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Controllability for neutral stochastic functional integrodifferential equations with infinite delay

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Abstract

In this work, we study the controllability for a class of nonlinear neutral stochastic functional integrodifferential equations with infinite delay in a real separable Hilbert space. Sufficient conditions for the controllability are established by using Nussbaum fixed point theorem combined with theories of resolvent operators. As an application, an example is provided to illustrate the obtained result.

Keywords: C_0-semigroup; Controllability; Hilbert space; Neutral stochastic functional integrodifferential equations; Nussbaum fixed point theorem.
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1 Introduction

Qualitative properties such as existence, uniqueness, stability and controllability for various types of stochastic differential equations have been extensively studied by many researchers (see [4, 6, 8, 17] and references therein). Many fundamental problems of control theory such as pole-assignment, stabilizability and optimal control may be solved under the assumption that the system is controllable. The controllability problem for an evolution equation also consists of driving the solution of the system to a prescribed final target state (exactly or in some approximate way) in a finite time interval. As an area of application oriented mathematics, the control problem has been studied extensively in the fields of infinite dimensional nonlinear systems [10]. The theory

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of semigroups of bounded linear operators is closely related to the solution of differential equations. In recent years, this theory has been applied to a large class of nonlinear differential equations in Banach spaces. Using the method of semigroups, various types of solutions of semilinear evolution equations have been discussed by Pazy in [20]. Semigroup theory gives a unified treatment of a wide class of stochastic parabolic, hyperbolic and functional differential equations, and much effort has been devoted to the study of controllability results for such evolution equations.

Motivated by the above works, in this paper we address sufficient conditions to ensure the controllability of neutral stochastic integrodifferential equations with infinite delays in a Hilbert space described by

$$\begin{cases} d[x(t)+F(t,x_t)] = \Big[A[x(t)+F(t,x_t)] + \int_0^t B(t-s)[x(s)+F(s,x_s)]ds + Cu(t) + h(t,x_t)\Big]\, dt \\ + \int_{-\infty}^t g(t,s,x_s)dw(s), \ t \in J := [0,b], \\ x(0) = \xi, \end{cases} \tag{1}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t), t \geq 0$, on a separable Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let K be another separable Hilbert space with inner product $(\cdot,\cdot)_K$ and norm $\|\cdot\|_K$. Suppose $\{w(t)\}_{t\geq 0}$ is a given K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q \geq 0$. We are also employing the same notation $\|\cdot\|$ for the norm $L(K,H)$, where $L(K,H)$ denotes the space of all bounded linear operators from K into H. The histories x_t belongs to some abstract phase space $\mathfrak{B}$ defined axiomatically (see Section 2); $F, h : J \times \mathfrak{B} \to H$ are the measurable mappings in H-norm, and $G : J \times J \times \mathfrak{B} \to L_Q(K,H)(L_Q(K,H)$ denotes the space of all Q-Hilbert-Schmidt operators from K into H, which is going to be defined below) is a measurable mapping in $L_Q(K,H)$-norm. The control function $u(\cdot)$ taking values in $L_2(J,U)$ of admissible control functions for a separable Hilbert space U, C is a bounded linear operator from U into H, and $\phi(t)$ is a $\mathfrak{B}$-valued random variable independent of Brownian motion $\{w(t)\}$ with finite second moment.

The aim of our paper is to present some results on the controllability of (1) based on the Nussbaum fixed point theorem combined with theories of resolvent operators for integrodifferential equations. Our main results concerning (1), rely essentially on techniques using strongly continuous family of operators $\{R(t),\ t \geq 0\}$, defined on the Hilbert space H and called their resolvent. The resolvent operator is similar to the semigroup operator for abstract differential equations in Banach spaces. However, the resolvent operator does not satisfy semigroup properties (see, for instance, [11]), and our objective in this paper is to apply the theories of resolvent operators, which was proposed by Grimmer [2].

The rest of this paper is organized as follows. In Section 2, we recall some basic definitions, notations, and lemmas which will be needed in the sequel. In Section 3, the controllability of neutral stochastic integrodifferential equations with infinite delay is studied in Hilbert spaces. Section 4 is devoted to an application which illustrates the main results.

2 Preliminaries

2.1 Basic Concepts of Stochastic Analysis

For more details on this section, the reader is referred to Da Prato and Zabczyk [5], Gard [12], and the references therein. Throughout the paper, H and K denote real separable Hilbert spaces.

Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space furnished with a complete family of right continuous increasing sub σ-algebras $\{\mathfrak{F}_t, t \in J\}$ satisfying $\mathfrak{F}_t \subset \mathfrak{F}$. A H-valued random variable is an $\mathfrak{F}$-measurable function $x(t) : \Omega \to H$, and a collection of random variables $S = \{x(t,w) : \Omega \to H | t \in J\}$ is called a stochastic process. Usually, we suppress the dependence on $\omega \in \Omega$ and write $x(t)$ instead of $x(t,\omega)$ and $x(t) : J \to H$ in the place of S. Let $\beta_n(t)(n=1,2,...)$ be a sequence of real-valued one-dimensional standard Brownian motions mutually independent over $(\Omega,\mathfrak{F},P)$. Set

$$w(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\beta_n(t)\zeta_n, \quad t\geq 0,$$

where $\lambda_n \geq 0(n=1,2,...)$ are nonnegative real numbers and $\{\zeta_n\}(n=1,2,...)$ is complete orthonormal basis in K. Let $Q\in L(K,K)$ be an operator defined by $Q\zeta_n=\lambda_n\zeta_n$ with finite $TrQ=\sum_{n=1}^{\infty}\lambda_n<\infty$ (Tr denotes the trace of the operator). Then the above K-valued stochastic process $w(t)$ is called a Q-Wiener process. We assume that $\mathfrak{F}_t=\sigma(w(s):0\leq s\leq t)$ is the σ-algebra generated by w and $\mathfrak{F}_T=\mathfrak{F}$. Let $\varphi\in L(K,H)$ and define

$$\|\varphi\|_Q^2=Tr(\varphi Q\varphi^*)=\sum_{n=1}^{\infty}\|\sqrt{\lambda_n}\varphi\zeta_n\|^2.$$

If $\|\varphi\|_Q<\infty$, then φ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K,H)$ denote the space of all Q-Hilbert-Schmidt operators $\varphi:K\to H$. The completion $L_Q(K,H)$ of $L(K,H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$ where $\|\varphi\|_Q^2=\langle\varphi,\varphi\rangle$ is a Hilbert space with the above norm topology.

In this work, we will employ an axiomatic definition of the phase space $\mathfrak{B}$ introduced by Hale and Kato [13]. The axioms of the space $\mathfrak{B}$ are established for $\mathfrak{F}_0$-measurable functions from J_0 into H, endowed with a seminorm $\|\cdot\|_{\mathfrak{B}}$. We will assume that $\mathfrak{B}$ satisfies the following axioms:

(ai) If $x:(-\infty,a)\to H, a>0$, is continuous on $[0,a)$ and x_0 in $\mathfrak{B}$, then for every $t\in[0,a)$ the following conditions hold:

(a) x_t is in $\mathfrak{B}$,

(b) $\|x(t)\|\leq L\|x_t\|_{\mathfrak{B}}$,

(c) $\|x_t\|_{\mathfrak{B}}\leq\Gamma(t)\sup\{\|x(s)\|:0\leq s\leq t\}+N(t)\|x_0\|_{\mathfrak{B}}$, where $L>0$ is a constant; $\Gamma,N:[0,\infty)\to[0,\infty)$, Γ is continuous, N is locally bounded, and L,Γ,N are independent of $x(\cdot)$.

(aii) For the function $x(\cdot)$ in (ai), x_t is a $\mathfrak{B}$-valued function $[0,a)$.

(aiii) The space $\mathfrak{B}$ is complete.

Suppose $x(t):\Omega\to H, t\leq a$, is a continuous $\mathfrak{F}_t$-adapted H-valued stochastic process. We can associate with another process $x_t:\Omega\to\mathfrak{B}, t\geq 0$ by setting $x_t=\{x(t+s)(w):s\in(-\infty,0]\}$. This is regarded as a $\mathfrak{B}$-valued stochastic process.

The collection of all strongly measurable, square-integrable H-valued random variables, denoted by $L_2(\Omega,\mathfrak{F},P;H)\equiv L_2(\Omega;H)$, is a Banach space equipped with norm

$$\|x(\cdot)\|_{L_2}=\left(\mathbb{E}\|x(\cdot;\omega)\|_H^2\right)^{\frac{1}{2}},$$

where the expectation $\mathbb{E}$ is defined by $\mathbb{E}(h)=\int_\Omega h(\omega)dP$.
Let $J_1=(-\infty,b]$ and $C(J_1,L_2(\Omega;H))$ be the Banach space of all continuous maps from J_1 into $L_2(\Omega;H)$ satisfying the condition $\sup_{t\in J_1}\mathbb{E}\|x(t)\|^2<\infty$. An important subspace is given by $L_2^0(\Omega,H)=\{f\in L_2(\Omega,H): f \text{ is } \mathfrak{F}_0-\text{measurabale}\}$.

Let Z be the closed subspace of all continuous process x that belong to the space $C(J_1,L_2(\Omega;H))$ consisting of $\mathfrak{F}_t$-adapted measurable processes such that the $\mathfrak{F}_0$-adapted processes $\phi\in L_2(\Omega;\mathfrak{B})$. Let $\|\cdot\|_Z$ be a seminorm in Z defined by

$$\|x\|_Z=\left(\sup_{t\in J}\|x_t\|_{\mathfrak{B}}^2\right)^{\frac{1}{2}},$$

where

$$\|x_t\|_{\mathfrak{B}}\leq\bar{N}\mathbb{E}\|\phi\|_{\mathfrak{B}}+\overline{\Gamma}\sup\{\mathbb{E}\|x(s)\|:0\leq s\leq b\},$$

$\overline{N}=\sup_{t\in J}\{N(t)\},\overline{\Gamma}=\sup_{t\in J}\{\Gamma(t)\}$. It is easy to verify that Z furnished with the norm topology as defined above is a Banach space.

2.2 Resolvent operator for Eq. (1)

In the present section, we recall some definitions, notations and propreties needed in the sequel. In what follows, H will denote a Banach space, A and $B(t)$ are closed linear operators on H. Y represents the Banach space $\mathscr{D}(A)$, the domain of operator A, equiped with the graph norm

$$\|y\|_Y := \|Ay\| + \|y\| \text{ for } \quad y \in Y.$$

The notation $C([0,+\infty);Y)$ stands for the space of all continuous function from $[0,+\infty)$ into Y. We then consider the following Cauchy problem

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds \quad \text{for} \quad t \geq 0, \\ v(0) = v_0 \in H. \end{cases} \tag{2}$$

Definition 1. [2] A resolvent operator of Eq. (2) is a bounded linear operator valued function $R(t) \in \mathscr{L}(H)$ for $t \geq 0$, satisfying the following propreties:

1. $R(0) = I$ and $\|R(t)\| \leq \tilde{N}e^{\beta t}$ for some constant $\tilde{N}$ and β.
2. For each $x \in H$, $R(t)x$ is strongly continuous for $t \geq 0$.
3. For $x \in Y$, $R(.)x \in C^1([0,+\infty);H) \cap C([0,+\infty);Y)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)xds \\ &= R(t)Ax + \int_0^t R(t-s)xds \quad \text{for} \quad t \geq 0. \end{aligned}$$

For additional details on resolvent operators, we refer the reader to [2]. The resolvent operator plays an important role to study the existence of solutions and to establish a variation of constants formula for nonlinear systems. For this reason, we need to know when the linear system (2) possesses a resolvent operator. Theorem1 below provides a satisfactory answer to this problem.
In what follows we suppose the following assumptions:
(H1) A is the infinitesimal generator of a C_0- semigroup $(T(t))_{t\geq 0}$ on H
(H2) For all $t \geq 0$, $B(t)$ is a continuous linear operator from $(Y, \|.\|_Y)$ into $(H, \|.\|_H)$.
Moreover, there exists an integrable function $c : [0,+\infty) \to \mathbb{R}^+$ such that for any $y \in Y, t \mapsto B(t)y$ belongs to $W^{1,1}([0,+\infty),H)$ and

$$\|\frac{d}{dt}B(t)y\|_H \leq c(t)\|y\|_Y \text{ for } y \in Y \text{ and} t \geq 0.$$

We recall that $W^{k,p}(\mathbb{O}) = \{\tilde{\omega} \in L^p(\mathbb{O}) : D^\alpha \tilde{\omega} \in L^p(\mathbb{O}), \forall \|\alpha\| \leq k\}$, where $D^\alpha \tilde{\omega}$ is the weak α-th partial derivative of $\tilde{\omega}$.

Theorem 1. *[2] Assume that hypotheses **(H1)** and **(H2)** hold. Then the Eq. (2) admits a resolvent operator* $(R(t))_{t\geq 0}$.

Lemma 2. *[11] Let hypotheses **(H1)** and **(H2)** be satisfied. Then there exists a constant* $L = L(T)$ *such that*

$$\|R(t+\varepsilon) - R(\varepsilon)R(t)\| \leq L\varepsilon, \quad \forall 0 \leq \varepsilon \leq t \leq T.$$

Theorem 3. *[11] Assume that hypotheses **(H1)** and **(H2)** hold. Let T(t) be a compact operator for* $t > 0$*. Then, the corresponding resolvent operator* $R(t)$ *of Eq. (2) is continuous for* $t > 0$ *in the operator norm, namely, for* $t_0 > 0$*, it holds that* $\lim_{h\to 0} \|R(t_0+h) - R(t_0)\| = 0$.

In the sequel, we recall some results on the existence of solutions for the following integro-differentiel equation

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds + q(t) \quad \text{for} \quad t \geq 0, \\ v(0) = v_0 \in H. \end{cases} \tag{3}$$

where $q : [0,+\infty[\to H$ is continuous function.

Definition 2. A continuous function $v : [0,+\infty) \to H$ is said to be a strict solution of the Eq. (3) if

1. $v \in C^1([0,+\infty);H) \cap C([0,+\infty);Y)$,
2. v satisfies Eq. (3) for $t \geq 0$.

Remark 1. From this definition we deduce that $v(t) \in \mathscr{D}(A)$, and the function $B(t-s)v(s)$ is integrable, for all $t > 0$ and $s \in [0,+\infty)$.

Theorem 4. *[2] Assume that hypotheses **(H1)** and **(H2)** hold. If v is a stict solution of Eq. (3), then the following variation of constant formula holds*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds \quad for \quad t \geq 0. \tag{4}$$

Accordingly, we can establish the following definiton.

Definition 3. A function $v : [0,+\infty) \to H$ is called mild solution of Eq. (3), for $v_0 \in H$, if v satisfies the variation of constants formula (4).

The next theorem provides sufficient conditions ensuring the regularity of solutions of Eq.(3).

Theorem 5. *Let $q \in C^1([0,+\infty);H)$ and v be defined by (4). If $v_0 \in \mathscr{D}(A)$, then v is a strict solution of the Eq.(3).*

Definition 4. An $\mathfrak{F}_t$-adapted stochastic process $x(t) : J_1 \to H$ is a mild solution of the abstract Cauchy problem (1) if $x_0 = \phi \in \mathfrak{B}$ on J_0 satisfying $\|\phi\|^2_{\mathfrak{B}} < \infty$. The restriction of $x(\cdot)$ to the interval $[0,b)$ is a continuous stochastic process such that the following equation is satisfied

$$\begin{aligned} x(t) &= R(t)[\phi(0) + F(0,\phi)] - F(t,x_t) \\ &+ \int_0^t R(t-s)\left[h(s,x_s) + Cu(s) + \int_{-\infty}^s g(s,\tau,x_\tau)dw(\tau)\right]ds \quad \text{for a.e } t \in J. \end{aligned} \tag{5}$$

Definition 5. The nonlinear neutral stochastic integrodifferential equation (1) is said to be controllable on the interval J, if for every continuous initial stochastic process $\phi \in \mathfrak{B}$ defined on J_0, there exists a stochastic control $u \in L_2(J,U)$ that is adapted to the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ such that the solution $x(\cdot)$ of (1) satisfies $x(b) = x_1$, where $x(b)$ is a random variable which is $\mathscr{F}_b$-measurable, x_1 and b are preassigned terminal state and time, respectively.

As a key tool for developing the controllability in this work, the consideration of this paper is based on the following fixed point theorem due Nussbaum [17]. Throughout the paper, $B_r[x] \subset L_2(\Omega,\mathfrak{B})$ is the closed ball centered at x with radius $r > 0$.

Theorem 6. *(Nussbaum Fixed Point Theorem). Let S be a closed, bounded, and convex subset of a Banach space X. Let Φ_1, Φ_2 be continuous mappings from S into X such that*

(i) $(\Phi_1 + \Phi_2)S \subset S$.

(ii) $\|\Phi_1 x_1 - \Phi_1 x_2\| \leq k\|x_1 - x_2\|$ *for all $x_1, x_2 \in S$, where k is a constant and* $0 \leq k < 1$.

(iii) $\overline{\Phi_2(S)}$ *is compact.*

Then the operator $\Phi_1 + \Phi_2$ has a fixed point in S.

3 Main result

To investigate the controllability of system (1), we assume the following conditions:

(H3) the resolvent operator $R(t)$ is compact with $\|R(t)\| \leq M$, for all $t \geq 0$;

(H4) the linear operator W from $L_2(J,U)$ into $L_2(\Omega;H)$, defined by

$$W = \int_0^b R(b-s)(Cu)(s)ds$$

has an induced inverse operator W^{-1} that takes values in $L_2(J,U)/KerW$ (see Carmichael and Quinn [7]) and there exist positive constants M_C, M_W such that

$$\|C\| \leq M_C \text{ and } \|W^{-1}\| \leq M_W;$$

(H5) $F : J \times \mathfrak{B} \to H$ is a continuous function, and there exist a constant $M_F > 0$ such that the function F satisfies the Lipschitz condition:

$$\|F(s_1,\psi_1) - F(s_2,\psi_2)\| \leq M_F\left(|s_1 - s_2| + \|\psi_1 - \psi_2\|_{\mathfrak{B}}\right)$$

for $0 \leq s_1, s_2 \leq b, \psi_1, \psi_2 \in L_2(J,\mathfrak{B})$;

(H6) F and $h : J \times \mathfrak{B} \to H$ are continuous and there exists nonnegative constants $\overline{M}_F, M_h$ such that

$$\|F(t,\psi)\| \leq \overline{M}_F \text{ and } \|h(t,\psi)\| \leq M_h \tag{6}$$

for every $0 \leq s \leq t \leq b, \psi \in B_r[\phi]$;

(H7) the function $g : J \times J \times \mathfrak{B} \to L(K,H)$ is continuous and there exists $M_g \geq 0$ such that

$$\|g(t,s,\eta)\|_Q \leq M_g$$

for every $0 \leq s \leq t \leq b$ and $\eta \in B_r[\phi]$;

(H8) For each $\phi \in \mathfrak{B}$

$$l(t) = \lim_{a\to\infty} \int_{-a}^0 g(t,s,\phi(s))dw(s)$$

exists and it is continuous. Further, there exists $\overline{M}_g$ such that $\|l(t)\|_Q \leq \overline{M}_g$;

Theorem 7. *In addition to hypotheses **(H1)-(H8)**, assume that the following conditions are also satisfied*

1. $q = \frac{r-4\varepsilon}{4\overline{\Gamma}^2}$ *and* $\rho = 64\left[\left\{1 + (6M_W M_C M b)^2\right\}(M_F\overline{\Gamma})^2\right]$,

2. $64\left[M^2\overline{M}_F^2+\overline{M}_F^2+(Mb)^2[M_C^2\mathbb{G}'+M_h^2+\overline{M}_g^2+Tr(Q)bM_g^2]\right]=(1-\rho)q,$

3. $L_0=2M_F^2<1.$

Then, system (1) is controllable on J.

Proof. Using hypothesis **(H4)** for an arbitrary $x(\cdot)$ and for a.e $t\in J$, define the control

$$u_x^b(t)=W^{-1}\left\{x_1-R(b)(\phi(0)+F(0,\phi))+F(b,x_b)-\int_0^b R(b-s)\right.$$
$$\left.\times\left[h(s,x_s)+l(s)+\int_0^s g(s,\tau,x_\tau)dw(\tau)\right]ds\right\}(t)$$

$$u_x^b(t)=W^{-1}\left\{x_1-R(b)[\phi(0)+F(0,\phi)]+F(b,x_b)-\int_0^b R(b-s)[h(s,x_s)+l(s)]ds\right.$$
$$\left.+\int_0^s g(s,\tau,x_\tau)dw(\tau)ds\right\}(t).$$

Let $\mathfrak{B}_b$ be the space of all functions $x:(-\infty,b]\to H$ such that $x_0\in\mathfrak{B}$ and the restriction $x:J\to H$ is continuous. Let $\|\cdot\|_b$ be the seminorm in $\mathfrak{B}_b$ defined by

$$\|x\|_b=\|x_0\|_{\mathfrak{B}}+\sup\{\|x(s)\|:0\le s\le b\},\quad x\in\mathfrak{B}_b$$

Let $Z_b=C(J_1,L_2(\Omega;\mathfrak{B}_b))$. Consider the map $\Phi:Z_b\to Z_b$ defined by Φx, by

$$\Phi x(t)=\begin{cases}\phi(t), & \text{if } t\in J_0,\\ R(t)[\phi(0)+F(0,\phi)]-F(t,x_t)+\int_0^t R(t-\eta)Cu_x^b(\eta)d\eta & \\ \quad+\int_0^t R(t-s)\left[h(s,x_s)+l(s)+\int_0^t g(s,\tau,x_\tau)\right]ds & \text{for a.e } t\in J.\end{cases}$$

We shall show that the operator Φ has a fixed point, which then is a solution of the system (1). Clearly, $(\Phi x)(b)=x_1$.

For $\phi\in Z$, let $y(\cdot):(-\infty,b)\to Z_b$ be the function defined by

$$y(t)=\begin{cases}\phi(t) & \text{if } t\in(-\infty,0]\\ R(t)\phi(0) & \text{if } t\in J.\end{cases}$$

Set $x(t)=z(t)+y(t),-\infty<t\le b$. It is clear that x satisfies (5) if and only if z satisfies $z_0=0$ and

$$z(t)=R(t)F(0,\phi)-F(t,z_t+y_t)+\int_0^t R(t-\eta)Cu_{z+y}^b(\eta)d\eta$$
$$+\int_0^t R(t-s)\left[h(s,z_s+y_s)+l(s)+\int_0^s g(s,\tau,x_\tau)dw(\tau)\right]ds,\quad t\in J,$$

where

$$u_{z+y}^b(t)=W^{-1}\left\{x_1-R(b)(\phi(0)+F(0,\phi))+F(b,z_b+y_b)-\int_0^b R(b-s)\right.$$
$$\left.\times\left[h(s,z_s+y_s)+l(s)+\int_0^s g(s,\tau,z_\tau+y_\tau)dw(\tau)\right]ds\right\}(t).$$

Let

$$\mathfrak{B}_b^0 = \{z \in \mathfrak{B}_b : z_0 = 0 \in \mathfrak{B}\}$$

For any $z \in \mathfrak{B}_b^0$, we have

$$\|z\|_b = \|z_0\|_{\mathfrak{b}} + \sup\{\|z(s)\| : 0 \le s \le b\} = \{\sup\{\|z(s)\| : 0 \le s \le b\}$$

Thus if $Z_0^b = C(J_1, L_2(\Omega; \mathfrak{B}_0^b)$, then $(Z_0^b, \|\cdot\|_b)$ is a Banach space. Set

$$B_q = \{z \in Z_b^0 : \|z\|_b^2 \le q\} \quad \text{for some } \; q \ge 0;$$

then $B_q \subseteq Z_b^0$ is uniformly bounded. For $z(\cdot) \in B_q$, from axiom (ai) and hypothesis **(H8)**, we remark that

$$\begin{aligned}\|z_t + y_t - \phi\|_{\mathfrak{B}}^2 &\le 4\left(\|z_t\|_{\mathfrak{B}}^2 + \|y_t - \phi\|_{\mathfrak{B}}^2\right)\\ &\le 4\left(\overline{\Gamma}^2 q + \varepsilon\right) := r,\end{aligned} \tag{7}$$

where $\varepsilon = \|y_t - \phi\|_{\mathfrak{B}}^2$.
Thus, $z_t + y_t \in B_r[\phi]$ for all $0 \le t \le b$. Let the operator $\mathcal{Q} : Z_b^0 \to Z_b^0$ be defined by $\mathcal{Q}z$, by

$$\mathcal{Q}z(t) = \begin{cases} 0 \qquad t \in J_0 \\ R(t)F(0,\phi) - F(t, z_t + y_t) + \int_0^t R(t-\eta) C u_{z+y}^b(\eta) d\eta \\ \quad + \int_0^t R(t-s)\left[h(s, z_s + y_s) + l(s) + \int_0^s g(s,\tau, z_\tau + y_\tau) dw(\tau)\right] ds, \quad t \in J. \end{cases}$$

Obviously, the operator Φ has a fixed point is equivalently to prove that $\mathcal{Q}$ has a fixed point. For each positive number q, let

$$B_q = \{z \in Z_b^0 : z(0) = 0, \|z\|_b^2 \le q, 0 \le t \le b\} \quad \text{ for some } \; q \ge 0$$

then for each q, $B_q \subseteq Z_b^0$ is clearly a bounded closed convex set. In addition to the familiar Young, Hölder, and Minkowskii inequalities, the inequality of the form $\left(\sum_{i=1}^n a_i\right)^m \le n^m \sum_{i=1}^n a_i^m$ where a_i are nonnegative constants $(i = 1, 2, \dots, n)$ and $m, n \in \mathbb{N}$ is helpful to establishing various estimates. The Hölder inequality yields the following relation :

$$\left\| \int_0^t R(t-s) h(s, z_s + y_s) ds \right\|^2 \le (bMM_h)^2. \tag{8}$$

Similary from **(H7)** and together with the Ito's formula, a computation can be performed to obtain the following:

$$\begin{aligned} &\mathbb{E}\left\| \int_0^t R(t-s)\left[\int_0^s g(s,\tau, z_\tau + y_\tau) dw(\tau)\right] ds \right\|^2 \\ &\le Tr(Q) M^2 b \int_0^t \int_0^s \mathbb{E}\|g(s,\tau,z_\tau + y_\tau)\|_Q^2 d\tau ds \\ &\le Tr(Q)(M + M_g)^2 b^3. \end{aligned} \tag{9}$$

Thus, $\mathcal{Q}$ is well defined on B_q. Further noting that

$$\begin{aligned} \mathbb{E}\|u_{z+y}^b\|^2 \le{}& (6M_W)\left[\|x_1\|^2 + M^2[\mathbb{E}\|\phi(0)\|^2 + \mathbb{E}\|F(0,\phi)\|^2] + \mathbb{E}\|F(b, z_b + y_b) - F(b, y_b)\|^2\right. \\ &+ \mathbb{E}\|F(b,y_b)\|^2 + (3M)^2 b \int_0^t \left\{\mathbb{E}\|h(s, z_s + y_s)\|^2 + \mathbb{E}\|l(s)\|^2\right. \\ &\left.\left. + \int_0^s \mathbb{E}\|g(s,\tau, z_\tau + y_\tau)\|^2 d\tau\right\}\right] \\ \le{}& (6M_W)^2\left[\|x_1\|^2 + M^2\|\phi(0)\|_{\mathfrak{B}}^2 + (M_F\overline{\Gamma})^2\|z_b\|_Z^2 + \overline{M}_F^2\right. \\ &\left. + (3Mb)^2(M_h^2 + \overline{M}_g^2 + Tr(Q)M_g^2 b)\right]. \end{aligned}$$

Thus

$$\|u^b_{z+y}\|^2_Z \le (6M_W)^2\left[\|x_1\|^2 + M^2\|\phi(0)\|^2_Z + (M_F\overline{\Gamma})^2 q + \overline{M}^2_F + (3Mb)^2(M^2_h + \overline{M}^2_g + M^2_g b^2)\right] := \mathbb{G}. \tag{10}$$

Also let

$$(6M_W)^2\left[\|x_1\|^2 + M^2\|\phi(0)\|^2_Z + \overline{M}^2_F + (3Mb)^2(M^2_h + \overline{M}^2_g + M^2_g b^2)\right] := \mathbb{G}'. \tag{11}$$

Next, we will show that the operator $\mathscr{Q}$ has a fixed point on B_q, which implies equation (1) has a mild solution. To this end, we decompose $\mathscr{Q}$ as $\mathscr{Q} = \mathscr{Q}_1 + \mathscr{Q}_2$, where the operators $\mathscr{Q}_1, \mathscr{Q}_2$ are defined on B_q, respectively, by

$$(\mathscr{Q}_1 z)(t) = R(t)F(0,\phi) + F(t, z_t + y_t)$$

and

$$\begin{aligned}(\mathscr{Q}_2 z)(t) = &\int_0^t R(t-\eta)Cu^b_{z+y}(\eta)d\eta \\ &+ \int_0^t R(t-s)\left\{h(s, z_s + y_s) + l(s) + \int_0^s g(s,\tau,z_\tau + y_\tau)dw(\tau)\right\}ds,\end{aligned}$$

for $t \in J$. In order to apply the Nussbaum fixed point theorem for the operator $\mathscr{Q}$, we prove the following assertions:

(i) $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are well defined;

(ii) $\mathscr{Q}_1$ satisfies contractive condition;

(iii) $\mathscr{Q}_2$ is relatively compact;

(iv) $\mathscr{Q}B_q \subset B_q$

Now, for $0 \le t \le b$,

$$\begin{aligned}\mathbb{E}\|\mathscr{Q}_1 z)(t)\|^2 &\le 16\mathbb{E}\left\{\|R(t)F(0,\phi)\|^2 + \|F(t,y_t) - F(t, z_t + y_t)\|^2 + \|F(t,y_t)\|2\right\} \\ &\le 16\left\{M^2M^2_F + M^2_F\|z_t\|^2_{\mathfrak{B}} + \overline{M}^2_F\right\} \\ &\le 16\left\{M^2M^2_F + M^2_F\overline{\Gamma}^2 q + \overline{M}^2_F\right\}\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\|(\mathscr{Q}_2 z)(t)\|^2 \le 16b\Bigg\{&\int_0^t \|R(t-\eta)\|^2\|C\|^2\mathbb{E}\|u^b_{z+y}\|^2 d\eta \\ &+ \int_0^t \|R(t-s)\|^2\mathbb{E}\|h(s, z_s + y_s)\|^2 ds \\ &+ \int_0^t \|R(t-s)\|^2\mathbb{E}\|l(s)\|^2 \\ &+ Tr(Q)\int_0^t \|R(t-s)\|^2\int_0^s \mathbb{E}\|g(s,\tau,z_\tau + y_\tau)\|^2 d\tau ds\Bigg\} \\ \le (4Mb)^2[M^2_C\mathbb{G}' &+ M^2_h + \overline{M}^2_g + Tr(Q)bM^2_g + (6M_W M_C M_F\overline{\Gamma})^2 q].\end{aligned}$$

Thus, we have

$$\begin{aligned}\|(\mathscr{Q}z)(t)\|^2_Z &\le 4\mathbb{E}\|(\mathscr{Q}_1 z)(t)\|^2 + 4\|(\mathscr{Q}_2 z)(t)\|^2 \\ &\le 64\Big[\left\{1 + (6M_W M_C Mb)^2\right\}(M_F\overline{\Gamma})^2 q + M^2\overline{M}^2_F + \overline{M}^2_F \\ &\quad + (Mb)^2[M^2_C\mathbb{G}' + M^2_h + \overline{M}^2_g + Tr(Q)bM^2_g]\Big].\end{aligned}$$

Hence $\mathscr{Q}B_q \subseteq B_q$. Next, we shall prove that the operator $\mathscr{Q}_1$ satisfies the Lipschitz condition, we take $z^{(1)}, z^{(2)} \in B_q$, then for each $t \in J$ and by condition **(H3)**, equations (3.3) and (3.5), we have

$$\begin{aligned}
&\mathbb{E}\|(\mathscr{Q}_1 z^{(1)})(t) - (\mathscr{Q}_1 z^{(2)})(t)\|^2 \\
&\le \mathbb{E}\|F(t, z_t^{(1)} + y_t) - F(t, z_t^{(2)} + y_t)\|^2 \\
&\le M_F^2 \mathbb{E}\|z_t^{(1)} - z_t^{(2)}\|_{\mathfrak{B}}^2 \\
&\le L_0 \sup_{0\le s\le b} \|z^{(1)}(s) - z^{(2)}(s)\|_{\mathfrak{B}}^2.
\end{aligned}$$

Thus,

$$\|\mathscr{Q}_1 z^{(1)} - \mathscr{Q}_1 z^{(2)}\|_Z^2 \le L_0 \|z^{(1)} - z^{(2)}\|_Z^2$$

and so $\mathscr{Q}_1$ satisfies Lipschitz condition with $L_0 < 1$.

Finally, we prove that $\mathscr{Q}_2$ is relatively compact in B_q. To prove this, first we shall show that $\mathscr{Q}_2$ maps B_q into a precompact subset of $\mathscr{Q}$. We now show that for every fixed $t \in J$ the set $V(t) = \{(\mathscr{Q}_2 z)(t) : z \in B_q\}$ is precompact in H.

Obviously for $t = 0, V(0) = \{\mathscr{Q}(0)\}$. Let $0 < t \le b$ be fixed and ε be a real number satisfying $\varepsilon \in (0,t)$. For $z \in B_q$, we define the operators

$$\begin{aligned}
(\mathscr{Q}_2^{*\varepsilon} z)(t) = R(\varepsilon)\int_0^{t-\varepsilon} R(t-\varepsilon-\eta) C u_{z+y}^b(\eta) d\eta + R(\varepsilon)\int_0^{t-\varepsilon} R(t-\varepsilon-s) \\
\times \left[h(s, z_s + y_s) + l(s) + \int_0^s g(s,\tau,z_\tau + y_\tau) dw(\tau)\right] ds.
\end{aligned}$$

and

$$\begin{aligned}
(\tilde{\mathscr{Q}}_2^{\varepsilon} z)(t) &= \int_0^{t-\varepsilon} R(t-\eta) C u_{z+y}^b(\eta) d\eta \\
&\quad + \int_0^{t-\varepsilon} R(t-s)\left[h(s, z_s + y_s) + l(s) + \int_0^s g(s,\tau,z_\tau + y_\tau) dw(\tau)\right] ds \\
&= R(\varepsilon)\int_0^{t-\varepsilon} R(t-\varepsilon-\eta) C u_{z+y}^b(\eta) d\eta + R(\varepsilon)\int_0^{t-\varepsilon} R(t-\varepsilon-s) \\
&\quad \times \left[h(s, z_s + y_s) + l(s) + \int_0^s g(s,\tau,z_\tau + y_\tau) dw(\tau)\right] ds.
\end{aligned}$$

By Lemma 2 and the compactness of the operator $R(\varepsilon)$, the set $V_\varepsilon^*(t) = \{(\mathscr{Q}_2^{*\varepsilon} z)(t) : z \in B_q\}$ is relatively compact in H, for every ε, $\varepsilon \in (0,t)$. Moreover, also by Lemma 2, Hölder's inequality, for each $z \in B_q$, we obtain

$$\begin{aligned}
&\mathbb{E}\|(\mathscr{Q}_2^{*\varepsilon} z)(t) - (\tilde{\mathscr{Q}}_2^{\varepsilon} z)(t)\|^2 \\
&\quad \le 4b \int_0^{t-\varepsilon} \|R(\varepsilon)R(t-\eta-\varepsilon) - R(t-\eta)\|^2 \mathbb{E}\|u_{z+y}^b(\eta)\|^2 d\eta \\
&\qquad + 36b \int_0^{t-\varepsilon} \|R(\varepsilon)R(t-s-\varepsilon) - R(t-s)\|^2 \Big\{\mathbb{E}\|h(s, z_s + y_s)\|^2 \\
&+ \mathbb{E}\|l(s)\|^2 + Tr(Q)\int_0^s \mathbb{E}\|g(s,\tau,z_\tau + y_\tau)\|^2 d\tau\Big\} ds \\
&\le 4b(\varepsilon L)^2 \int_0^{t-\varepsilon} \mathbb{E}\|u_{z+y}^b(\eta)\|^2 d\eta \\
&+ 36(\varepsilon L)^2 \int_0^{t-\varepsilon} \Big\{\mathbb{E}\|h(s, z_s + y_s)\|^2 \\
&+ \mathbb{E}\|l(s)\|^2 + Tr(Q)\int_0^s \mathbb{E}\|g(s,\tau,z_\tau + y_\tau)\|^2 d\tau\Big\} ds.
\end{aligned}$$

We obtain that the set $\tilde{V}^*_\varepsilon(t) = \{(\tilde{\mathscr{Q}}^{*\varepsilon}_2 z)(t) : z \in B_q\}$ is precompact in H by using the total boundedness. Applying this idea again, we obtain

$$\begin{aligned}\mathbb{E}\|(\mathscr{Q}_2 z)(t) - (\tilde{\mathscr{Q}}^{\varepsilon}_2 z)(t)\| &\le (2MM_C\varepsilon)^2\mathbb{G} + 4\varepsilon M^2\int_{t-\varepsilon}^{t} 9\{M_h^2 + \overline{M}_g^2 + Tr(Q)bM_g^2\}\\ &\le (2M\varepsilon)^2\left[\mathbb{G}M_C^2 + 9(M_h^2 + \overline{M}_g^2 + Tr(Q)bM_g^2)\right] \longrightarrow 0,\end{aligned}$$

when $\varepsilon \to 0$, and there are precompact sets arbitrarily close to the set $\{(\mathscr{Q}_2 z)(t) : z \in B_q\}$. Thus the set $\{(\mathscr{Q}_2 z)(t) : z \in B_q\}$ is precompact in H.
We now show that the image of $B_q, \mathscr{Q}(B_q) = \{\mathscr{Q}z : z \in B_q\}$ is an equicontinuous family of functions. To do this, let $\varepsilon > 0$ small, $0 < t_1 < t_2$, then from (10), we have

$$\begin{aligned}&\mathbb{E}\|(\mathscr{Q}_2 z)(t_1) - (\mathscr{Q}_2 z)(t_2)\|^2\\ &\le 36\Bigg\{b\int_0^{t-\varepsilon} \|R(t_2-\eta) - R(t_2-\eta)\|^2\mathbb{E}\|Cu^b_{z+y}\|^2 d\eta\\ &+\varepsilon\int_{t-\varepsilon}^{t_1} \|R(t_2-\eta) - R(t_1-\eta)\|^2\mathbb{E}\|Cu^b_{z+y}\|^2 d\eta + (t_2-t_1)\int_{t_1}^{t_2} \|R(t_2-\eta)\|^2\mathbb{E}\|Cu^b_{z+y}\|^2 d\eta\\ &+9b\int_0^{t_1-\varepsilon} \|R(t_2-s) - R(t_1-s)\|^2\mathbb{E}\|Cu^b_{z+y}\|^2\left[\mathbb{E}\|h(s,x_s) + l(s) + Tr(Q)\int_0^s \mathbb{E}\|g(s,\tau,x_\tau)d\tau\right] ds\\ &+9\varepsilon\int_{t_1-\varepsilon}^{t_1} \|R(t_2-s) - R(t_1-s)\|^2\mathbb{E}\|Cu^b_{z+y}\|^2\left[\mathbb{E}\|h(s,x_s) + l(s) + Tr(Q)\int_0^s \mathbb{E}\|g(s,\tau,x_\tau)d\tau\right] ds\\ &+9(t_2-t_1)\int_{t_1}^{t_2} \|R(t_2-s)\|^2\mathbb{E}\|Cu^b_{z+y}\|^2\left[\mathbb{E}\|h(s,x_s) + l(s) + Tr(Q)\int_0^s \mathbb{E}\|g(s,\tau,x_\tau)d\tau\right] ds.\Bigg\}\end{aligned}$$

That is,

$$\begin{aligned}&\|(\mathscr{Q}_2 z)(t_1) - (\mathscr{Q}_2 z)(t_2)\|^2_Z\\ &36\Bigg\{bM_C^2\mathbb{G}\int_0^{t_1-\varepsilon} \|R(t_2-\eta) - R(t_1-\eta)\|^2 d\eta + \varepsilon M_C^2\mathbb{G}\int_{t_1-\varepsilon}^{t_1} \|R(t_2-\eta) - R(t_1-\eta)\|^2 d\eta\\ &+(t_2-t_1)M_C^2\mathbb{G}\int_{t_1}^{t_2} \|R(t_2-\eta)\|^2 d\eta\\ &+9b\left[M_h^2 + \overline{M}_g^2 + Tr(Q)bM_g^2\right]\int_0^{t_1-\varepsilon} \|R(t_2-s) - R(t_1-s)\|^2 ds\\ &+9\varepsilon\left[M_h^2 + \overline{M}_g^2 + Tr(Q)bM_g^2\right]\int_{t_1-\varepsilon}^{t_1} \|R(t_2-s) - R(t_1-s)\|^2 ds\\ &+9(t_2-t_1)\left[M_h^2 + \overline{M}_g^2 + Tr(Q)bM_g^2\right]\int_{t_1}^{t_2} \|R(t_2-s) - R(t_1-s)\|^2 ds\Bigg\}.\end{aligned}$$

we see that $\|(\mathscr{Q}_2 z)(t_1) - (\mathscr{Q}_2 z)(t2)\|^2_Z$ tends to zero independently of $z \in B_q$ as $t_2 \to t_1$, with ε sufficiently small since the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology. Hence, $\mathscr{Q}_2$ maps B_q into a equicontinuous family of functions.

Also $\mathscr{Q}_2(B_q)$ is bounded in Z and so by the Arzela–Ascoli theorem, $\mathscr{Q}_2(B_q)$ is precompact. Hence it follows from the Nussbaum fixed point theorem there exists a fixed point $z(\cdot)$ for $\mathscr{Q}$ on B_q such that $\mathscr{Q}z(t) = z(t)$. Since we have $x(t) = z(t) + y(t)$, it follows that $x(t)$ is a mild solution of (1) on J satisfying $x(b) = x_1$. Thus the system (1) is controllable on J.

4 Example

In this section an example is presented for the controllability results to the following partial neutral stochastic integrodifferential equation:

$$\begin{aligned}
& d\left[v(t,x)+\int_{-\infty}^{t}\int_{0}^{\pi}\mu_1(s-t,y,x)v(s,y)dyds\right] \\
& =\frac{\partial^2}{\partial x^2}\left[v(t,x)+\int_{-\infty}^{t}\int_{0}^{\pi}\mu_1(s-t,y,x)v(s,y)dyds\right]dt \\
& +\int_{0}^{t}\gamma(t-s)\left[v(s,x)+\int_{-\infty}^{s}\int_{0}^{\pi}\mu_1(\tau-s,y,x)v(\tau,y)dyd\tau\right]ds \qquad (12)\\
& +\mu_3(x)v(t,x)+c(x)u(t)dt+\int_{-\infty}^{t}\mu_2(s-t)v(s,x)dw(s), \quad o\le x\le\pi,\ t\in J=[0,b], \\
& v(t,0)=v(t,\pi)=0, \quad t\ge 0, \\
& v(t,x)=\phi(t,x), \quad t\in J_0, \quad 0\le x\le\pi.
\end{aligned}$$

where $w(t)$ denotes an $\mathbb{R}$-valued Brownian motion.

To rewrite (12) into the abstract form of (1), we consider $H=K=U=L^2([0,\pi])$ with the norm $\|.\|$. Let $e_n:=\sqrt{\frac{2}{\pi}}\sin(nx)$, $(n=1,2,3,\cdots)$ denote the completed orthonormal basis in H and $w:=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\beta_n(t)e_n$ $(\lambda_n>0)$, where $\beta_n(t)$ are one dimensional standard Brownian motion mutually independent on a usual complete probability space $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\ge0},\mathbb{P})$.
Define $A:H\to H$ by $A=\frac{\partial^2}{\partial z^2}$, with domain $D(A)=H^2([0,\pi])\cap H_0^1([0,\pi])$ where

$$H_0^1([0,\pi]=\{\delta\in L^2([0,\pi]):\frac{\partial\delta}{\partial l}\in L^2([0,\pi]),\delta(0)=\delta(\pi)=0\}$$

and

$$H^2([0,\pi]=\{\delta\in L^2([0,\pi]):\frac{\partial\delta}{\partial l},\frac{\partial^2\delta}{\partial l}\in L^2([0,\pi]),\delta(0)=\delta(\pi)=0\}.$$

Then $Ah=-\sum_{n=1}^{\infty}n^2<h,e_n>e_n$, $h\in D(A)$, where e_n, $n=1,2,3,\cdots$, is also the orthonormal set of eigenvectors of A.
It is well-known that A is the infinitesimal generator of a strongly continuous semigroup on H, thus **(H1)** is true .

Let $B:D(A)\subset H\to H$ be the operator defined by $B(t)(z)=\gamma(t)Az$ for $t\ge0$ and $z\in D(A)$.

Here we take the phase space $\mathfrak{B}=C_0\times L^2(q;H)$, which contains all classes of functions $\phi:J_0\to H$ such that ϕ is $\mathfrak{F}_0$-measurable and $q(\cdot)\|\phi(\cdot)\|^2$ is integrable on J_0 where $q:(-\infty,0)\to\mathbb{R}$ is a positive integrable function. The seminorm in $\mathfrak{B}$ is defined by

$$\|\phi\|=\|\phi(0)\|+\left(\int_{-\infty}^{t}q(\theta)\|\phi(\theta)\|^2d\theta\right)^{\frac{1}{2}}.$$

The general form of phase space $\mathfrak{B}=C_r\times L^p(q;H), r\ge0, 1\le p<\infty$ has been discussed in Hino et al. [15] (here in particular, we are taking r=0, p= 2). From Hino et al. [15], under some conditions, $(\mathfrak{B},\|\phi\|_{\mathfrak{B}})$ is a Banach space that satisfies **(i)-(iii)** with

$$\Gamma(t)=1+\left(\int_{-t}^{0}q(\theta)d\theta\right)^{\frac{1}{2}}.$$

We assume the following conditions hold for system (12)

(i) The function $\mu_1(\cdot)$ is $\mathfrak{F}_t$-measurable and

$$\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}\mu_1^2(\theta,y,x)/q(\theta))dyd\theta dx<\infty$$

(ii) The function $\mu_2(\cdot)$ is $\mathfrak{F}_t$-measurable with

$$\int_{-\infty}^{0}\left(\mu_2^2(\theta)/q(\theta)\right)d\theta<\infty.$$

(iii) The function ϕ defined by $\phi(\theta)(x)=\phi(\theta,x)$ belongs to $\mathfrak{B}$.

(iv) $C:L^2([0,\pi])\to H$ is a bounded linear operator defined by

$$Cu(x)=c(x)u,\quad 0\leq x\leq\pi,\quad u\in\mathbb{R},\quad u\in L_2([0,\pi]).$$

(v) The linear operator $W:L^2(J,U)\to H$ defined by

$$Wu=\int_0^b R(b-s)c(x)u(s)ds$$

has an induced inverse operator W^{-1} defined on $L^2(J,\mathbb{R})/kerW$ and satisfies condition **(H4)**.

Now, define the operators $F:[0,\infty)\times\mathfrak{F}\to H, G:[0,\infty)\times\mathfrak{B}\to L(K,H)$ and $h:[0,\infty)\times\mathfrak{B}\to H$ respectively, as

$$\begin{aligned}
F(t,\phi)&=\Psi_1(\phi)=\int_{-\infty}^{0}\int_0^{\pi}\mu_1(\theta,y,x)\phi(\theta,y)dyd\theta,\\
G(t,\phi)&=\Psi_2(\phi)=\int_{-\infty}^{0}\mu_2(\theta)\phi(\theta,x)d\theta\\
h(t,\phi)&=\Psi_3(\phi)=\mu_3(x)\phi(\theta,x).
\end{aligned}$$

If we put
$\begin{cases} x(t) = v(t,\xi) \text{ for } t\geq 0 \text{ and } x\in[0,\pi]\\ \varphi(\theta)(\xi)=v_0(\theta,\xi) \text{ for } \theta\in]-\infty,0] \text{ and } x\in[0,\pi].\end{cases}$
then, system (12) is the abstract formulation of the system (1).

$$\begin{cases}
d[x(t)+F(t,x_t)]=\left[A[x(t)+F(t,x_t)]+\int_0^t B(t-s)[x(s)+F(s,x_s)]ds+Cu(t)+h(t,x_t)\right]dt\\
+\int_{-\infty}^{t}g(t,s,x_s)dw(s),\ \ t\in J:=[0,b],\\
x(0)=\xi,
\end{cases}\tag{13}$$

We suppose γ is a bounded and C^1 function such that γ' is bounded and uniformly continuous, which implies that the operator $B(t)$ satisfies **(H2)** . Consequently by Theorem 1, we deduce that Eq. (2) has a resolvent operator $(R(t))_{t\geq 0}$ on H. Moreover, for $0\leq s_1,s_2\leq b,\psi_1,\psi_2\in L_2(J,\mathfrak{B})$, we have from (i) by using Hölder inequality the following estimation

$$\|F(s_1,\psi_1)-F(s_2,\psi_2)\|\leq\left[\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}\mu_1^2(\theta,y,x)/q(\theta))dyd\theta dx\right]^{\frac{1}{2}}(|s_1-s_2|+\|\psi_1-\psi_2\|_{\mathfrak{B}}).$$

Similarly we can verify under conditions (ii) that F, G and h satisfy respectively the hypotheses **(H6)**-**(H8)**. Therefore, under the above assumptions, the stated conditions of Theorem 7 are satisfied, the system (12) is controllable on J .

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