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Monotonicity preserving representations of curves and surfaces

Jorge Delgado †, Juan Manuel Peña.

† Departamento de Matemática Aplicada, Universidad de Zaragoza, Escuela Universitaria Politécnica de Teruel, 44003 Teruel, SPAIN

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Abstract

In this paper we revisit the problem of monotonicity preservation of curves and surfaces and we provide some new proofs and open problems. In particular, we prove a new formula for the derivation of rational Bézier curves. We also deal with the rational monotonicity preservation of rational Bézier surfaces and a related conjecture is presented.

Keywords: Monotonicity preservation; axial monotonicity; rational surfaces; rational bilinear patch; shape parameter; shape preservation.
AMS 2010 codes: 65D17, 68U07.

1 Introduction

Recently there have been important advances on the stability and accuracy of algorithms in Computer-Aided Geometric Design (CAGD), as can be seen in [7,8,10–12]. For the computation of curves and surfaces in CAGD, another very important topic is shape preservation. One of the simpler shape properties is monotonicity, to which we devote this paper and for which important problems are still open, as will be shown. These open problem arise in particular for surface design. In contrast to the variation diminishing properties of Bézier curves, tensor product Bézier surfaces and Bézier triangles do not satisfy simple extensions of these properties, as shown in [21]. Several definitions of monotonicity preserving properties for these surfaces were introduced in [16] and [17], including the simplest of them: axial monotonicity preservation. As for monotonicity preserving properties for rational Bézier surfaces, it has been proved only the axial monotonicity preservation of surfaces generated by the tensor product of two univariate rational Bernstein bases (see [5, 6, 16]).

†Corresponding author.
Email address: jorgedel@unizar.es

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Monotonicity preservation for curves has been deeply studied and we recall in Section 2 some basic results. In Section 2, we also include a new formula for the derivation of rational Bézier curves, which has its own interest and will be used here to give a direct proof of the monotonicity preservation of these curves. Section 3 is devoted to the monotonicity preservation for surfaces, where there are still open problems, as we had announced before and we shall comment at the end of the paper. In Subsection 3.1 we consider triangular patches and in Subsections 3.2 and 3.3 rectangular patches, considering the rational case in Subsection 3.3. In Subsection 3.3 we conjecture that there are no more axially monotonicity preserving surfaces, that is, a rational bilinear patch is axially monotonicity preserving if and only if it corresponds to the tensor product of two univariate rational Bernstein bases.

2 Monotonicity preservation of curves

In this section we revisit the main results on the monotonicity preserving representation of curves and we also include a new formula for the derivation of rational Bézier curves, which has its own interest and further potential applications in addition to its application in this section.

Let $\mathscr{U}$ be a vector space of real functions defined on $[a,b] \subseteq \mathbb{R}$ and $(u_0,\dots,u_n)$ a basis of $\mathscr{U}$. If a control polygon $P_0 \cdots P_n$ is given then we define a parametric curve

$$\gamma(t) = \sum_{i=0}^{n} P_i u_i(t), \quad t \in [a,b]. \tag{1}$$

In CAGD the functions $u_0,\dots,u_n$ are usually nonnegative and $\sum_{i=0}^{n} u_i(t) = 1 \; \forall t \in [a,b]$ (i.e. the system $(u_0,\dots,u_n)$ is *normalized*) and in this case we say that $(u_0,\dots,u_n)$ is a *blending system*. The *convex hull property* is an important property for curve design: for any control polygon, the curve always lies in the convex hull of the control polygon. The convex hull property holds if and only if $(u_0,\dots,u_n)$ is a blending system. In interactive design we want that the shape of a parametrically defined curve mimics the shape of its control polygon; thus we can predict or manipulate the shape of the curve by choosing or changing the control polygon suitably. One of the simplest shape properties is monotonicity, which will be now described.

In the design of curves it is required that the sense of the path tracing of the curve and the polygon agree. Let us draw the control polygon starting from P_0 and finishing in P_n, and the curve $\gamma(t)$ taking all values of t starting from a and finishing at b. Let $P_0,\cdots,P_n$ be control points in $\mathbb{R}^k$ and let γ be the curve defined by (1). A surjective affine mapping $T:\ \mathbb{R}^k \to \mathbb{R}$ can be interpreted as the projection of the space onto some line. In the case that the projection of the control polygon onto this line is increasing, that is, $T(P_0) \le \cdots \le T(P_n)$, then the projection of the corresponding curve $\gamma(t)$ onto that line $T(\gamma(t))$ must also be increasing. So, we see that this shape preserving property is essentially 1-dimensional. We can see a graphical example of this property in Figure 1. A system satisfying this property is said to be a *monotonicity preserving system*. This shape preserving property can be formalized as follows:

Definition 1. A system of functions $(u_0,\dots,u_n)$ is *monotonicity preserving* (resp., *strictly monotonicity preserving*) if for any $\alpha_0 \le \alpha_1 \le \dots \le \alpha_n$ (resp., $\alpha_0 < \alpha_1 < \dots < \alpha_n$) in $\mathbb{R}$, the function $\sum_{i=0}^{n} \alpha_i u_i$ is increasing (resp., strictly increasing).

Some properties and applications of monotonicity preserving systems can be seen in [2–4]. A proof of the following result, which characterizes monotonicity preserving systems, appears in Proposition 2.3 of [4].

Proposition 1. *Let $(u_0,\dots,u_n)$ be a system of functions defined on an interval $[a,b]$. Let $v_i := \sum_{j=i}^{n} u_j$ for $i \in \{0,1,\dots,n\}$. Then:*

1. *$(u_0,\dots,u_n)$ is monotonicity preserving if and only if v_0 is a constant function and the functions v_i are increasing for $i = 1,\dots,n$.*

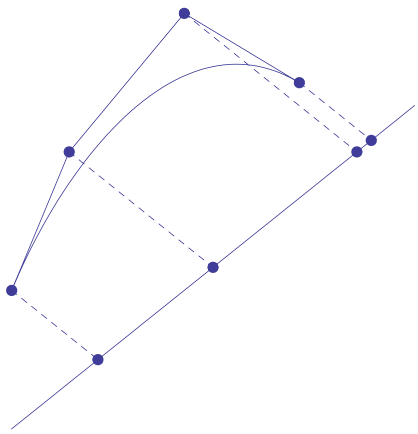

Fig. 1 Monotonicity preservation of a representation of curves

2. $(u_0,\dots,u_n)$ *is strictly monotonicity preserving if and only if* $(u_0,\dots,u_n)$ *is monotonicity preserving and the function* $\sum_{j=1}^{n} v_j$ *is strictly increasing.*

2.1 Nonrational curves

Bézier curves provide the most usual representation of curves in CAGD. These curves are represented in terms of Bernstein polynomials. The Bernstein polynomials of degree n are defined by

$$b_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad x \in [0,1],$$

for all $i = 0,1,\dots,n$. The system $(b_0^n,\dots,b_n^n)$ forms a basis of the space of polynomials of degree at most n, Π_n. For more details in Bernstein polynomials and applications see [13] and [14]. The following result is well known and an argument for its proof can be found in p. 381 of [4].

Proposition 2. *The Bernstein basis* $(b_0^n,\dots,b_n^n)$ *preserves monotonicity strictly.*

Bézier curves are polynomial curves. But when working with polynomials some problems arise. For example, polynomial curves have a global behavior, that is, if one modifies, even slightly, one of its control points, then the modification affects to the whole function. In addition, polynomial curves with high degree can present big oscillations. Piecewise polynomials are the solution to avoid these poblems in diverse fields of mathematics including CAGD. In this field, the role played by Bernstein polynomials in Béizer cuves is played by B-splines in the case of piecewise polyomial curves. Let us consider $d \in \mathbb{N}_0$ and a sequence of nondecreasing real numbers $\mathbf{x} = (x_j)_{j=1}^{n+d+1}$ with at least $d+2$ elements. The j-th B-spline of degree d with nodes $\mathbf{x}$ is defined by

$$N_{j,d,\mathbf{x}}(x) = \frac{x - x_j}{x_{j+d} - x_j} N_{j,d-1,\mathbf{x}}(x) + \frac{x_{j+d+1} - x}{x_{j+d+1} - x_{j+1}} N_{j+1,d-1,\mathbf{x}}(x), \tag{2}$$

for all $x \in \mathbb{R}$, with

$$N_{j,0,\mathbf{x}}(x) = \begin{cases} 1, & \text{if } x_j \le x < x_{j+1}, \\ 0, & \text{in other case.} \end{cases} \tag{3}$$

For more details in B-splines and applications see [13] and [20]. The following result is also well known and an argument for its proof can be found in pp. 381–382 of [4].

Proposition 3. *The B-spline basis* $(N_{i,d,\mathbf{x}})_{i=1}^{n}$ *associated to a sequence of nodes* $\mathbf{x} = (x_j)_{j=1}^{n+d+1}$ *preserves monotonicity.*

2.2 Rational curves

In this subsection, we include a new formula for the derivation of rational Bézier curves, which has its own interest and will be used here to give a direct proof of the monotonicity preservation of these curves.

In CAGD given a system of functions $U = (u_0,\dots,u_n)$ defined in $[a,b]$, it is usual to construct a rational system of functions $(r_0,\dots,r_n)$ from a sequence of positive weights $(w_i)_{i=0}^n$ defined by

$$r_i(t) = \frac{w_i u_i(t)}{\sum_{i=0}^n w_i u_i(t)}, \quad t \in [a,b]. \tag{4}$$

The weights act as shape parameters.

Rational Bernstein bases arise taking $U = (b_0^n,\dots,b_n^n)$ in (4) and the corresponding curve is called *rational Bézier curve*. It is known, from the results of [4], that rational bases constructed from the Bernstein basis of Π_n are also monotonicity preserving. Now let us prove this result from a new and different approach. We will use a new formula for the derivative of a rational Bézier curve presented in the folowing result. This formula is not only important for proving that rational Bernstein bases are monotonicity preserving, but it can be also useful for the problems studied in [15, 18, 22, 24].

Proposition 4. *Let us consider a rational Bézier curve γ given by*

$$\gamma(t) = \sum_{i=0}^n P_i \frac{w_i b_i^n(t)}{\sum_{i=0}^n w_i b_i^n(t)}, \quad t \in [0,1],$$

with $(w_i)_{0\le i\le n}$ a sequence of positive weights and $P_0 \cdots P_n$ the control polygon. Then we have

$$\gamma'(t) = n \frac{\sum_{i=0}^{n-1}\sum_{j=i}^{n-1} \frac{(j+1-i)n}{(j+1)(n-i)} \left(\sum_{k=i}^{j}(\Delta P_k)\right) b_i^{n-1}(t)\, b_j^{n-1}(t)}{\left(\sum_{i=0}^n w_i b_i^n(t)\right)^2},$$

where $\Delta P_k := P_{k+1} - P_k$.

Proof. Taking into account that

$$\gamma(t) = \frac{N(t)}{D(t)},$$

where $N(t) := \sum_{i=0}^n P_i w_i b_i^n(t)$ and $D(t) := \sum_{i=0}^n w_i b_i^n(t)$, we deduce

$$\gamma'(t) = \frac{N'(t)D(t) - N(t)D'(t)}{(D(t))^2}. \tag{5}$$

Since

$$N'(t) = n \sum_{i=0}^{n-1} \Delta(P_i w_i)\, b_i^{n-1}(t),$$

$$D'(t) = n \sum_{i=0}^{n-1} (\Delta w_i)\, b_i^{n-1}(t),$$

we can write (5) in the following way

$$\gamma'(t) = n \frac{\sum_{i=0}^{n-1}\sum_{j=0}^{n} [\Delta(P_i w_i) w_j - P_j w_j (\Delta w_i)]\, b_i^{n-1}(t)\, b_j^n(t)}{(D(t))^2}. \tag{6}$$

From the previous formula, since $\Delta(P_i w_i) = w_{i+1}(\Delta P_i) + (\Delta w_i) P_i$, we deduce

$$\gamma'(t)\,(D(t))^2 = n \sum_{i=0}^{n-1} \sum_{j=0}^{n} [w_{i+1}\, w_j\,(\Delta P_i) \\ +(\Delta w_i)\, w_j\, P_i - (\Delta w_i)\, w_j\, P_j]\; b_i^{n-1}(t)\, b_j^n(t).$$

Applying the univariate de Casteljau algorithm to the right-hand of the previous formula we derive

$$\gamma'(t)\;(D(t))^2 = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{1} \left[w_{i+1}\, w_{j+k}\,(\Delta P_i) \\ +(\Delta w_i)\, w_{j+k}\, P_i - (\Delta w_i)\, w_{j+k}\, P_{j+k}\right]\; b_i^{n-1}(t)\, b_j^{n-1}(t)\, b_k^1(t). \tag{7}$$

Now, rearranging the right-hand of the previous formula, we have

$$\begin{aligned}\gamma'(t)\;(D(t))^2 = {} & \sum_{i=0}^{n-1}\sum_{j=0}^{i-1}\sum_{k=0}^{1} \left[w_{i+1}\, w_{j+k}\,(\Delta P_i) + (\Delta w_i)\, w_{j+k}\, P_i \right.\\ & \left. -(\Delta w_i)\, w_{j+k}\, P_{j+k}\right]\; b_i^{n-1}(t)\, b_j^{n-1}(t)\, b_k^1(t) \\ & + \sum_{i=0}^{n-1}\sum_{k=0}^{1} [w_{i+1}\, w_{i+k}\,(\Delta P_i) + (\Delta w_i)\, w_{i+k}\, P_i \\ & -(\Delta w_i)\, w_{i+k}\, P_{i+k}]\; b_i^{n-1}(t)\, b_i^{n-1}(t)\, b_k^1(t) \\ & + \sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\sum_{k=0}^{1} \left[w_{i+1}\, w_{j+k}\,(\Delta P_i) + (\Delta w_i)\, w_{j+k}\, P_i \right. \\ & \left. -(\Delta w_i)\, w_{j+k}\, P_{j+k}\right]\; b_i^{n-1}(t)\, b_j^{n-1}(t)\, b_k^1(t).\end{aligned}$$

Then we can deduce that the previous expression can be written as

$$\begin{aligned}\gamma'(t)\;(D(t))^2 = {} & \sum_{i=0}^{n-1}\sum_{j=0}^{i-1}\sum_{k=0}^{1} \left[w_{i+1}\, w_{j+k} \sum_{l=j+k}^{i} (\Delta P_l) \right. \\ & \left. - w_i\, w_{j+k} \sum_{l=j+k}^{i-1} (\Delta P_l)\right]\; b_i^{n-1}(t)\, b_j^{n-1}(t)\, b_k^1(t) \\ & + \sum_{i=0}^{n-1}\sum_{k=0}^{1} w_i\, w_{i+1}\,(\Delta P_i)\, b_i^{n-1}(t)\, b_i^{n-1}(t)\, b_k^1(t) \\ & + \sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\sum_{k=0}^{1} \left[-w_{i+1}\, w_{j+k} \sum_{l=i+1}^{j+k-1} (\Delta P_l) \right. \\ & \left. + w_i\, w_{j+k} \sum_{l=i}^{j+k-1} (\Delta P_l)\right]\; b_i^{n-1}(t)\, b_j^{n-1}(t)\, b_k^1(t).\end{aligned}$$

By performing an index change and reordering, we have

$$\begin{aligned}\gamma'(t)\,(D(t))^2 = &\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\sum_{k=0}^{1}\Bigg[w_{j+1}\,w_{i+k}\sum_{l=i+k}^{j}(\Delta P_l)\\ &-w_j\,w_{i+k}\sum_{l=i+k}^{j-1}(\Delta P_l)\Bigg]\,b_i^{n-1}(t)\,b_j^{n-1}(t)\,b_k^1(t)\\ &+\sum_{i=0}^{n-1}\sum_{k=0}^{1}w_i\,w_{i+1}\,(\Delta P_i)\,b_i^{n-1}(t)\,b_i^{n-1}(t)\,b_k^1(t)\\ &+\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\sum_{k=0}^{1}\Bigg[-w_{i+1}\,w_{j+k}\sum_{l=i+1}^{j+k-1}(\Delta P_l)\\ &+w_i\,w_{j+k}\sum_{l=i}^{j+k-1}(\Delta P_l)\Bigg]\,b_i^{n-1}(t)\,b_j^{n-1}(t)\,b_k^1(t).\end{aligned}$$

Extending the sum on k and operating, we deduce that

$$\begin{aligned}\gamma'(t)\,(D(t))^2 = &\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\Bigg[w_i\,w_{j+1}\sum_{l=i}^{j}(\Delta P_l)\\ &-w_{i+1}\,w_j\sum_{l=i+1}^{j-1}(\Delta P_l)\Bigg]\,b_i^{n-1}(t)\,b_j^{n-1}(t)\\ &+\sum_{i=0}^{n-1}w_i\,w_{i+1}\,(\Delta P_i)\,b_i^{n-1}(t)\,b_i^{n-1}(t).\end{aligned}$$

From the previous formula we can derive in a straightforward way

$$\begin{aligned}\gamma'(t)\,(D(t))^2 = &\sum_{i=0}^{n-1}\sum_{j=i}^{n-1}\Bigg[w_i\,w_{j+1}\sum_{l=i}^{j}(\Delta P_l)\Bigg]\,b_i^{n-1}(t)\,b_j^{n-1}(t)\\ &-\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\Bigg[w_{i+1}\,w_{j+1}\sum_{l=i+1}^{j-1}(\Delta P_l)\Bigg]\,b_i^{n-1}(t)\,b_j^{n-1}(t).\end{aligned}$$

Then, arranging the indexes and operating, we have

$$\begin{aligned}\gamma'(t)\,(D(t))^2 = \sum_{i=0}^{n-1}\sum_{j=i}^{n-1}\Bigg[w_i\,w_{j+1}\sum_{l=i}^{j}(\Delta P_l)\Bigg]\,&(b_i^{n-1}(t)\,b_j^{n-1}(t)\\ &-b_{i-1}^{n-1}(t)\,b_{j+1}^{n-1}(t)).\end{aligned}$$

So the result follows from the last expression.

As a consequence of the previous proposition we derive the following result.

Corollary 5. *A rational Bézier basis* $(r_0,\ldots,r_n)$ *defined by*

$$r_i(t) = \frac{w_i b_i^n(t)}{\sum_{i=0}^n w_i b_i^n(t)},$$

with $(w_i)_{0\le i\le n}$ *a sequence of positive weights, is strictly monotonicity preserving.*

Proof. Let us consider the function

$$\gamma(t) = \sum_{i=0}^{n} \alpha_i \frac{w_i b_i^n(t)}{\sum_{i=0}^{n} w_i b_i^n(t)}$$

with $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. Differentiating the previous formula we have

$$\gamma'(t) = n \frac{\sum_{i=0}^{n-1} \sum_{j=i}^{n-1} \frac{(j+1-i)n}{(j+1)(n-i)} \left(\sum_{k=i}^{j} (\Delta\alpha_k)\right) b_i^{n-1}(t)\, b_j^{n-1}(t)}{\left(\sum_{i=0}^{n} w_i b_i^n(t)\right)^2} > 0, \quad t \in (0,1),$$

and so γ is strictly increasing. Therefore, the rational basis is strictly monotonicity preserving.

3 Monotonicity preservation of surfaces

In [16] and [17], some generalizations of the monotonicity preservation of curves have been extended for surfaces defined on rectangular and triangular patches, respectively. Subsection 3.1 considers triangular patches, and rectangular patches are analyzed in the remaining subsections. In Subsection 3.3 we present a conjecture on axially monotonicity preserving rational Bézier surfaces defined on rectangular patches.

3.1 Monotonicity preservation of surfaces defined in triangular patches

Any point τ in a plane can be expressed in terms of its barycentric coordinates with respect to any nondegenerate triangle $\mathscr{T}$ in that plane with vertices T_0, T_1 and T_2:

$$\tau = \sum_{i=0}^{2} \tau_i T_i, \quad \sum_{i=0}^{2} \tau_i = 1.$$

If $\tau \in \mathscr{T}$, then $\tau_i \geq 0$, $i = 0,1,2$. Let $\mathbf{i} = (i_0, i_1, i_2)$ denote a multi-index where $i_0, i_1, i_2 \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ and let us denote by $|\mathbf{i}|$ the sum $i_0 + i_1 + i_2$.

Given $n \geq 1$, let us consider for each $\mathbf{i}$ such that $|\mathbf{i}| = n$ a function $\phi_{\mathbf{i}} : \mathscr{T} \to \mathbb{R}$. We shall refer to them as a system and write $(\phi_{\mathbf{i}})_{|\mathbf{i}|=n}$. Then, given $(c_{\mathbf{i}})_{|\mathbf{i}|=n}$ a sequence of coefficients in $\mathbb{R}$, we can define a function $F : \mathscr{T} \to \mathbb{R}$, as

$$F(\tau) = \sum_{|\mathbf{i}|=n} c_{\mathbf{i}}\, \phi_{\mathbf{i}}(\tau), \quad \tau \in \mathscr{T}.$$

Let us consider the following points $x_{\mathbf{i}} = \frac{i_0}{n} T_0 + \frac{i_1}{n} T_1 + \frac{i_2}{n} T_2$, with $|\mathbf{i}| = n$. Then we define the control net of F as the function

$$p : \mathscr{T} \to \mathbb{R},$$

which is linear on each subtriangle of $\mathscr{T}$ and satisfies

$$p(x_{\mathbf{i}}) = c_{\mathbf{i}}, \quad |\mathbf{i}| = n.$$

The control net is important in interactive design because it is a mesh of points used to control the shape of the surface. So, in [17] Peña and Floater provided several generalizations of the concept of monotonicity preservation of curves to surfaces.

Definition 2. A system $(\phi_{\mathbf{i}})_{|\mathbf{i}|=n}$ is *axially monotonicity preserving* (AMP) if the function F is increasing in the direction $T_1 - T_0$, $T_2 - T_1$ or $T_0 - T_2$ whenever the control net p is increasing in the same direction.

In [17] it was proved that the Bernstein polynomials of degree n on a triangle, $(b_{\mathbf{i}}^n)_{|\mathbf{i}|=n}$ defined by

$$b_{\mathbf{i}}^n(\tau) = \frac{n!}{i_0!\, i_1!\, i_2!} \tau_0^{i_0}\, \tau_1^{i_1}\, \tau_2^{i_2}, \quad |\mathbf{i}| = n,$$

are AMP and even satisfy stronger monotonicity preserving properties.

Now let us consider the rational Bernstein basis of order n $(\phi_{\mathbf{i}})_{|\mathbf{i}|=n}$ given by $\phi_{\mathbf{i}} = \frac{w_{\mathbf{i}} b^n_{\mathbf{i}}}{\sum_{|\mathbf{i}|=n} w_{\mathbf{i}} b^n_{\mathbf{i}}}$, where $(w_{\mathbf{i}})_{|\mathbf{i}|=n}$ is a sequence of positive weights. In [6] it was proved that the Bernstein basis on a triangle is the unique rational Bernstein basis which is AMP.

Theorem 6. *(see Theorem 2 of [6]) If a rational Bernstein basis on a triangle with positive weights is AMP, then $w_{\mathbf{i}} = w_{\mathbf{j}}$ for all $\mathbf{i},\mathbf{j}$ such that $|\mathbf{i}| = |\mathbf{j}| = n$.*

3.2 Monotonicity preservation of nonrational surfaces defined in rectangular patches

Given a normalized nonnegative system of bivariate functions $U = (u_{ij}(x,y))_{0\le i\le m}^{0\le j\le n}$ defined on $[a_1,b_1] \times [a_2,b_2]$ and a sequence of values in $\mathbb{R}$, $(c_{ij})_{0\le i\le m}^{0\le j\le n}$, let us consider the corresponding generated bivariate function

$$F(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}\, u_{ij}(x,y), \quad (x,y) \in [a_1,b_1] \times [a_2,b_2]. \tag{8}$$

Now we shall associate a control net p with the function F. Given two strictly increasing sequences of abscissae

$$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m) \quad \text{and} \quad \beta = (\beta_0, \beta_1, \ldots, \beta_n),$$

we define the control net

$$p : [\alpha_0, \alpha_m] \times [\beta_0, \beta_n] \to \mathbb{R}$$

to be the unique function which satisfies the interpolation conditions

$$p(\alpha_i, \beta_j) = c_{ij} \quad \text{for all } i = 0,1\ldots,m \text{ and } j = 0,1,\ldots,n,$$

and is bilinear on each rectangle

$$R_{ij} = [\alpha_i, \alpha_{i+1}] \times [\beta_j, \beta_{j+1}].$$

As in the triangular case, the control net is used to control the shape of the surface in interactive design. A bivariate function g is increasing in a direction $d = (d_1,d_2) \in \mathbb{R}^2$, if

$$g(x+\lambda d_1, y+\lambda d_2) \ge g(x,y),\ \lambda > 0.$$

In particular, the control net p can be increasing in a direction d. In [16] Floater and Peña characterized this situation as follows:

Lemma 7. *The control net p is increasing in the direction $d = (d_1,d_2)$ in $\mathbb{R}^2$ if and only if for $i = 0,1,\ldots,m-1$ and $j = 0,1,\ldots,n-1$,*

$$d_1\,\Delta_1 c_{i,j+l} + d_2\,\Delta_2 c_{i+k,j} \ge 0, \quad k,l \in \{0,1\},$$

where $\Delta_1 c_{ij} := (c_{i+1,j} - c_{ij})/(\alpha_{i+1} - \alpha_i)$ and $\Delta_2 c_{ij} := (c_{i,j+1} - c_{ij})/(\beta_{j+1} - \beta_j)$.

Given a sequence $(c_{ij})_{0\le i\le m}^{0\le j\le n}$, $\Lambda_1 c_{ij} := c_{i+1,j} - c_{ij}$ for $i = 0,1,\ldots,m-1$ and $j = 0,1,\ldots,n$, and $\Lambda_2 c_{ij} := c_{i,j+1} - c_{ij}$ for $i = 0,1,\ldots,m$ and $j = 0,1,\ldots,n-1$.

Remark 1. As a consequence of Lemma 7 we have that the control net p is increasing in the X-axis direction $d = (1,0)$ if and only if for $i = 0,1,\ldots,m-1$ and $j = 0,1,\ldots,n-1$, $\Lambda_1 c_{ij} \ge 0$. Analogously, the control net p is increasing in the Y-axis direction $d = (0,1)$ if and only if for $i = 0,1,\ldots,m-1$ and $j = 0,1,\ldots,n-1$, $\Lambda_2 c_{ij} \ge 0$

In [16] several concepts of monotonicity preservation for rectangular patches were introduced.

Definition 3.

- The system U *preserves monotonicity* with respect to the abscissae α and β if when the control net p of the function F in (8) is increasing in any direction d in $\mathbb{R}^2$ then so is F.
- The system U *preserves axial monotonicity* if, for any abscissae α and β, when p is increasing in the direction $d=(1,0)$ or $d=(0,1)$ then so is F.

Now let us consider the particular case of tensor product surfaces. So let us consider two systems of univariate functions $U^1=(u_0^1,u_1^1,\dots,u_m^1)$ and $U^2=(u_0^2,u_1^2,\dots,u_n^2)$ defined on $[a_1,b_1]$ and $[a_2,b_2]$, respectively. Then we consider the system of tensor-product functions

$$U=U^1\otimes U^2=(u_i^1\cdot u_j^2)_{i=0,1,\dots,m}^{j=0,1,\dots,n} \tag{9}$$

defined on the rectangle $[a_1,b_1]\times[a_2,b_2]$. If the systems U^1 and U^2 are blending then the system U is also blending. Given $c_{ij}\in\mathbb{R}$ and taking $u_{ij}=u_i^1\cdot u_j^2$ in (8), $i\in\{0,1,\dots,m\}$ and $j\in\{0,1,\dots,n\}$, the system U generates the following parametric function:

$$F(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}c_{ij}u_i^1(x)u_j^2(y),\quad (x,y)\in[a_1,b_1]\times[a_2,b_2].$$

The next result characterizes axial monotonicity preservation:

Proposition 8. *(Proposition 2.3 of [16]) The blending system U in (9) preserves axial monotonicity if and only if the functions*

$$v_i^1:=\sum_{k=i}^{m}u_k^1,\quad i=1,\dots,m,\quad \text{and}\quad v_j^2:=\sum_{k=j}^{n}u_k^2,\quad j=1,\dots,n,$$

are increasing.

As a consequence, we can derive the following result.

Corollary 9. *If the blending univariate systems U^1 and U^2 are monotonicity preserving, then the blending bivariate system U preserves axial monotonicity.*

Taking into account that systems of Bernstein polynomials and of B-splines are monotonicity preserving its corresponding tensor products are axially monotonicity preserving.

Proposition 10.

- *The tensor product of Bernstein bases*

$$(b_i^m(x)\cdot b_j^n(y))_{0\le i\le m}^{0\le j\le n}$$

preserves monotonicity axially.

- *Given two sequence of nodes $\mathbf{x}=(x_i)_{i=1}^{m+d_1+1}$ and $\mathbf{y}=(y_j)_{j=1}^{n+d_2+1}$, the tensor product of the corresponding B-spline bases*

$$(N_{i,d_1,\mathbf{x}}(x)\cdot N_{j,d_2,\mathbf{y}}(y))_{1\le i\le m}^{1\le j\le n}$$

preserves monotonicity axially.

3.3 Monotonicity preservation of rational Bézier surfaces defined in rectangular patches.

Let F be a *rational Bézier surface* defined as

$$F(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}\frac{w_{ij}b_i^m(x)b_j^n(y)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{ij}b_i^m(x)b_j^n(y)}, \quad (x,y)\in[0,1]\times[0,1], \tag{10}$$

where $(w_{ij})_{0\le i\le m}^{0\le j\le n}$ is a sequence of positive weights and $b_i^k(t)$, $i=0,1,\dots,k$, are the Bernstein polynomials of degree k. In [6] it was proved that rational Bézier surfaces are not, in general, even axially monotonicity preserving. In addition, in that paper a particular case of rational Bézier surfaces was considered, the surfaces

$$F(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}\frac{w_i b_i^m(x)}{\sum_{i=0}^{m} w_i b_i^m(x)}\frac{\overline{w}_j b_j^n(y)}{\sum_{j=0}^{n}\overline{w}_j b_j^n(y)}, \qquad (x,y)\in[0,1]\times[0,1], \tag{11}$$

generated by the bases formed by the tensor product of univariate rational Bernstein bases

$$\left(\frac{w_i b_i^m(x)}{\sum_{i=0}^{m} w_i b_i^m(x)}\right)_{i=0}^{m} \otimes \left(\frac{\overline{w}_j b_j^n(y)}{\sum_{j=0}^{n}\overline{w}_j b_j^n(y)}\right)_{j=0}^{n}, \tag{12}$$

where $(w_i)_{i=0}^m$ and $(\overline{w}_j)_{j=0}^n$ are two sequences of strictly positive weights. Let us observe that taking $w_{ij} = w_i\overline{w}_j$, $i=0,1\dots,m$ and $j=0,1\dots,n$, in (10) we obtain the surface in (11). By Corollary 9 and Corollary 5 these rational bases preserve monotonicity axially as it was pointed in [5]. We conjecture that the converse also holds. A particular case of this problem has been considered in [9]. Let us consider the system

$$B = \left(\frac{w_{ij}b_i^m(x)b_j^n(y)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{ij}b_i^m x)b_j^n(y)}\right)_{0\le i\le m}^{0\le j\le n} \tag{13}$$

Now we present our conjecture:

Conjecture 11. *The system B of (13) is axially monotonicity preserving if and only if it corresponds to the tensor product of two univariate rational Bernstein systems.*

If $m=n=1$, the surface (10) is called a *rational bilinear patch* (see [23]) and we shall see that the conjecture holds in this case. Rational bilinear patches have been considered recently in [19] and in [23]. The quotient

$$W := \frac{w_{00}w_{11}}{w_{01}w_{10}} \tag{14}$$

is called in [23] *shape parameter* of the rational bilinear patch. In general, it is desirable that W to be sufficiently close to 1, as remarked in page 3 of [23]. If $W=1$, it corresponds to the case when the patch exhibits minimal curvature along a diagonal of the control net.

Now we characterize in several ways for the case $m=n=1$ the rational bases preserving the axial monotonicity.

Theorem 12. *Let us consider the system*

$$B = \left(\frac{w_{ij}b_i^1(x)b_j^1(y)}{\sum_{i=0}^{1}\sum_{j=0}^{1} w_{ij}b_i^1(x)b_j^1(y)}\right)_{0\le i\le 1}^{0\le j\le 1} \tag{15}$$

and the rational bilinear patch given by (10) with $m=n=1$. Then the following properties are equivalent:

(i) B is axially monotonicity preserving.

(ii) The optimal shape parameter W of (14) is 1.

(iii) B can be expressed as in (12) for $m = n = 1$.

Proof. Let us consider a rational Bézier surface (10) with $m = n = 1$.

(i) $\Leftrightarrow$ (ii). Differentiating the surface respect to x we obtain

$$\frac{\partial F(x,y)}{\partial x} = \frac{F_1(y)+F_2(y)}{\left(\sum_{i=0}^{1}\sum_{j=0}^{1} w_{ij} b_i^1(x) b_j^1(y)\right)^2},$$

where

$$\begin{aligned} F_1(y) &= w_{00}w_{10}(\Lambda_1 c_{00}) b_0^2(y) \\ &\quad + \frac{1}{2}\left[w_{00}w_{11}(\Lambda_1 c_{01}) + w_{01}w_{10}(\Lambda_1 c_{00})\right] b_1^2(y) \\ &\quad + w_{01}w_{11}(\Lambda_1 c_{01}) b_2^2(y), \\ F_2(y) &= (w_{00}w_{11} - w_{01}w_{10})(\Lambda_2 c_{00}) b_1^2(y). \end{aligned}$$

By Remark 1, a control net p is increasing in the X-axis direction $(1,0)$ if and only if $\Lambda_1 c_{ij} \geq 0$ for $i = 0,1,\ldots,m-1$ and $j = 0,1,\ldots,n-1$. Taking $c_{00} = c_{10} = c \neq 0$ and $c_{01} = c_{11} = 2c$ we have that $\Lambda_1 c_{00} = \Lambda_1 c_{01} = 0$ and so the control net is increasing in the X-axis direction. With the previous choice of the coefficients we have that $F_1(y) = 0$ and $F_2(y) = c(w_{00}w_{11} - w_{01}w_{10}) b_1^1(y)$. Taking into account that $\left(\sum_{i=0}^{1}\sum_{j=0}^{1} w_{ij} b_i^1(x) b_j^1(y)\right)^2 > 0$, F is increasing in the X-axis direction if and only if $F_2(y) = c(w_{00}w_{11} - w_{01}w_{10}) b_1^1(y) \geq 0$ for all $(x,y) \in [0,1]^2$. Since c can be any real number $F_2(y) \geq 0$ if and only if $w_{00}w_{11} - w_{01}w_{10} = 0$, which is equivalent to

$$\operatorname{rank}\begin{pmatrix} w_{00} & w_{01} \\ w_{10} & w_{11} \end{pmatrix} = 1. \tag{16}$$

Analogously, it can be proved that monotonicity preservation in the Y-axis direction is also equivalent to (16). Finally, (16) is clearly equivalent to (ii).

(ii) $\Leftrightarrow$ (iii). Since (ii) is clearly equivalent to (16), it is sufficient to observe that a rank one positive matrix can be written as the product of a positive column vector and a positive row vector:

$$\begin{pmatrix} w_{00} & w_{01} \\ w_{10} & w_{11} \end{pmatrix} = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \begin{pmatrix} \overline{w}_0 & \overline{w}_1 \end{pmatrix}$$

The following corollary is a reformulation of the equivalence of (i) and (iii) in the previous theorem.

Corollary 13. *The system B of* (15) *is axially monotonicity preserving if and only if it corresponds to the tensor product of two univariate rational Bernstein systems, that is, if and only if* $B = U \otimes \overline{U}$, *where*

$$U = \left(\frac{w_0 b_0^1(x)}{\sum_{i=0}^{1} w_i b_i^1(x)}, \frac{w_1 b_1^1(x)}{\sum_{i=0}^{1} w_i b_i^1(x)}\right) \quad \text{and} \quad \overline{U} = \left(\frac{\overline{w}_0 b_0^1(y)}{\sum_{i=0}^{1} \overline{w}_i b_i^1(y)}, \frac{\overline{w}_1 b_1^1(y)}{\sum_{i=0}^{1} \overline{w}_i b_i^1(y)}\right).$$

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