1}{2} + (\frac{3}{8})^{\alpha} + 8^{-\alpha}, d1_{13} = d1_{31} = \frac{-3}{4} + (\frac{5}{8})^{\alpha} + 8^{-\alpha}$$

$$d1_{14} = d_{41} = -1 + (\frac{7}{8})^{\alpha} + 8^{-\alpha},\ d1_{23} = d1_{32} = -1 + (\frac{3}{8})^{\alpha} + (\frac{5}{8})^{\alpha}$$

$$d1_{24} = d_{42} = \frac{-5}{4} + (\frac{3}{8})^{\alpha} + (\frac{5}{8})^{\alpha}, d1_{34} = d1_{43} = \frac{-3}{2} + (\frac{5}{8})^{\alpha} + (\frac{7}{8})^{\alpha}.$$

$$D_2^{\alpha} = [d2_{ij}]$$

$$d2_{11} = \frac{-1}{4} + 2^{1-3\alpha} + 2^{-1-\alpha}, d2_{22} = \frac{-3}{4} + 2^{1-3\alpha}(3)^{\alpha} + 3(2)^{-1-\alpha},$$

$$d2_{33} = 2^{-2-3\alpha}(16 - 5\,(2)^{1+2\alpha} - 8\ (5)^{\alpha} + 5(8)^{\alpha},$$

$$d2_{44} = 2^{-1-3\alpha}(4 - 3\,(2)^{1+2\alpha} + 4\ (3)^{\alpha} - 2\,(5)^{\alpha} - 2\,(7)^{\alpha} + 3\,(8)^{\alpha}),$$

$$d2_{12} = d2_{21} = \frac{-1}{2} + (\frac{3}{8})^{\alpha} + 3\,(2)^{-2-\alpha} + 8^{-\alpha}, d2_{13} = d2_{31} = \frac{-1}{2} + (\frac{5}{8})^{\alpha} + (2)^{-\alpha} + 3(8)^{-\alpha}$$

$$d2_{14} = d2_{21} = 2^{-2-3\alpha}(-4 + 3\,(2)^{1+2\alpha} - 8\ (3)^{\alpha} + 4\,(7)^{\alpha} - 3\,(8)^{\alpha}),$$

$$d2_{23} = d2_{32} = 2^{-2-3\alpha}(-8 + (2)^{1+2\alpha} - 4\ (3)^{\alpha} + 4\,(5)^{\alpha} - (8)^{\alpha}),$$

$$d2_{24} = d2_{42} = \frac{-1}{2} + (\frac{7}{8})^{\alpha} + 2^{-\alpha} + 3^{1+\alpha}\, 8^{-\alpha}\quad ,$$

$$d2_{34} = d2_{43} = 2^{-1-3\alpha}(4 - 3\,(2)^{1+2\alpha} + 4\ (3)^{\alpha} - 2\,(5)^{\alpha} - 2\,(7)^{\alpha} + 3(8)^{\alpha}),$$

$$D_3^{\alpha} = \ [d3_{ij}]$$

$$d3_{11} = 2^{-2-3\alpha}(8+(3)^{\alpha}(2)^{1+\alpha} - \ (4)^{\alpha} - (8)^{\alpha}), d3_{22} = 2^{-2-3\alpha}(16-8(3)^{\alpha}+3\ (4)^{\alpha} - \left((6)^{1+\alpha}+3(8)^{\alpha}\right), d3_{33} = d3_{44} = 0$$

,

$$d3_{12} = d3_{21} = 2^{-2-3\alpha}(-12+4\ (3)^{\alpha}+(3)^{\alpha}(2)^{1+\alpha} - \left((4)^{\alpha} - (8)^{\alpha}\right),$$

$$d3_{13} = d3_{31} = 2^{-3(1+\alpha)}(8-16(3)^{\alpha}+5(3)^{\alpha}(2)^{1+\alpha} - 5\left((4)^{\alpha}+8(5)^{\alpha} - 5(8)^{\alpha}\right),$$

$$d3_{14} = d3_{41} = 2^{-3(1+\alpha)}(8(3)^{\alpha}+7(3)^{\alpha}(2)^{1+\alpha} - 7\left((4)^{\alpha} - 16(5)^{\alpha}+8(7)^{\alpha} - 7(8)^{\alpha}\right)$$

,

$$d3_{23} = d3_{32} = 2^{-3(1+\alpha)}(-8+16\,(3)^{\alpha} - 5(3)^{\alpha}(2)^{1+\alpha}+5\left((4)^{\alpha} - 8(5)^{\alpha}+5(8)^{\alpha}\right),$$

$$d3_{24} = d3_{42} = 2^{-3(1+\alpha)}(-8\,(3)^{\alpha} - 7\,(3)^{\alpha}(2)^{1+\alpha}+7\left((4)^{\alpha}+16(5)^{\alpha} - 8(7)^{\alpha}+7\,(8)^{\alpha}\right),$$

$$d3_{34} = d3_{43} = 0$$

$$D_4^{\alpha} = [d4_{ij}]$$

$$d4_{11} = d4_{22} = 0, d4_{33} = 2^{-2-3\alpha}(8+5(2)^{1+\alpha} - 5\ (4)^{\alpha}), d4_{44} = 2^{-2-3\alpha}(16-7(2)^{1+\alpha} - 8\ (3)^{\alpha}+7(4)^{\alpha} d4_{12} = d4_{21} = 0,$$

$$d4_{13} = d4_{31} = d4_{14} = d4_{41} = -2^{-3-2\alpha}(-2+(2)^{\alpha}) d4_{23} = d4_{32} = -3(2)^{-3-2\alpha}(-2+\ 2^{\alpha})$$

$$d4_{24} = d4_{42} = 3(2)^{-3-2\alpha}(-2+\ 2^{\alpha}), d4_{34} = d4_{43} = (2)^{-3-2\alpha}(-12+\ 2^{\alpha+1}+4(3)^{\alpha} - 4^{\alpha}$$

$$A_1^{\alpha} = [a1_{ij}]$$

$$a1_{11} = 2^{-2-3\alpha}(8+4^{\alpha} - 8^{\alpha}), a1_{21} = -\frac{1}{2}+(\frac{3}{8})^{\alpha}+2^{-2-\alpha}+8^{-\alpha}$$

$$a1_{31} = 2^{-2-3\alpha}(4+4^{\alpha}+4(5^{\alpha}) - 3(8^{\alpha})), a1_{41} = -1+(\frac{7}{8})^{\alpha}+2^{-2-\alpha}+8^{-\alpha}$$

$$a1_{12} = -\frac{1}{2}+(\frac{3}{8})^{\alpha}+3(2)^{-2-\alpha}+8^{-\alpha}, a1_{22} = 2^{-2-3\alpha}(8(3)^{\alpha}+3(4)^{\alpha} - 3(8)^{\alpha})$$

$$a1_{32} = -1+(\frac{3}{8})^{\alpha}+(\frac{5}{8})^{\alpha}+3(2)^{-2-\alpha}, a1_{42} = 2^{-2-3\alpha}(4(3)^{\alpha}+3(4)^{\alpha}+4(7)^{\alpha} - 5(8)^{\alpha})$$

$$a1_{13} = 2^{-2-3\alpha}(-12-2^{1+3\alpha}+5(4)^{\alpha}+4(5)^{\alpha}), a1_{23} = 2^{-2-3\alpha}(-8-4(3)^{\alpha}+5(4)^{\alpha}+4(5)^{\alpha} - (8)^{\alpha})$$

$$a1_{33} = 2^{-2-3\alpha}(-8+5(4)^{\alpha}), a1_{34} = 2^{-2-3\alpha}(-8+5(4)^{\alpha}+4(5)^{\alpha} - 4(7)^{\alpha}+(8)^{\alpha})$$

$$a1_{41} = 2^{-2-3\alpha}(-4-8(3)^{\alpha}+7(4)^{\alpha}+4(7)^{\alpha} - 3(8)^{\alpha}),$$

$$a1_{42} = 2^{-2-3\alpha}(-2^{1+3\alpha} - 4(3)^{1+\alpha}+7(4)^{\alpha}+4(7)^{\alpha}),$$

$$a1_{43} = 2^{-2-3\alpha}(-8(3)^{\alpha}+7(4)^{\alpha} - 4(5)^{\alpha}+4(7)^{\alpha} - (8)^{\alpha}), a1_{44} = 2^{-2-3\alpha}(-8(3)^{\alpha}+7(4)^{\alpha}),$$

$$A_2^{\alpha} = [a2_{ij}]$$

$$a2_{11} = -2^{-3(1+\alpha)}\left(-16+2^{1+3\alpha} - 2^{1+\alpha}3^{\alpha}+4^{\alpha}\right), a2_{21} = -2^{-3(1+\alpha)}(-8+2^{2+3\alpha} - 8(3)^{\alpha} - 2^{1+\alpha}3^{\alpha}+4^{\alpha}),$$

$$a2_{31} = 8^{-1-\alpha}(8-3(2)^{1+3\alpha}+2^{1+\alpha}(3)^{\alpha} - 4^{\alpha}+8(5)^{\alpha}), a2_{41} = 2^{-3(1+\alpha)}(8+2^{1+\alpha}3^{\alpha} - 4^{\alpha}+8(7)^{\alpha} - 8^{1+\alpha}),$$

$$a2_{12} = 8^{-1-\alpha}(-24-2^{1+3\alpha}+8(3)^{\alpha} - 3(4)^{\alpha}+6^{1+\alpha}), a2_{22} = 8^{-1-\alpha}(-16-3(4)^{\alpha}+6^{1+\alpha}),$$

$$a2_{32} = 2^{-3(1+\alpha)}(-16+2^{1+3\alpha}+8(3)^{\alpha} - 3(4)^{\alpha} - 8(5)^{\alpha}+6^{1+\alpha}),$$

$$a2_{42} = 2^{-3(1+\alpha)}(-16+2^{2+3\alpha}+8(3)^{\alpha}-3(4)^{\alpha}+6^{1+\alpha}-8(7)^{\alpha}),$$

$$a2_{13} = a2_{23} = a2_{33} = a2_{43} = 2^{-3(1+\alpha)}(8-163^{\alpha}+5(2^{1+\alpha})3^{\alpha}-5(4)^{\alpha}+8(5)^{\alpha}-5(8)^{\alpha}),$$

$$a2_{14} = a2_{24} = a2_{34} = a2_{44} = 2^{-3(1+\alpha)}(8(3^{\alpha})+7(2^{1+\alpha})3^{\alpha}-7(4^{\alpha})-16(5^{\alpha})+8(7^{\alpha})-7(8^{\alpha})),$$

$$A_3^{\alpha} = [a3_{ij}]$$

$$a3_{11} = a3_{22} = a3_{31} = a3_{41} = -2^{-3-2\alpha}(-2+2^{\alpha}), a3_{12} = a3_{22} = a3_{23} = a3_{24} = -3(2^{-3-2\alpha})(-2+2^{\alpha}),$$

$$a3_{13} = 2^{-3(1+\alpha)}(16+5(2)^{1+\alpha}-5(4)^{\alpha}-8^{\alpha}), a3_{23} = 2^{-3(1+\alpha)}(8+5(2)^{1+\alpha}+8(3)^{\alpha}-5(4)^{\alpha}-3(8)^{\alpha}),$$

$$a3_{33} = 2^{-3(1+\alpha)}(8+5(2)^{1+\alpha}-5(4)^{\alpha}+8(5)^{\alpha}-5(8)^{\alpha}), a3_{43} = 2^{-3(1+\alpha)}(8+5(2)^{1+\alpha}-5(4)^{\alpha}+8(7)^{\alpha}-7(8)^{\alpha}),$$

$$a3_{14} = 2^{-3(1+\alpha)}(8+5(2)^{1+\alpha}-5(4)^{\alpha}+8(7)^{\alpha}-7(8)^{\alpha}), a3_{24} = 2^{-3(1+\alpha)}(-16+7(2)^{1+\alpha}-7(4)^{\alpha}+3(8)^{\alpha}),$$

$$a3_{34} = 2^{-3(1+\alpha)}(-16+7(2)^{1+\alpha}+8(3)^{\alpha}-7(4)^{\alpha}-8(5)^{\alpha}+5(8)^{\alpha}),$$

$$a3_{44} = 2^{-3(1+\alpha)}(-16+7(2)^{1+\alpha}+8(3)^{\alpha}-7(4)^{\alpha}-8(7)^{\alpha}+7(8)^{\alpha}),$$

$$B_1^{\alpha} = [b1_{ij}]$$

$$b1_{11} = 8^{-1-\alpha}(16-2^{1+3\alpha}+2^{1+\alpha}(3)^{\alpha}+4^{\alpha}), b1_{21} = 8^{-1-\alpha}(8-2^{2+3\alpha}+8(3)^{\alpha}+2^{1+\alpha}(3^{\alpha})+5(4)^{\alpha})$$

$$b1_{31} = -2^{-3(1+\alpha)}(24+2^{2+3\alpha}+2^{1+\alpha}(3)^{\alpha}-11(4)^{\alpha}-8(5)^{\alpha}),$$

$$b2_{13} = 2^{-3(1+\alpha)}(16+5(2)^{1+\alpha}-3(4)^{\alpha}-(8)^{\alpha}), b2_{23} = 2^{-3(1+\alpha)}(8+5(2)^{1+\alpha}+8(3)^{\alpha}+4^{\alpha}-3(8)^{\alpha}),$$

$$b2_{33} = 8^{-1-\alpha}(-24-5(2)^{1+\alpha}+15(4)^{\alpha}+8(5)^{\alpha}-5(8)^{\alpha}),$$

$$b2_{43} = -2^{-3(1+\alpha)}(8+5(2)^{1+\alpha}+16(3)^{\alpha}-19(4)^{\alpha}-8(7)^{\alpha}+7(8)^{\alpha}),$$

$$b2_{14} = 2^{-3(1+\alpha)}(-24+7(2)^{1+\alpha}+8(3)^{\alpha}-9(4)^{\alpha}+(8)^{\alpha}), b2_{24} = 2^{-3(1+\alpha)}(-16+7(2)^{1+\alpha}-13(4)^{\alpha}+3(8)^{\alpha}),$$

$$b2_{34} = 2^{-3(1+\alpha)}(32-7(2)^{1+\alpha}-8(3)^{\alpha}-3(4)^{\alpha}-8(5)^{\alpha}+5(8)^{\alpha}),$$

$$b2_{44} = 2^{-3(1+\alpha)}(16-7(2)^{1+\alpha}+8(3)^{\alpha}-7(4)^{\alpha}-8(7)^{\alpha}+7(8)^{\alpha}),$$

$$C_1^{\alpha} = [c1_{ij}]$$

$$c_{11} = -c_{21} = -2^{-3-2\alpha}(-2+2^{\alpha}), c_{31} = c_{41} = 0,\ c_{12} = -c_{22} = -3(2^{-3-2\alpha})(-2+2^{\alpha}), c_{32} = c_{42} = 0$$

$$c_{13} = 8^{-1-\alpha}(16+5(2)^{1+\alpha}-3(2)^{1+2\alpha}+(2^{1+\alpha})3^{\alpha}-8^{\alpha}),$$

$$c_{23} = 2^{-3(1+\alpha)}(4+3(2)^{\alpha})(-6+2^{1+\alpha}+2(3)^{\alpha}-4^{\alpha}),$$

$$c_{33} = 2^{-3(1+\alpha)}(8-16(3)^{\alpha}+5(2^{1+\alpha})3^{\alpha}-5(4)^{\alpha}+8(5)^{\alpha}-5(8)^{\alpha}),$$

$$c_{43} = 2^{-3(1+\alpha)}(8(3)^{\alpha}+7(2^{1+\alpha})3^{\alpha}-7(4)^{\alpha}-16(5)^{\alpha}+8(7)^{\alpha}-7(8)^{\alpha}),$$

$$c_{14} = 8^{-1-\alpha}(-4+2^{\alpha})(6-2^{1+\alpha}-2(3)^{\alpha}+4^{\alpha}),$$

$$c_{24} = 2^{-3(1+\alpha)}(32-7(2)^{1+\alpha}+5(2)^{1+2\alpha}-16(3)^{\alpha}-6^{1+\alpha}+3(8)^{\alpha}),$$

$$c_{34} = 2^{-3(1+\alpha)}(-8+16(3)^{\alpha}-5(2^{1+\alpha})3^{\alpha}+5(4)^{\alpha}-8(5)^{\alpha}+5(8)^{\alpha}),$$

$$c_{44} = 2^{-3(1+\alpha)}(-8(3)^{\alpha}-7(2^{1+\alpha})3^{\alpha}+7(4)^{\alpha}+16(5)^{\alpha}-8(7)^{\alpha}+7(8)^{\alpha}),$$

References

[1] Chun-Lin,Liu, (2010), A Tutorial of the Wavelet Transform

[2] Gilbert Strang, (1993), Waveletransfors versus fourier transforms, *appeared in bulletin of the American mathematical society*, **28(2)**, 288-305. Doi: 10.1090/S0273-0979-1993-00390-2

[3] Gilbert Strang, (1989), Wavelets and Dilation Equations: A Brief Introduction *SIAM Review*, **31(4)**, 614-627. Doi: 10.1137/1031128

[4] Shi, Z., Tao Liu, Bo Gao, (2010), Haar wavelet Method for Solving Wave Equation, *International Journal Conference on Computer Application and System Modeling*.

[5] I. K. Youssef, A. R. A. Ali, (2016), Memory Effects in Diffusion Like Equation Via HaarWavelets *Pure and Applied Mathematics Journal*, **5(4)**, 130-140. Doi: 10.11648/j.pamj.20160504.17

[6] I. K. Youssef, A. R. A. Ali, (2016), Memory Effects Due to Fractional Time Derivative and Integral Space in Diffusion Like Equation Via Haar Wavelets *Applied and Computational Mathematics*, **5(4)**, 177-185.Doi: 10.11648/j.acm.20160504.12

[7] Youssef, I. K. ,Eldewaik, M.H., (2017), Haar Wavelet Solution of Poisson's Equation and Their Block Structures, *American Journal of Mathematics and Computer Modelling*. Doi: 10.11648/j.ajmcm.20170203.11

[8] UloLepik, Helle Hein, (2014), Haar wavelets with applications, *Springer international Publishing Switzerland*. DOI: 10.1007/978-3-319-04295-4

[9] C.F.Chen, C.H.Hsiao, (1997), Haar wavelet method for solving lumped and distributed-parameter systems, *IEE Proc.-Control Theory Appl.*,**144(1)**. DOI: 10.1049/ip-cta:19970702

[10] Zhi Shi, Li-Youan Deng, Qing-Jiang Chen, (2007), Numerical solution of differential equation by using Haar wavelet, *Proceedings of the 2007 Int. Conf. on Wavelet Analysis and Pattern Recognition*, *Beijing* .

[11] Sumana R. Shesha, Tejuswini M. and Achala L. Nargund, (2016) Haar wavelet Methods for the solution of elliptic partial differential equations, *International Journal of Mathematics and Computer Research*, **4(6)**, 1481-1492.

[12] Youssef, I. K. ,Shukur,A. M., (2014) Precondition for Discretized Fractional Boundary Value Problem, *Pure and Applied Mathematics Journal*,**3(1)**. Doi: 10.11648/j.pamj.20140301.11

[13] G.D.Smith, (1986) Numerical Solution of partial differential equations, *Clarendon Press, Oxford* .

[14] David M. Young, (1971) Iterative solution of large linear system, *Academic Press*. Doi.org/10.2307/3616228.

[15] AbdollahBorhanifar and SohrabValizadeh, (2013) A fractional finite difference method for solving the fractional Poisson equation based on the shifted Grunwald estimate, *Walailak J Sci & Tech*, **10(5)**.