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Applied Mathematics and Nonlinear Sciences 1(2) (2016) 437–472

Applied Mathematics and Nonlinear Sciences

http://journals.up4sciences.org

A survey on fractal dimension for fractal structures

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Submission Info

Communicated by Juan L.G. Guirao
Received 12th April 2016
Accepted 8th September 2016
Available online 8th September 2016

Abstract

Along the years, the foundations of Fractal Geometry have received contributions starting from mathematicians like Cantor, Peano, Hilbert, Hausdorff, Carathéodory, Sierpiński, and Besicovitch, to quote some of them. They were some of the pioneers exploring objects having self-similar patterns or showing anomalous properties with respect to standard analytic attributes. Among the new tools developed to deal with this kind of objects, fractal dimension has become one of the most applied since it constitutes a single quantity which throws useful information concerning fractal patterns on sets. Several years later, fractal structures were introduced from Asymmetric Topology to characterize self-similar symbolic spaces. Our aim in this survey is to collect several results involving distinct definitions of fractal dimension we proved jointly with Prof.M.A. Sánchez-Granero in the context of fractal structures.

Keywords: Fractals; fractal structures; fractal dimension; Hausdorff dimension; box-counting dimension; fractal geometry; fractal stochastic processes; Hurst exponent; applications of fractal dimension.
AMS 2010 codes: 37F35, 28A78, 28A80, 11K55, 54E35, 54E99

1 Introduction

The word *fractal*, which derives from the Latin term *frangere* (that means "to break"), has led to a very important concept in mathematics since Mandelbrot first introduced it in early eighties [1]. In fact, both the study and analysis of fractal patterns have become more and more important in the last years due to the large number of applications to diverse scientific fields where fractals have been identified. They include economics, physics, and statistics (c.f. [2, 3]). In addition, there has also been a special interest in the application of fractals to social sciences (c.f. [4] and the references therein).

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The main tool used in these areas to deal with fractals is the fractal dimension since it is their main invariant which throws some useful information about the complexity and irregularitires that a certain set presents once it has been explored with enough level of detail. It is worth mentioning that fractal dimension theory has also been applied in several scientific fields including the study of dynamical and mechanical systems [5, 6], diagnosis of diseases (such as osteoporosis [7] or cancer [8]), ecology [9], earthquakes [10], detection of eyes in human face images [11], analysis of the human retina [12], and brain computer interface systems [13], to name a few.

Usually, they are used both the Hausdorff and box dimensions, which can be defined on any metric space. Thus, while the former is "better" from a theoretical approach (since its definition is based on a measure), the latter is "better" from the viewpoint of applications, since it becomes easier to be empirically calculated or estimated. In this way, we should mention here that most empirical applications involving fractal dimension have been carried out in the context of Euclidean spaces through the box dimension.

The idea consisting of defining measures by means of coverings of certain subsets was first introduced by Carathéodory (c.f. [14]). Afterwards, Hausdorff applied this method to define the measures that now bear his name and showed that the middle third Cantor set has positive and finite measure of dimension equal to $\log 2/\log 3$ (c.f. [15]). Some properties and technical aspects regarding Hausdorff measures and dimensions have been developed mainly by Besicovitch [16], Besicovitch and his pupils [17], Falconer [2, 18], Feder [3], and Rogers [19].

On the other hand, it seems that the origins of box dimensions go back to the twenties, when they were first explored by pioneers of Hausdorff measure and dimension. Nevertheless, they were rejected for being less appropriate from a theoretical viewpoint. In this way, Bouligand adapted the Minkowski content to non-integral dimensions (c.f. [20]), and the classical definition of box dimension was provided by Pontrjagin and Schnirelman (c.f. [21]). Popularity of box dimension is mainly due to the possibility of its effective calculation and empirical estimation. Box dimension is also known as Kolmogorov entropy, entropy dimension, capacity dimension, metric dimension, information dimension, logarithmic density, . . . , etc.

The introduction of fractal structures, which were first sketched in [22] and then formally defined and applied in [23] to characterize non-Archimedeanly quasi-metrizable spaces, has allowed to formalize some topics on Fractal Geometry from both theoretical and applied viewpoints. A fractal structure is a countable collection of coverings of a given set which provides better approximations to it as deeper stages (called *levels* of the fractal structure) are explored. Accordingly, if we analyze the standard definition of the box dimension, then we can observe that fractal structures provide a perfect context where new models of fractal dimension can be provided. It is worth pointing out that the use of fractal structures allows to connect diverse interesting topics on Topology like transitive quasi-uniformities, non-Archimedean quasi-metrization, metrization, topological and fractal dimensions, self-similar sets, and even space-filling curves (c.f. [24]).

Moreover, self-similar sets constitute a kind of fractals which can be always endowed with a fractal structure on a natural manner (first introduced in [25]). Along this survey, we shall provide some results allowing to calculate the fractal dimension of self-similar sets throughout an easy equation only involving the similarity ratios associated with the corresponding iterated function system.

First, we shall motivate each definition of fractal dimension and provide useful expressions to deal with its effective calculation. We collect some connections of each definition of fractal dimension with the classical definitions of fractal dimension, namely, both the box and the Hausdorff dimensions. In addition, we also provide some links to other fractal dimensions defined from a fractal dimension approach. Interestingly, we shall generalize the box dimension throughout the so-called fractal dimensions I, II, and III, whereas we shall generalize the Hausdorff dimension by means of fractal dimensions V and VI. It is also worth mentioning that fractal dimension IV constitutes a middle definition between Hausdorff and box dimensions.

Next, we summarize the content of each section in this survey.

In Section 2, we recall some concepts, results, and notations that become useful to develop a new theory of fractal dimension for fractal structures. This section is focused on the following topics: quasi-pseudometrics, fractal structures, iterated function systems, box and Hausdorff dimensions.

In both Sections 3 and 4, we formally introduce fractal dimension I and II models to calculate the fractal dimension of a set with respect to a fractal structure.They extend the classical box dimension theory to the more general context of fractal structures. Thus, if it is selected the so-called natural fractal structure which any Euclidean set can be always endowed with, then the box dimension remains a particular case. This idea allows us to consider a wide range of fractal structures to calculate the fractal dimension of any set. Unlike it happens with the classical theory of fractal dimension, the definitions we provide along this section can be computed in contexts where the box dimension can lack sense or cannot be calculated. In fact, the new models can be applied to calculate the fractal dimension of any space admitting a fractal structure as easy as the boxdimension in empirical applications. The results contained in this section were first contributed in [26].

In Section 5, it is provided a new model to calculate the fractal dimension of a set with respect to a fractal structure which generalizes the box dimension in Euclidean spaces. This has been carried out by means of a suitable discretization regarding both the Hausdorff measure and dimension. Thus, we shall provide some connections among this middle definition and the classical ones as well as with both fractal dimensions I and II explored in previous Sections 3 and 4. In this way, we shall generalize them and provide an easy expression to calculate the fractal dimension of strict self-similar sets not required to satisfy the so-called open set condition. The results appeared along this section were first contributed in [27].

In Section 6, we study how to generalize the Hausdorff dimension throughout three new models of fractal dimension for a fractal structure: two of them consist of discretizations of the Hausdorff dimension (fractal dimensions IV and V), while the remaining one becomes a new continuous approach to Hausdorff dimension from a fractal structure approach. We shall collect several results where the three new definitions are connected among them and also with fractal dimensions I, II, and III as well as with classical dimension. It is worth noting that the analytic construction of fractal dimension VI is based on a measure as it is the case of Hausdorff dimension. Additionally, we shall generalize Hausdorff dimension by means of fractal dimensions V and VI in the context of Euclidean sets endowed with their natural fractal structures. The results appeared in this section first appeared in [28].

2 Preliminaries

The main goal in this section is to recall some notations, definitions, and notations that will result useful to tackle with a new theory of fractal dimension for fractal structures. In this way, we shall be focused on quasi-pseudometrics, fractal structures, iterated function systems, and Hausdorff and box dimension topics.

2.1 Quasi-pseudometrics

A quasi-pseudometric on a set X is a non-negative real-valued function ρ defined on $X \times X$ such that for all $x, y, z \in X$, the two following conditions are satisfied:

1. $\rho(x,x) = 0$.

2. $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$.

Moreover, if ρ satisfies also the next one:

3. $\rho(x,y) = \rho(y,x) = 0$ if and only if $x = y$,

then ρ is called a quasi-metric. In particular, a non-Archimedean quasi-pseudometric is a quasi-pseudometric which also satisfies that $\rho(x,y) \leq \max\{\rho(x,z), \rho(z,y)\}$ for all $x,y,z \in X$. Each quasi-pseudometric ρ on X generates a quasi-uniformity $\mathcal{U}_\rho$ on X which has as a base the family of sets of the form $\{(x,y) \in X \times X : \rho(x,y) < 1/2^n\} : n \in \mathbb{N}$. The topology $\tau(\mathcal{U}_\rho)$ induced by the quasi-uniformity $\mathcal{U}_\rho$ will be denoted $\tau(\rho)$, merely. Therefore, a topological space (X, τ) is said to be (non-Archimedeanly) quasi-pseudometrizable if there exists a (non-Archimedean) quasi-pseudometric ρ on X such that $\tau = \tau(\rho)$. The theory of quasi-uniform spaces is covered in detail in [29].

Let (X,ρ) be a (quasi-)metric space. Then we shall denote the diameter of a subset $A \subseteq X$ by $\mathrm{diam}(A) = \sup\{\rho(x,y) : x,y \in A\}$, as usual. In addition, $B_\rho(x,\varepsilon)$ will denote the ball centred in $x \in X$ with radius $\varepsilon > 0$ with respect to the metric (resp. quasi-metric) ρ, namely, $B_\rho(x,\varepsilon) = \{y \in X : \rho(x,y) < \varepsilon\}$.

2.2 Fractal structures

The concept of fractal structure was first introduced in [23] to characterize non-Archimedeanly quasi-metrizable spaces though it can be also used to deal with fractals. For instance, in [25] it was used to study attractors of iterated function systems.

Fractal structures constitute a powerful tool to introduce new models for a fractal dimension definition since it is a natural context where the concept of fractal dimension can be developed. Further, they will allow to calculate the fractal dimension in new spaces and situations.

A family Γ of subsets of a given space X is called a covering if $X = \cup\{A : A \in \Gamma\}$. Let Γ be a covering of X. Then we shall denote $\mathrm{St}(x,\Gamma) = \cup\{A \in \Gamma : x \in A\}$ and $U_{x\Gamma} = X \setminus \cup\{A \in \Gamma : x \notin A\}$. Furthermore, if $\mathbf{\Gamma} = \{\Gamma_n\}_{n\in\mathbb{N}}$ is a countable family of coverings of X, then we shall denote $U_{xn} = U_{x\Gamma_n}$, $\mathscr{U}_x^{\mathbf{\Gamma}} = \{U_{xn}\}_{n\in\mathbb{N}}$, and $\mathrm{St}(x,\mathbf{\Gamma}) = \{\mathrm{St}(x,\Gamma_n)\}_{n\in\mathbb{N}}$.

Next, we provide a first approach to define a fractal structure on a set X. Indeed, let Γ_1 and Γ_2 be two coverings of X. Thus, we shall denote $\Gamma_1 \prec \Gamma_2$ that Γ_1 is a refinement of Γ_2, namely, for all $A \in \Gamma_1$ there exists $B \in \Gamma_2$ such that $A \subseteq B$. In addition to that, the notation $\Gamma_1 \prec\prec \Gamma_2$ means that Γ_1 is a strong refinement of Γ_2, namely, $\Gamma_1 \prec \Gamma_2$, and in addition, for all $B \in \Gamma_2$ we can write $B = \cup\{A \in \Gamma_1 : A \subseteq B\}$. Thus, a fractal structure on a set X can be defined as a countable family of coverings of X, $\mathbf{\Gamma} = \{\Gamma_n\}_{n\in\mathbb{N}}$, such that $\Gamma_{n+1} \prec\prec \Gamma_n$ for all $n \in \mathbb{N}$. Next, we provide the definition of a fractal structure on a topological space.

Definition 1. (c.f. [23, Definition 3.1]) Let X be a topological space.

1. A pre-fractal structure on X is a countable family of coverings, $\mathbf{\Gamma} = \{\Gamma_n\}_{n\in\mathbb{N}}$ such that $\mathscr{U}_x^{\mathbf{\Gamma}}$ is an open neighborhood base for each $x \in X$.
2. Moreover, if Γ_{n+1} is a refinement of Γ_n such that for all $x \in A$ with $A \in \Gamma_n$, there exists $B \in \Gamma_{n+1}$ such that $x \in B \subseteq A$, then we will say that $\mathbf{\Gamma}$ is a fractal structure on X.
3. If $\mathbf{\Gamma}$ is a (pre-)fractal structure on X, then we will say that $(X,\mathbf{\Gamma})$ is a generalized (pre-)fractal space or merely a (pre-)GF-space. If there is no doubt concerning the fractal structure $\mathbf{\Gamma}$, then we will say that X is a (pre-)GF-space.

It is worth noting that covering Γ_n is called level n of the fractal structure $\mathbf{\Gamma}$.

Remark 1. (c.f. . [27, Remark 2.2]) To simplify the theory, the levels of any fractal structure $\mathbf{\Gamma}$ will not be coverings in the usual sense. Thus, we shall allow that a set may appear more than once in any level of $\mathbf{\Gamma}$, instead. For instance, $\Gamma_1 = \{[0,1/2],[1/2,1],[0,1/2]\}$ can be the first level of a fractal structure defined on $[0,1]$.

If $\mathbf{\Gamma}$ is a pre-fractal structure, then any of its levels is a closure-preserving closed covering (c.f. [30, Proposition 2.4]). Let $\mathbf{\Gamma}$ be a fractal structure on X such that $\mathrm{St}(x,\mathbf{\Gamma})$ is a neighborhood base for all $x \in X$. Then $\mathbf{\Gamma}$ is called a starbase fractal structure. Interestingly, starbase fractal structures are connected with metrizability (c.f. [30, 31]). A fractal structure $\mathbf{\Gamma}$ is finite provided that all its levels are finite coverings. A fractal structure $\mathbf{\Gamma}$ is said to be locally finite if for each level Γ_n of the fractal structure $\mathbf{\Gamma}$, it holds that any point $x \in X$ belongs to a finite number of elements $A \in \Gamma_n$. Additionally, a fractal structure $\mathbf{\Gamma}$ is said to be $\mathbf{\Gamma}-$Cantor-complete if for each decreasing sequence $\{A_n\}_{n\in\mathbb{N}}$ (namely, $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$) of subsets of X with $A_n \in \Gamma_n$, we have $\cap_{n\in\mathbb{N}} A_n \neq \emptyset$. In general, if Γ_n has the property $\mathscr{P}$ for all $n \in \mathbb{N}$ and $\mathbf{\Gamma} = \{\Gamma_n\}_{n\in\mathbb{N}}$ is a fractal structure on X, then we will say that $\mathbf{\Gamma}$ is a fractal structure with the property $\mathscr{P}$ and also that $(X,\mathbf{\Gamma})$ is a GF-space under such a property $\mathscr{P}$.

2.3 Iterated function systems

Self-similar sets are a kind of fractals that can be always endowed with a fractal structure in a natural way. Along this section, we recall a standard approach to construct attractors of iterated function systems. In addition, we shall also describe their natural fractal structure as self-similar sets. The results and properties described below are essential to understand some results appeared in upcoming sections.

Firstly, let $f : X \longrightarrow X$ be a mapping defined on a metric space (X,ρ). Recall that f is said to be Lipschitz if it satisfies the following condition: $\rho(f(x), f(y)) \leq c \cdot \rho(x,y)$ for all $x,y \in X$, where $c > 0$ is the Lipschitz constant associated with f. In particular, if $c < 1$, then f is said to be a contraction and c is called the contraction ratio associated with f. In addition, if it is reached the equality in the previous expression, namely, $\rho(f(x), f(y)) = c \cdot \rho(x,y)$ for all $x,y \in X$, then we have a similarity and its Lipschitz constant is called its similarity ratio.

Next, let us recall the standard construction of self-similar sets provided by Hutchinson (c.f. [32]). Assume that (X,ρ) is a complete metric space and let $\{f_i\}_{i=1}^m$ be a finite set of contractions defined on X. The scheme $(X, \{f_i\}_{i=1}^m)$ is called an iterated function system (IFS). Thus, let us consider that IFS in order to define the map $\mathscr{W} : \mathbb{H} \longrightarrow \mathbb{H}$ by $\mathscr{W}(A) = \cup_{i=1}^m f_i(A)$ for all $A \in \mathbb{H}$, where $\mathbb{H}$ denotes the hyperspace of X, namely, the set consisting of all non-empty compact subsets of X.

It can be proved that $\mathscr{W}$ is a contraction with respect to the Hausdorff metric d_{H} on $\mathbb{H}$ (c.f. [2, Section 9.1]), with associated contraction ratio $c < 1$. In addition, since (X,ρ) is a complete metric space, then $(\mathbb{H}, d_{\mathrm{H}})$ also is (due to Zenor-Morita's Theorem). Hence, since we have a contraction on a complete metric space, then the Banach fixed-point Theorem guarantees that there exists a unique non-empty compact subset $\mathscr{K} \subseteq X$ such that $\mathscr{K} = \mathscr{W}(\mathscr{K})$. $\mathscr{K}$ is said to be the attractor of the corresponding IFS. A strict self-similar set is an attractor of an IFS such that all its contractions are similarities. Along the sequel, by an IFS-attractor, we shall understand the attractor of an IFS whose contractions are similarities.
The next example describes analytically the so-called Sierpiński gasket, first defined in [33].

Example 1. *Let $I = \{1,2,3\}$ and $\{f_i\}_{i\in I}$ be a finite set of similarities defined from the Euclidean plane into itself as follows:*

$$f_i(x,y) = \begin{cases} \left(\frac{x}{2}, \frac{y}{2}\right) & \text{if } i = 1 \\ f_1(x,y) + \left(\frac{1}{2}, 0\right) & \text{if } i = 2 \\ f_1(x,y) + \left(\frac{1}{4}, \frac{1}{2}\right) & \text{if } i = 3 \end{cases}$$

for all $(x,y) \in \mathbb{R}^2$. Thus, the Sierpiński gasket is fully determined as the unique non-empty compact subset $\mathscr{K}$ satisfying the following Hutchinson's equation: $\mathscr{K} = \cup_{i\in I} f_i(\mathscr{K})$. It is worth noting that each component $f_i(\mathscr{K})$ is a self-similar copy of the whole Sierpiński gasket.

As it was mentioned previously, attractors of IFSs can be always endowed with a natural fractal structure, first sketched in [22] and formally defined later in [25]. It is worth pointing out that the latter provides the definition of a fractal structure as a mathematical concept, whereas the former deals with the (natural) fractal structure of a self-similar set. Next, we recall the description of such a fractal structure as provided in [25, Definition 4.4].

Definition 2. (c.f. [23, Definition 4.4]) Let $I = \{1, \ldots, m\}$ be a finite index set, $(X, \{f_i\}_{i\in I})$ an IFS, and $\mathscr{K}$ the attractor of that IFS. The natural fractal structure on $\mathscr{K}$ as a self-similar set is the countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n\}_{n\in\mathbb{N}}$, where $\Gamma_n = \{f_\omega(\mathscr{K}) : \omega \in I^n\}$ for each $n \in \mathbb{N}$. Here, for $n \in \mathbb{N}$ and each word $\omega = \omega_1\, \omega_2 \cdots \omega_n \in I^n$, we shall denote $f_\omega = f_{\omega_1} \circ \cdots \circ f_{\omega_n}$.

Remark 2. (c.f. [26, Remark 2.5]) Another suitable description concerning the levels of such a fractal structure is as follows: $\Gamma_1 = \{f_i(\mathscr{K}) : i \in I\}$ and $\Gamma_{n+1} = \{f_i(A) : A \in \Gamma_n, i \in I\}$ for all $n \in \mathbb{N}$.

In Example 1, it was analytically described the IFS whose attractor is the Sierpiński gasket. Next, we describe the natural fractal structure that can be defined on this strict self-similar set.

Example 2. (c.f. [26, Example 2]) The natural fractal structure on the Sierpiński gasket as a self-similar set is the countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n\}_{n\in\mathbb{N}}$, where Γ_1 is the union of three equilateral "triangles" with sides equal to $1/2$, Γ_2 consists of the union of 3^2 equilateral "triangles" with sides equal to $1/2^2$, and in general, Γ_n is the union of 3^n equilateral "triangles" whose sides are equal to $1/2^n$ for each natural number n. In addition, this is a finite starbase fractal structure.

2.4 The box dimension

Fractal dimension is one of the main tools applied to study fractals since it is a single quantity which throws useful information regarding their complexity when being examined with enough level of detail. It is worth noting that fractal dimension is usually understood as the classical box-counting dimension (box dimension along the sequel, for short) which is also known as information dimension, Kolmogorov entropy, capacity dimension, entropy dimension, metric dimension, logarithmic density, ..., etc. (c.f. [2, Section 3.1]).

Though the Hausdorff dimension can be also considered a fractal dimension, in practical applications it is always used the box dimension since it can be easily calculated for a finite range of scales, which is the case of empirical applications.Popularity of the box dimension is mainly due to the possibility of its effective calculation and empirical estimation in Euclidean contexts. It is worth pointing out that, concerning applications of fractal dimension, the box dimension can be estimated as the slope of the regression line of a $\log-\log$ graph plotted for a suitable discrete range of scales.

The basic theory on box dimension is covered in detail in [2]. Next, we recall the definition of the standard box dimension.

Definition 3. (c.f. [2, Section 3.1] and [2, Equivalent definitions 3.1]) The (lower/upper) box-counting dimension of a subset $F \subseteq \mathbb{R}^d$ is given by the following (lower/upper) limit:

$$\dim_{\mathrm{B}}(F) = \lim_{\delta \to 0} \frac{\log \mathcal{N}_\delta(F)}{-\log \delta}, \tag{1}$$

where δ is the scale and $\mathcal{N}_\delta(F)$ can be calculated equivalently throughout any of the following quantities (see [2, Equivalent Definitions 3.1]):

1. The number of $\delta-$cubes that intersect F, where a $\delta-$cube in $\mathbb{R}^d$ is a set of the form $[k_1\delta, (k_1+1)\delta] \times \cdots \times [k_d\delta, (k_d+1)\delta]$, where $k_1, \ldots, k_d \in \mathbb{Z}$.
2. The number of $\delta = 1/2^n-$cubes that intersect F with $n \in \mathbb{N}$.
3. The smallest number of sets of diameter at most δ that cover F.
4. The largest number of disjoint balls of radius δ with centres in F.

Notice also that the limit in Eq. (1) can be discretized by taking, for instance, $\delta = 1/2^n$. This is formalized in the next remark.

Remark 3. (c.f. [2, Section 3.1] and c.f. [26, Remark 2.7]) To calculate the (lower/upper) box dimension of any subset F of a Euclidean space $\mathbb{R}^d$, it suffices with taking limits as $\delta \to 0$ through any decreasing sequence $\{\delta_n\}_{n\in\mathbb{N}}$ satisfying that $c \cdot \delta_n \leq \delta_{n+1}$ for all $n \in \mathbb{N}$, where $c \in (0,1)$ is a suitable constant. In particular, it holds for $\delta_n = 1/2^n$.

2.5 The Hausdorff dimension

The main purpose of this section is to include a sketch about the construction of both Hausdorff measure and dimension whose definitions and properties can be found out in [2, Chapter 2].

The first to define a measure by means of coverings of sets was Carathéodory in [14]. Later (1919), Hausdorff used this method to define the measures that now bear his name, and showed that the middle third Cantor set has positive and finite measure of dimension equal to $\log 2/\log 3$ [15]. A detailed study regarding the analytical properties of both Hausdorff measure and dimension was mainly developed by Besicovitch and his pupils.

The Hausdorff dimension, which is the oldest definition of fractal dimension, presents the best analytical properties. Indeed, note that this fractal dimension can be defined for any subset of a Euclidean (resp. metrizable) space and its definition is based on a measure which makes it quite appropriate from a mathematical viewpoint. Nevertheless, it presents some disadvantages, especially from the viewpoint of applications, since it can be hard to calculate or to estimate.

Thus, while this fractal dimension is “better” from a theoretical approach, the box-counting dimension is “better” for a wide range of applications. Next, let us recall the analytical construction of the Hausdorff dimension. Let (X,ρ) be a metric space and δ be a positive real number. For any subset F of X, a $\delta-$cover of F is a countable family of subsets $\{U_i\}_{i\in I}$ such that $F \subseteq \cup_{i\in I} U_i$ with $\operatorname{diam}(U_i) \leq \delta$ for all $i \in I$. Let $\mathscr{C}_\delta(\mathscr{F})$ denote the collection of all $\delta-$covers of F. The underlying idea to define the Hausdorff measure consists of minimizing the sum of the $s-$powers of the diameters of all the subsets for any $\delta-$cover, where s is the fractal dimension to be calculated. In this way, the following quantity can be defined:

$$\mathscr{H}_\delta^s(F) = \inf\left\{\sum_{i\in I} \operatorname{diam}(U_i)^s : \{U_i\}_{i\in I} \in \mathscr{C}_\delta(F)\right\}. \tag{2}$$

Note that when δ decreases, then the class $\mathscr{C}_\delta(F)$ of all $\delta-$covers of F is reduced and hence, the measure of F increases. Accordingly, the next limit always exists:

$$\mathscr{H}_{\mathrm{H}}^s(F) = \lim_{\delta\to 0} \mathscr{H}_\delta^s(F) \tag{3}$$

which is called the $s-$dimensional Hausdorff measure of F.

It is worth mentioning that Hausdorff measure generalizes the classical Lebesgue measure for Euclidean subspaces. Indeed, if F is a Borel subset of $\mathbb{R}^d$, then we have $\mathscr{H}_{\mathrm{H}}^d(F) = c_d \cdot \operatorname{vol}^d(F)$, where the constant $c_d = \pi^{\frac{d}{2}}/(2^d \cdot (\frac{d}{2})!)$ is the volume of a $d-$dimensional ball of diameter equal to 1. In particular, $\mathscr{H}_{\mathrm{H}}^0(F) = \operatorname{cd}(F)$, namely, the number of points in F; $\mathscr{H}_{\mathrm{H}}^1(F)$ is the length of a smooth curve F; $\mathscr{H}_{\mathrm{H}}^2(F) = \frac{\pi}{4} \cdot \operatorname{area}(F)$, if F is a smooth surface; $\mathscr{H}_{\mathrm{H}}^3(F) = \frac{4}{3}\pi \cdot \operatorname{vol}(F)$; and in general, $\mathscr{H}_{\mathrm{H}}^m(F) = c_m \cdot \operatorname{vol}^m(F)$, if F is a smooth $m-$dimensional submanifold of $\mathbb{R}^d$, namely, an $m-$dimensional surface in the classical sense.

From Eq. (2), it becomes clear that for any set F and any scale $\delta \in (0,1)$, the quantity $\mathscr{H}_\delta^s(F)$ is non-increasing with s. Accordingly, $\mathscr{H}_{\mathrm{H}}^s(F)$ is also non-increasing with s (c.f. Eq. (3)). Let $t > s$ and let $\{U_i\}_{i\in I}$ be a δ-cover of F. Then $\sum_{i\in I} \operatorname{diam}(U_i)^t \leq \delta^{t-s} \cdot \sum_{i\in I} \operatorname{diam}(U_i)^s$, and taking infima in the last expression on the class $\mathscr{C}_\delta(F)$, it holds that $\mathscr{H}_\delta^t(F) \leq \delta^{t-s} \cdot \mathscr{H}_\delta^s(F)$. Letting $\delta \to 0$, we have $\mathscr{H}_{\mathrm{H}}^t(F) \leq \delta^{t-s} \cdot \mathscr{H}_{\mathrm{H}}^s(F)$. If $\mathscr{H}_{\mathrm{H}}^s(F) < \infty$ when $t > s$, then $\mathscr{H}_{\mathrm{H}}^t(F) = 0$. Thus, the point s where $\mathscr{H}_{\mathrm{H}}^s(F)$ “jumps” from ∞ to 0 is called the Hausdorff dimension of F (also called the Hausdorff-Besicovitch dimension). In fact, the Hausdorff dimension of F can be described in the following terms:

$$\dim_{\mathrm{H}}(F) = \inf\{s \geq 0 : \mathscr{H}_{\mathrm{H}}^s(F) = 0\} = \sup\{s \geq 0 : \mathscr{H}_{\mathrm{H}}^s(F) = \infty\},$$

or equivalently,

$$\mathscr{H}_{\mathrm{H}}^s(F) = \begin{cases} \infty \text{ if } s < \dim_H(F) \\ 0 \text{ if } s > \dim_H(F). \end{cases} \tag{4}$$

In particular, if $s = \dim_{\mathrm{H}}(F)$, then $\mathscr{H}_{\mathrm{H}}^s(F)$ can be equal to $0,\infty$, and even it can happen that $\mathscr{H}_{\mathrm{H}}^s(F) \in (0,\infty)$.

Next theorem collects several properties that are satisfied by Hausdorff dimension as a dimension function. Such a result will be referred to afterwards to compare these properties for the different models of fractal dimension for fractal structures we shall introduce along upcoming sections.

Theorem 3. (c.f. [2, Section 2.2])

1. *Monotonicity*: if $E \subseteq F$, then $\dim_{\mathrm{H}}(E) \leq \dim_{\mathrm{H}}(F)$.

2. *Countable stability*: if $\{F_i\}_{i\in I}$ is a countable collection of sets, then

$$\dim_{\mathrm{H}}(\cup_{i\in I}F_i) = \sup\{\dim_{\mathrm{H}}(F_i) : i \in I\}. \tag{5}$$

3. *Countable sets*: if F is a countable set, then $\dim_{\mathrm{H}}(F) = 0$.

It is also worth mentioning that the finite stability property consists of $\dim(E \cup F) = \max\{\dim(E), \dim(F)\}$. Clearly, if countable stability implies finite stability. In addition, we say that a dimension function satisfies the closure dimension property if there exists a subset F of X such that $\dim(F) \neq \dim(\overline{F})$, where $\overline{F}$ denotes the closure of F, namely, the smallest closed subset of $\mathbb{R}^d$ containing F.

The countable stability property satisfied by the Hausdorff dimension (c.f. Eq. (5)) is the key for the next result.

Remark 4. There exists a Euclidean subset F such that $\dim_{\mathrm{H}}(F) \neq \dim_{\mathrm{H}}(\overline{F})$.

Proof. Let $F = \mathbb{Q} \cap [0,1]$ considered as a Euclidean subset of the closed unit interval. Since F is countable, then we have $\dim_{\mathrm{H}}(F) = 0$ (due to Theorem 3 (3)). On the other hand, since $\overline{F} = [0,1]$, then it is clear that $\dim_{\mathrm{H}}(\overline{F}) = 1$. Accordingly, $\dim_{\mathrm{H}}(F) \neq \dim_{\mathrm{H}}(\overline{F})$.

It is also possible to calculate the Hausdorff dimension of a Euclidean subset F throughout $\delta-$covers of F consisting of open balls. In fact, if we define

$$\mathscr{B}^s_\delta(F) = \inf\left\{\sum_{i\in I} \operatorname{diam}(B_i)^s : \{B_i\}_{i\in I} \text{ is a } \delta-\text{cover of } F \text{ by balls}\right\},$$

then we obtain the measure $\mathscr{B}^s(F) = \lim_{\delta\to 0}\mathscr{B}^s_\delta(F)$ and also a dimension (we have to find out the point s where $\mathscr{B}^s(F)$ "jumps" from ∞ to 0) which agrees with the classical Hausdorff dimension described above (c.f. [2, Section 2.4]).

3 A box dimension type model

The main goal in this section is to generalize the classical box dimension in the broader context of fractal structures. We state that whether the so-called *natural fractal structure* (which any Euclidean subset can be always endowed with) is selected, then the box dimension remains as a particular case of the generalized fractal dimension model we shall explore along this section. That idea allows to consider a wide range of fractal structures to calculate the fractal dimension. Interestingly, unlike it happens with the classical box dimension, the new model of fractal dimension can be further calculated in non-Euclidean contexts, where the classical definitions of fractal dimension may lack sense. Another advantage of this new model of fractal dimension regards the possibility of its effective calculation or estimation for any space admitting a fractal structure. To calculate such a fractal dimension, we can proceed as easy as to estimate the box dimension in Euclidean applications.

3.1 The natural fractal structure on Euclidean subsets

Let F be a subset of $\mathbb{R}^d$ and $\mathscr{N}_\delta(F)$ be the number of $1/2^n-$cubes that intersect F (c.f. Definition 3 (2)). Consider also Remark 3. First, we shall define a fractal structure which every Euclidean set can be always endowed with. Such a fractal structure is locally finite and starbase.

Definition 4. (c.f. [26, Definition 3.1]) The natural fractal structure on every Euclidean space $\mathbb{R}^d$ is the countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n\}_{n\in\mathbb{N}}$ whose levels are given by

$$\Gamma_n = \left\{\left[\frac{k_1}{2^n}, \frac{k_1+1}{2^n}\right] \times \cdots \times \left[\frac{k_d}{2^n}, \frac{k_d+1}{2^n}\right] : k_1, \ldots, k_d \in \mathbb{Z}\right\}.$$

Next, we highlight that natural fractal structures can be induced on Euclidean subsets.

Remark 5. (c.f. [26, Remark 3.2]) Natural fractal structures can be always defined on Euclidean subsets from previous Definition 4. For instance, the natural fractal structure (induced) on the closed unit interval $[0,1]$ is defined as the family of coverings $\mathbf{\Gamma} = \{\Gamma_n\}_{n\in\mathbb{N}}$ whose levels are

$$\Gamma_n = \left\{ \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] : k = 0,1,\ldots,2^n - 1 \right\}.$$

The natural fractal structure on $\mathbb{R}^d$ is the tiling consisting of $1/2^n-$cubes on $\mathbb{R}^d$. Hence, note that both Definition 3 (2) of $\mathcal{N}_\delta(F)$ and Definition 4 allow to affirm that $\mathcal{N}_{1/2^n}(F)$ equals the number of elements in level n of the natural fractal structure that intersect a given subset F of $\mathbb{R}^d$. Following the analogy with the Euclidean case, we will define $\mathcal{N}_n(F)$ as the number of elements in level n of the fractal structure that intersect F.

3.2 Generalizing box dimension throughout fractal dimension I

Let us introduce our first generalized box dimension type model of fractal dimension.

Definition 5. (c.f. [26, Definition 3.3]) Let F be a subset of X, $\mathbf{\Gamma}$ be a fractal structure on X, and $\mathcal{N}_n(F)$ be the number of elements in level n of that fractal structure that intersect F. The (lower/upper) fractal dimension I of F is defined as the following (lower/upper) limit:

$$\dim_{\mathbf{\Gamma}}^1(F) = \lim_{n\to\infty} \frac{1}{n} \cdot \log_2 \mathcal{N}_n(F).$$

The following remark becomes especially appropriate to deal with empirical applications involving the calculation of fractal dimensions.

Remark 6. (c.f. [26, Remark 3.4]) Fractal dimension I can be estimated in empirical applications throughout the slope of a regression line comparing level n vs. $\log_2 \mathcal{N}_n(F)$, just like with box dimension estimation.

The first theoretical result we provide in this section establishes that fractal dimension I generalizes classical box dimension in the context of Euclidean subsets endowed with their natural fractal structures.

Theorem 4. *(c.f. [26, Theorem 3.5]) Let F be a subset of a Euclidean space $\mathbb{R}^d$ and $\mathbf{\Gamma}$ be the natural fractal structure on $\mathbb{R}^d$. Then the (lower/upper) fractal dimension I of F equals the (lower/upper) box dimension of F, namely:*

$$\dim_{\mathrm{B}}(F) = \dim_{\mathbf{\Gamma}}^1(F).$$

3.3 Theoretical properties of fractal dimension I

Recall that Hausdorff dimension constitutes the main theoretical model of fractal dimension that we should be mirrored in when providing new definitions of fractal dimension. In this way, our next goal is to explore some theoretical properties from those listed in both Theorem 3 and Remark 4 for fractal dimension I.

Proposition 5. *(c.f. [26, Proposition 3.6]) Let $\mathbf{\Gamma}$ be a fractal structure on X. The following statements hold.*

1. *Both the lower fractal dimension I and the upper fractal dimension I are monotonic.*
2. *The upper fractal dimension I is finitely stable.*
3. *There exist a countable subset F of X and a fractal structure $\mathbf{\Gamma}$ on X such that $\dim_{\mathbf{\Gamma}}^1(F) \neq 0$.*
4. *Neither the lower fractal dimension I nor the upper fractal dimension I are countably stable.*
5. *There exists a locally finite starbase fractal structure $\mathbf{\Gamma}$ defined on a certain subset $F \subseteq X$ such that $\dim_{\mathbf{\Gamma}}^1(F) \neq \dim_{\mathbf{\Gamma}}^1(\overline{F})$.*

3.4 Linking fractal dimension I to box dimension

Next step is to explore how box dimension and fractal dimension I are theoretically connected for any generalized-fractal space. To deal with, first we shall define the diameter of any level of a fractal structure and the diameter of any subset in a level of a fractal structure. Recall that a distance is a non-negative map $\rho : X \times X \longrightarrow \mathbb{R}$ such that $\rho(x,x) = 0$ for all $x \in X$.

Definition 6. (c.f. [26, Definition 3.7]) Let $\mathbf{\Gamma}$ be a fractal structure on a distance space (X,ρ) and F be a subset of X.

1. The diameter of level n of $\mathbf{\Gamma}$ is given by $\delta(\Gamma_n) = \sup\{\mathrm{diam}\,(A) : A \in \Gamma_n\}$.

2. The diameter of F in level n of $\mathbf{\Gamma}$ is calculated throughout the following expression:

$$\delta(F,\Gamma_n) = \sup\{\mathrm{diam}\,(A) : A \in \mathscr{A}_n(F)\},$$

where $\mathscr{A}_n(F) = \{A \in \Gamma_n : A \cap F \neq \emptyset\}$.

A feasible condition to be satisfied by a fractal structure consists of a geometric decrease regarding the sequence of diameters $\{\delta(F,\Gamma_n)\}_{n\in\mathbb{N}}$. That assumption leads to an upper bound to the box dimension of F in terms of its fractal dimension I (up to a constant).

Theorem 6. *(c.f. [26, Theorem 3.9]) Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ), F be a subset of X, and assume that there exists a constant $c \in (0,1)$ such that the following condition stands:*

$$\delta(F,\Gamma_{n+1}) \leq c \cdot \delta(F,\Gamma_n).$$

The three following hold:

1. $\overline{\dim}_B(F) \leq \gamma_c \cdot \overline{\dim}^1_{\mathbf{\Gamma}}(F)$.

2. $\underline{\dim}_B(F) \leq \gamma_c \cdot \underline{\dim}^1_{\mathbf{\Gamma}}(F)$.

3. *Additionally, if there exist both the fractal dimension I of F and the box dimension of F, then*

$$\dim_{\mathrm{B}}(F) \leq \gamma_c \cdot \dim^1_{\mathbf{\Gamma}}(F),$$

where the constant γ_c depends on c.

3.5 Fractal dimension I for IFS-attractors

It is worth noting that Theorem 6 can be extended to deal with IFS-attractors. In fact, this is due to the fact that the sequence of diameters $\{\delta(\Gamma_n)\}_{n\in\mathbb{N}}$ for this kind of sets decrease geometrically. With this aim, we shall select their natural fractal structures as self-similar sets.

Theorem 7. *(c.f. [26, Corollary 3.10]) Let $(X,\mathscr{F})$ be an IFS where X is a complete metric space and $\mathscr{K}$ is its IFS-attractor. Moreover, let $\mathbf{\Gamma}$ be the natural fractal structure on $\mathscr{K}$ as a self-similar set. Then*

$$\dim_{\mathrm{B}}(\mathscr{K}) \leq \gamma_c \cdot \dim^1_{\mathbf{\Gamma}}(\mathscr{K}),$$

where c is the maximum of the contraction factors associated with $\mathscr{F}$.

3.6 Dependence of fractal dimension I on the fractal structure

Next, we highlight how fractal dimension I depends on a selected fractal structure.

Remark 7. (c.f. [26, Remark 3.11]) There exists a Euclidean subset $\mathscr{C} \subset \mathbb{R}$ endowed with two distinct fractal structures, say $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$, such that $\dim^1_{\mathbf{\Gamma}_1}(\mathscr{C}) \neq \dim^1_{\mathbf{\Gamma}_2}(\mathscr{C})$.

Proof. Let $\mathbf{\Gamma}_1$ be the natural fractal structure on the standard middle third Cantor set $\mathscr{C}$. By Theorem 4, we have $\dim^1_{\mathbf{\Gamma}_1}(\mathscr{C}) = \dim_B(\mathscr{C})$ and such a value equals $\log 2/\log 3$ (c.f. [2, Example 3.3]). On the other hand, let $\mathbf{\Gamma}_2$ be the natural fractal structure on $\mathscr{C}$ as a self-similar set (c.f. Definition 2). Hence, $\dim^1_{\mathbf{\Gamma}_2}(\mathscr{C}) = 1$ since each level n of $\mathbf{\Gamma}_2$ contains 2^n "subintervals" with lengths equal to $1/3^n$.

A fractal structure is a kind of uniform structure. In fact, if there is no metric available in the space, the only way to "measure" a subset is by determining which level of the fractal structure contains that subset. In other words, it becomes quite natural that fractal dimension I depends on a fractal structure as well as box dimension depends on a metric.

4 A further step: fractal dimension II

Recall that fractal dimension I actually considers all the elements in level n of a fractal structure as having the same "size" (equal to $1/2^n$). In addition to a fractal structure, we can define a metric in the space to powerful effect. In fact, that metric can allow to "measure" the size of the elements in each level of the fractal structure. This is the case of any Euclidean subset, where they can be always considered both the natural fractal structure and the Euclidean metric.

Definition 7. (c.f. [26, Definition 4.1]) By a distance function (or a distance, for short), we shall understand a non-negative map $\rho : X \times X \longrightarrow \mathbb{R}$ such that $\rho(x,x) = 0$ for all $x \in X$.

Diameters of subsets, coverings, ... with respect to a distance are defined as in the case of a metric.
The second model for fractal dimension we shall provide with respect to a fractal structure is formulated in terms of a distance function.

Definition 8. (c.f. [26, Definition 4.2]) Let $\mathbf{\Gamma}$ be a fractal structure on a distance space (X,ρ), F be a subset of X, and $\mathscr{N}_n(F)$ be the number of elements in level n of $\mathbf{\Gamma}$ that intersect F. The (lower/upper) fractal dimension II of F is defined as the following (lower/upper) limit:

$$\dim^2_{\mathbf{\Gamma}}(F) = \lim_{n\to\infty} \frac{\log \mathscr{N}_n(F)}{-\log \delta(F,\Gamma_n)},$$

where $\delta(F,\Gamma_n)$ is the diameter of F in level n of $\mathbf{\Gamma}$.

4.1 A first connection between fractal dimensions I and II

In this subsection, we shall explore several conditions on the elements of each level of a fractal structure to reach the equality between fractal dimensions I and II. To tackle with, first we shall define the concepts of a semimetric on a topological space (c.f. [34, Definition 9.5]) and a semimetric associated with a starbase fractal structure.

Definition 9.

1. (c.f. [34, Definition 9.5]) A semimetric on a topological space X is a non-negative map $\rho : X \times X \longrightarrow \mathbb{R}$ satisfying the three following conditions:

   (a) $\rho(x,y) = 0$, if and only if, $x = y$.

(b) ρ is symmetric, namely, $\rho(x,y) = \rho(y,x)$ for all $x,y \in X$.
(c) The family $\{B_\rho(x,\varepsilon) : \varepsilon > 0\}$ is a neighborhood base for all $x \in X$. Equivalently, the topology induced by the semimetric ρ yields the starting topology.

2. (c.f. [25, Theorem 6.5]) Let $\mathbf{\Gamma}$ be a starbase fractal structure on X. The semimetric associated with $\mathbf{\Gamma}$ is defined as the non-negative map $\rho : X \times X \longrightarrow \mathbb{R}$ given by

$$\rho(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1/2^n & \text{if } y \in \mathrm{St}(x,\Gamma_n) \setminus \mathrm{St}(x,\Gamma_{n+1}) \\ 1 & \text{if } y \notin \mathrm{St}(x,\Gamma_1). \end{cases} \tag{6}$$

It is worth pointing out that Eq. (6) implies $B_\rho(x,1/2^n) = \mathrm{St}(x,\Gamma_{n+1})$ for all $n \in \mathbb{N}$ and each $x \in X$. Moreover, since $\mathbf{\Gamma}$ is starbase, we can affirm that the topology induced by the semimetric ρ matches with the topology induced by the fractal structure. Next, we contribute a condition concerning the levels of a (starbase) fractal structure to reach the equality between fractal dimensions I and II.

Theorem 8. *(c.f. [26, Theorem 4.6]) Let $\mathbf{\Gamma}$ be a starbase fractal structure on (X,ρ), where ρ is the semimetric associated with the fractal structure $\mathbf{\Gamma}$, and F be a subset of X. Moreover, assume that for all $n \in \mathbb{N}$ there exists $x \in F$ such that $\mathrm{St}(x,\Gamma_n) \neq \mathrm{St}(x,\Gamma_{n+1})$. The three following hold.*

1. $\overline{\dim}^1_{\mathbf{\Gamma}}(F) = \overline{\dim}^2_{\mathbf{\Gamma}}(F)$.
2. $\underline{\dim}^1_{\mathbf{\Gamma}}(F) = \underline{\dim}^2_{\mathbf{\Gamma}}(F)$.
3. *Additionally, if there exists either the fractal dimension I of F or the fractal dimension II of F, then* $\dim^1_{\mathbf{\Gamma}}(F) = \dim^2_{\mathbf{\Gamma}}(F)$.

Moreover, it can be also proved that fractal dimension II generalizes both fractal dimension I and box dimension in the context of Euclidean subsets endowed with their natural fractal structures. That result, which extends former Theorem 4, is stated next.

Theorem 9. *(c.f. [26, Theorem 4.7]) Let $\mathbf{\Gamma}$ be the natural fractal structure on $\mathbb{R}^d$ and $F \subseteq \mathbb{R}^d$. Then the (lower/upper) box dimension of F equals both the (lower/upper) fractal dimension I of F and the (lower/upper) fractal dimension II of F, namely:*

$$\dim_{\mathrm{B}}(F) = \dim^1_{\mathbf{\Gamma}}(F) = \dim^2_{\mathbf{\Gamma}}(F).$$

Notice that Theorem 9 allows to calculate the box dimension of any plane subset by counting triangles instead of squares, for instance. To deal with, we could define a fractal structure on $\mathbb{R}^2$ consisting of triangulations whose triangles have a diameter of $1/2^n-$order.

4.2 Theoretical properties of fractal dimension II

In this section, we theoretically explore the behavior of fractal dimension II as a dimension function similarly to Proposition 5 for fractal dimension I.

Proposition 10. *(c.f. [26, Remark 4.8 and Example 4]) Let $\mathbf{\Gamma}$ be a fractal structure on a distance space (X,ρ). The following statements hold.*

1. *Both the lower fractal dimension II and the upper fractal dimension II are monotonic.*
2. *Neither the lower fractal dimension II nor the upper fractal dimension II are finitely stable.*
3. *There exist a countable subset F of X and a fractal structure $\mathbf{\Gamma}$ on X such that* $\dim^2_{\mathbf{\Gamma}}(F) \neq 0$.
4. *Neither the lower fractal dimension II nor the upper fractal dimension II are countably stable.*
5. *There exists a locally finite starbase fractal structure $\mathbf{\Gamma}$ defined on a certain subspace $F \subseteq X$ such that* $\dim^2_{\mathbf{\Gamma}}(F) \neq \dim^2_{\mathbf{\Gamma}}(\overline{F})$.

4.3 Dependence of fractal dimension II on both a fractal structure and a metric

In Remark 7, we highlighted the dependence of fractal dimension I on the selected fractal structure. Next, we point out the additional dependence of fractal dimension II on a metric. In particular, we shall justify why the fractal dimension II of the standard middle third Cantor set (endowed with its natural fractal structure as a self-similar set) equals its box dimension.

Remark 8. (c.f. [26, Remark 4.9]) Let $\mathscr{C}$ denote the middle third Cantor set and $\mathbf{\Gamma}$ be the natural fractal structure on $\mathscr{C}$ as a self-similar set (c.f. Definition 2). Then the fractal dimension I of $\mathscr{C}$ does not equal its box dimension (c.f. Remark 7). More specifically, it holds that $\dim_{\mathrm{B}}(\mathscr{C}) = \log 2/\log 3$, whereas $\dim^1_{\mathbf{\Gamma}}(\mathscr{C}) = 1$. Observe that these fractal dimensions have been calculated with respect to distinct fractal structures. In fact, the natural fractal structure (induced) on $\mathscr{C}$ as a Euclidean subset is always chosen for box dimension calculation purposes. Nevertheless, if the natural fractal structure on $\mathscr{C}$ as a self-similar set is considered to calculate the fractal dimension II of $\mathscr{C}$, then we still have

$$\dim_{\mathrm{B}}(\mathscr{C}) = \dim^2_{\mathbf{\Gamma}}(\mathscr{C}) = \log 2/\log 3.$$

Proof. Indeed,

$$\dim^2_{\mathbf{\Gamma}}(\mathscr{C}) = \lim_{n\to\infty} \frac{\log 2^n}{-\log 3^{-n}} = \frac{\log 2}{\log 3} = \dim_{\mathrm{B}}(\mathscr{C}),$$

since level n of $\mathbf{\Gamma}$ consists of 2^n "subintervals" with diameters equal to $1/3^n$.

Even more, though the value obtained in Remark 7 for $\dim^1_{\mathbf{\Gamma}}(\mathscr{C})$ may seem counterintuitive at a first glance, it still becomes possible to justify it through its fractal dimension II value. Once again, the key reason lies in the fact that fractal dimension I only depends on the selected fractal structure. This is emphasized along the next remark.

Remark 9. (c.f. [26, Remark 4.10]) Fractal dimension I only depends on a fractal structure whereas fractal dimension II also depends on a distance.

Proof. To highlight that difference, we shall construct a family of spaces which are the same from the viewpoint of fractal structures. To deal with, let us consider slight modifications from the middle third Cantor set $\mathscr{C}$, which will we shall denote by $\mathscr{C}_i$. Let us assume that their similarity ratios are $c_i \in [\frac{1}{3}, \frac{1}{2})$ for each of the two similarities that yield $\mathscr{C}_i$. In addition, let $\mathbf{\Gamma}_i$ be the natural fractal structure on each space $\mathscr{C}_i$ as a self-similar set. Then $\delta(\mathscr{C}_i, \Gamma_n) = c_i^n$ and hence, easy calculations lead to (or apply upcoming Theorem 17)

$$\dim_{\mathrm{B}}(\mathscr{C}_i) = \dim^2_{\mathbf{\Gamma}_i}(\mathscr{C}_i) = -\frac{\log 2}{\log c_i} \longrightarrow 1 = \dim^1_{\mathbf{\Gamma}}(\mathscr{C}),$$

provided that $c_i \to 1/2$.

4.4 Linking fractal dimension II to box dimension

In this subsection, we provide an upper bound for both the Hausdorff and the box dimensions of any subset F in terms of its fractal dimension II.

Theorem 11. *(c.f. [26, Theorem 4.11]) Let $\mathbf{\Gamma}$ be a fractal structure on a distance space (X, ρ), F be a subset of X, and let us assume that $\delta(F, \Gamma_n) \to 0$. The three following hold.*

1. $\dim_{\mathrm{H}}(F) \leq \underline{\dim}_B(F) \leq \underline{\dim}^2_{\mathbf{\Gamma}}(F)$.

2. *If there exist both the box dimension and the fractal dimension II of F, then*

$$\dim_{\mathrm{H}}(F) \leq \dim_{\mathrm{B}}(F) \leq \dim^2_{\mathbf{\Gamma}}(F).$$

3. *If there exists a constant $c > 0$ such that $\delta(F,\Gamma_n) \leq c \cdot \delta(F,\Gamma_{n+1})$, then $\overline{\dim}_B(F) \leq \overline{\dim}^2_{\Gamma}(F)$.*

Theorem 11 also allows to achieve an upper bound for the box dimension of IFS-attractors throughout their fractal dimension II values. That result is stated next.

Theorem 12. *(c.f. [26, Corollary 4.12]) Let $\mathscr{F}$ be a Euclidean IFS whose IFS-attractor is $\mathscr{K}$. In addition, let F be a subset of $\mathscr{K}$ and $\mathbf{\Gamma}$ be the natural fractal structure on $\mathscr{K}$ as a self-similar set. The three following hold.*

1. $\underline{\dim}_B(F) \leq \underline{\dim}^2_{\Gamma}(F)$.
2. *If there exist both the box dimension and the fractal dimension II of F, then* $\dim_B(F) \leq \dim^2_{\Gamma}(F)$.
3. *Assume that f_i is a bi-Lipschitz function for some $i \in I$. Then $\overline{\dim}_B(F) \leq \overline{\dim}^2_{\Gamma}(F)$. In particular, this stands for strict self-similar sets.*

Our next goal is to explore which properties underlying the natural fractal structure on any Euclidean space (c.f. Definition 4) could allow to generalize Theorem 9. With this aim, observe that given a scale $\delta > 0$, it holds that any Euclidean subspace F of $\mathbb{R}^d$ such that $\mathrm{diam}\,(F) \leq \delta$ intersects at most to 3^d $\delta-$cubes. In this way, a similar property in the broader context of fractal structures would lead to an additional connection between fractal dimension II and box dimension.

Definition 10. Let $\mathbf{\Gamma}$ be a fractal structure on X and F be a subset of X. We shall understand that $\mathbf{\Gamma}$ is under the $\kappa-$condition if there exists a natural number κ such that for all $n \in \mathbb{N}$, every subset A of X with $\mathrm{diam}\,(A) \leq \delta(F,\Gamma_n)$ intersects at most to κ elements in level n of $\mathbf{\Gamma}$.

Theorem 13. *(c.f. [26, Theorem 4.13]) Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ), F be a subset of X, and assume that $\delta(F,\Gamma_n) \to 0$. In addition, assume that $\mathbf{\Gamma}$ is under the $\kappa-$condition. The three following hold.*

1. $\underline{\dim}_B(F) \leq \underline{\dim}^2_{\Gamma}(F) \leq \overline{\dim}^2_{\Gamma}(F) \leq \overline{\dim}_B(F)$.
2. *If there exists* $\dim_B(F)$, *then* $\dim_B(F) = \dim^2_{\Gamma}(F)$.
3. *If there exists a constant $c \in (0,1)$ such that $c \cdot \delta(F,\Gamma_n) \leq \delta(F,\Gamma_{n+1})$, then $\overline{\dim}_B(F) = \overline{\dim}^2_{\Gamma}(F)$ and $\underline{\dim}_B(F) = \underline{\dim}^2_{\Gamma}(F)$.*

It is worth pointing out that the main hypothesis in Theorem 13 to reach the equality between fractal dimension II and box dimension is necessary as the following counterexample highlights.

Counterexample 14. *(c.f. [26, Remark 4.14]) There exists a Euclidean IFS $\mathscr{F}$ whose IFS-attractor $\mathscr{K}$, endowed with its natural fractal structure as a self-similar set, satisfies that* $\dim_B(\mathscr{K}) \neq \dim^2_{\Gamma}(\mathscr{K})$.

Proof. Let $I = \{1,\ldots,8\}$ be a finite index set and $(\mathbb{R}^2,\mathscr{F})$ be a Euclidean IFS whose associated attractor is $\mathscr{K} = [0,1] \times [0,1]$. Further, define the contractions $f_i : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ as follows:

$$f_i(x,y) = \begin{cases} (\frac{x}{2},\frac{y}{4}) + (0,\frac{i-1}{4}) \text{ if } i = 1,2,3,4 \\ (\frac{x}{2},\frac{y}{4}) + (\frac{1}{2},\frac{i-5}{4}) \text{ if } i = 5,6,7,8. \end{cases}$$

In addition, let $\mathbf{\Gamma}$ be the natural fractal structure on $\mathscr{K}$ as a self-similar set. First, notice that the self-maps f_i are not similarities but affinities and all of them have the same contraction ratio, namely, $c_i = 1/2$. It is also clear that $\dim_B(\mathscr{K}) = 2$.
On the other hand, there are 8^n rectangles in level n of $\mathbf{\Gamma}$ whose dimensions are $\frac{1}{2^n} \times \frac{1}{2^{2n}}$. Hence,

$$\mathrm{diam}\,(A) = \delta(\mathscr{K},\Gamma_n) = \sqrt{\frac{1+2^{2n}}{2^{4n}}}$$

for all $A \in \Gamma_n$. Next, we calculate the fractal dimension II of $\mathscr{K}$.

$$\dim_{\Gamma}^{2}(\mathscr{K}) = \lim_{n\to\infty} \frac{\log N_n(\mathscr{K})}{-\log \delta(\mathscr{K},\Gamma_n)} = \lim_{n\to\infty} \frac{3n\log 2}{-\frac{1}{2}\log \frac{1+2^{2n}}{2^{4n}}} = \lim_{n\to\infty} \frac{3n\log 2}{n\log 2} = 3.$$

We also provide lower bounds for the ratios between $\delta(\mathscr{K},\Gamma_n)$ and the sides of each $\frac{1}{2^n} \times \frac{1}{2^{2n}}$−rectangle:

$$\frac{\sqrt{\frac{1+2^{2n}}{2^{4n}}}}{\frac{1}{2^{2n}}} = \sqrt{1+2^{2n}} > 2^n, \qquad \frac{\sqrt{\frac{1+2^{2n}}{2^{4n}}}}{\frac{1}{2^n}} = \sqrt{1+\frac{1}{2^{2n}}} \geq \frac{1}{2^n}.$$

Accordingly, each subset $A \subset \mathscr{K}$ whose diameter is at most $\sqrt{\frac{1+2^{2n}}{2^{4n}}}$ intersects at most to $3 \cdot 2^{n+1}$ elements in level n of $\mathbf{\Gamma}$. Since that quantity depends on each $n \in \mathbb{N}$, then the $\kappa-$condition is not satisfied.

4.5 Generalizing fractal dimension I by fractal dimension II

Let us recall when two sequences of positive real numbers are said to be of the same order.

Definition 11. (c.f. [26, Section 4]) Let $f, g : \mathbb{N} \longrightarrow \mathbb{R}$ be two sequences of positive real numbers. It is said that f and g are of the same order, namely, $\mathscr{O}(f) = \mathscr{O}(g)$, if and only if, the following condition stands:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \in (0,\infty).$$

Thus, if it is assumed that all the elements in each family $\mathscr{A}_n(F)$ have a diameter of $1/2^n-$order, then it can be proved that fractal dimension II equals fractal dimension I.

Theorem 15. *(c.f. [26, Theorem 4.15]) Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ), F be a subset of X, and assume that* $\operatorname{diam}(A) = \delta(F,\Gamma_n)$ *for all $A \in \mathscr{A}_n(F)$. If $\mathscr{O}(\delta(F,\Gamma_n)) = \mathscr{O}(1/2^n)$, then the (lower/upper) fractal dimension I of F equals the (lower/upper) fractal dimension II of F, namely,*

$$\dim_{\Gamma}^{1}(F) = \dim_{\Gamma}^{2}(F).$$

4.6 Fractal dimension II for IFS-attractors

Fractal dimension II provides an upper bound concerning the box dimension of any Euclidean IFS-attractor (c.f. Corollary 12). Going beyond, it is even possible to reach that equality under certain conditions on the corresponding IFS. More specifically, this kind of result stands provided that the elements in each level of the fractal structure do not overlap "too much". Hence, due to the shape of the elements in the natural fractal structure which any IFS-attractor can be endowed with, this restriction will rely on the similarities of the IFS. In this context, the so-called *open set condition* (OSC) plays a key role.

Definition 12. Let $\mathscr{F}$ be an IFS and $\mathscr{K}$ be its IFS-attractor.

1. (c.f. [32, Section 5.2]) We understand that $\mathscr{F}$ is under the OSC if there exists a (non-empty) bounded open subset $\mathscr{V} \subseteq X$ such that $\cup_{i\in I} f_i(\mathscr{V}) \subset \mathscr{V}$, where $f_i(\mathscr{V}) \cap f_j(\mathscr{V}) = \emptyset$ for all $i \neq j$.

2. (c.f. [35]) Additionally, if $\mathscr{V} \cap \mathscr{K} \neq \emptyset$, then $\mathscr{F}$ is said to satisfy the strong open set condition (SOSC).

Schief proved that both the OSC and the SOSC are equivalent for Euclidean IFSs (c.f. [36, Theorem 2.2]). In 1946, P.A.P. Moran contributed a strong result allowing the calculation of both the box and the Hausdorff dimensions for a certain class of Euclidean IFS-attractors throughout the (unique) solution of an equation only involving the similarity ratios associated with each similarity of the IFS (c.f. [37, Theorem III] and [2, Theorem 9.3]). That classical result is stated next.

Theorem 16 (Moran, 1946)**.** *Let $\mathscr{F} = \{f_1, \ldots, f_k\}$ be a Euclidean IFS under the OSC whose IFS-attractor is $\mathscr{K}$ and c_i be the similarity ratio associated with each similarity $f_i \in \mathscr{F}$. Then $\dim_H(\mathscr{K}) = \dim_B(\mathscr{K}) = s$, where s is the unique solution of the following expression:*

$$\sum_{i=1}^{k} c_i^s = 1. \tag{7}$$

Additionally, for that value of s, $\mathscr{H}_H^s(\mathscr{K}) \in (0, \infty)$.

Accordingly, under the OSC, the box dimension of any IFS-attractor equals its Hausdorff dimension, and that common value can be easily calculated from Eq. (7). The next result we provide guarantees the equality between the box dimension and the fractal dimension II of IFS-attractors lying under the OSC. Indeed, the calculation of these dimensions follows immediately from the number of similarities in the IFS and their common similarity ratio, as in [37, Theorem II].

Theorem 17. *(c.f. [26, Theorem 4.19]) Let $\mathscr{F} = \{f_1, \ldots, f_m\}$ be a Euclidean IFS under the OSC whose IFS-attractor is $\mathscr{K}$ and let $\mathbf{\Gamma}$ be the natural fractal structure on $\mathscr{K}$ as a self-similar set. Moreover, assume that all the similarities $f_i \in \mathscr{F}$ have a common similarity ratio $c \in (0,1)$. Then*

$$\dim_B(\mathscr{K}) = \dim_{\mathbf{\Gamma}}^2(\mathscr{K}) = -\frac{\log m}{\log c}.$$

We would like to point out that the hypothesis consisting of equal similarity ratios in Theorem 17 is necessary. Recall that Counterexample 14 implies that all the contractions involved in Theorem 17 must be similarities. Further, the following counterexample justifies why all the similarity ratios must be equal.

Counterexample 18. *(c.f. [26, Remark 4.20]) There exists a Euclidean IFS $\mathscr{F}$ under the OSC whose IFS-attractor $\mathscr{K}$, endowed with its natural fractal structure as a self-similar set, satisfies that $\dim_B(\mathscr{K}) < \dim_{\mathbf{\Gamma}}^2(\mathscr{K})$.*

Proof. Let $\mathscr{F} = \{f_1, f_2\}$ be a Euclidean IFS with similarities $f_1, f_2 : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f_i(x) = \begin{cases} \frac{x}{2} & \text{if } i = 1 \\ \frac{x+3}{4} & \text{if } i = 2. \end{cases}$$

It is clear that their associated contraction ratios are $c_1 = 1/2$ and $c_2 = 1/4$, respectively. Moreover, it holds that $\mathscr{K}$ is a strict self-similar set. It is also possible to justify that $\mathscr{F}$ is under the OSC. In fact, let $\mathscr{V} = (0,1) \subset \mathbb{R}$. Thus, Moran's Theorem allows to affirm that the box dimension of $\mathscr{K}$ equals the solution of the equation $\frac{1}{2^s} + \frac{1}{4^s} = 1$. Hence, $\dim_B(\mathscr{K}) = \log(\frac{1+\sqrt{5}}{2}) / \log 2$. Finally, observe that there are 2^n "subintervals" of $[0,1]$ in level n of the fractal structure $\mathbf{\Gamma}$, where the diameter of the largest of them equals $1/2^n$. Accordingly, $\dim_{\mathbf{\Gamma}}^2(\mathscr{K}) = 1 > \dim_B(\mathscr{K})$.

5 A middle definition between Hausdorff and box dimensions

In both Sections 3 and 4, two novel definitions of fractal dimension for a fractal structure have been explored. Recall that fractal dimension I allows a selection involving a larger collection of fractal structures than box dimension. In fact, the natural fractal structure on any Euclidean subset (c.f. Definition 4) throws the classical box dimension as a particular case (c.f. Theorem 4).

On the other hand, though the fractal dimension II model allows the possibility that different diameter sets could appear in a level of a fractal structure, it does not actually distinguish among different diameter sets (c.f. Remark 9). Recall that we have to count the number of elements in each level of a fractal structure that intersect a given set F to calculate its fractal dimensions I and II. Then we have to weigh these quantities by a discrete scale: either a fixed quantity for each level (in the case of fractal dimension I) or the "largest" diameter of all the elements in each family $\mathscr{A}_n(F) = \{A \in \Gamma_n : A \cap F \neq \emptyset\}$ (in the case of fractal dimension II). Both ideas give rise to suitable discretizations regarding the classical box dimension definition. Nevertheless, the Hausdorff dimension still constitutes the most accurate model to calculate the fractal dimension in metrizable spaces. Being based on this classical definition of dimension, our main goal along this section is to analytically construct a new definition of fractal dimension with respect to a fractal structure which someway seems the Hausdorff dimension.

5.1 Analytical construction of a new fractal dimension

Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X, ρ), F be a subset of X, and $s \geq 0$. We shall "measure" the size of each element in any collection $\mathscr{A}_n(F)$ by its diameter. Notice that the sum of the $s-$powers of the diameters of all the elements in each family $\mathscr{A}_n(F)$ allows to "measure" the level of irregularity and complexity of F provided that it is explored by a whole range of scales. In this way, we shall define the following expression for each natural number n:

$$\mathscr{H}_n^s(F) = \sum\{\text{diam}\,(A)^s : A \in \mathscr{A}_n(F)\}, \tag{8}$$

as well as its asymptotic behavior,

$$\mathscr{H}^s(F) = \lim_{n\to\infty} \mathscr{H}_n^s(F). \tag{9}$$

Recall that Hausdorff dimension is fully determined throughout the (unique) value of s satisfying the equality $\sup\{s : \mathscr{H}_{\text{H}}^s(F) = \infty\} = \inf\{s : \mathscr{H}_{\text{H}}^s(F) = 0\}$, where $\mathscr{H}_{\text{H}}^s(F)$ denotes the $s-$dimensional Hausdorff measure. Following the above, our next goal is to verify a property of this kind for the set function $\mathscr{H}^s(F)$ just defined in Eq. (9) with respect to a fractal structure. To deal with, let t be another non-negative real number and consider Eq. (8). Then

$$\sum_{A\in\mathscr{A}_n(F)} \text{diam}\,(A)^t \leq \delta(F, \Gamma_n)^{t-s} \cdot \sum_{A\in\mathscr{A}_n(F)} \text{diam}\,(A)^s. \tag{10}$$

Observe that Eq. (10) is equivalent to

$$\mathscr{H}_n^t(F) \leq \mathscr{H}_n^s(F) \cdot \delta(F, \Gamma_n)^{t-s}.$$

Letting $n \to \infty$, we have

$$\mathscr{H}^t(F) \leq \mathscr{H}^s(F) \cdot \lim_{n\to\infty} \delta(F, \Gamma_n)^{t-s}.$$

Hence, if $\mathscr{H}^s(F) < \infty$ and $\delta(F, \Gamma_n) \to 0$ for all $t > s$, then it holds that $\mathscr{H}^t(F) = 0$. Accordingly, under the natural hypothesis consisting of the sequence of diameters $\{\delta(F, \Gamma_n)\}_{n\in\mathbb{N}}$ goes to 0 (such a condition regarding the elements in each level of $\mathbf{\Gamma}$ makes the fractal structure being starbase, c.f. [26, Proposition 3.8]), our new theoretical method to calculate the fractal dimension of a subset with respect to a fractal structure establishes that this value is exactly the unique critical point where $\mathscr{H}^s(F)$ "jumps" from ∞ to 0. Formally, the new fractal dimension for a fractal structure can be described as follows:

$$\sup\{s \geq 0 : \mathscr{H}^s(F) = \infty\} = \inf\{s \geq 0 : \mathscr{H}^s(F) = 0\},$$

provided that $\delta(F, \Gamma_n) \to 0$. Going beyond, that hypothesis, which is only a natural constraint concerning the size of the elements in each level of the involved fractal structure, becomes necessary as the next counterexample highlights.

Counterexample 19. *(c.f. [27, Remark 4.1]) There exist a fractal structure $\mathbf{\Gamma}$ on a metric space (X, ρ) and a subset F of X with $\delta(F, \Gamma_n) \nrightarrow 0$, satisfying that*

$$\inf\{s \geq 0 : \mathscr{H}^s(F) = 0\} \neq \sup\{s \geq 0 : \mathscr{H}^s(F) = \infty\}.$$

Proof. Indeed, let $F=[0,1]\times[0,1]$ and $\mathbf{\Gamma}$ be the natural fractal structure on the unit square as a Euclidean subset but adding F itself to each level of $\mathbf{\Gamma}$. We shall apply Eqs. (8) and (9) to calculate the fractal dimension of F. In this case, we have $\delta(F,\Gamma_n)=\sqrt{2}$ for all $n\in\mathbb{N}$, leading to $\delta(F,\Gamma_n)\nrightarrow 0$. Moreover, the next expression holds:

$$\mathscr{H}_n^s(F)=2^{\frac{s}{2}}\cdot\left(1+\frac{1}{2^{n(s-2)}}\right).$$

Hence, it becomes clear that

$$\mathscr{H}^s(F)=\begin{cases}\infty & \text{if } s<2\\ 2^{\frac{s}{2}} & \text{if } s>2,\end{cases}$$

which implies $\inf\{s\geq 0:\mathscr{H}^s(F)=0\}\neq\sup\{s\geq 0:\mathscr{H}^s(F)=\infty\}$.

However, unlike it happens with the $s-$dimensional Hausdorff measure $\mathscr{H}_{\mathrm{H}}^s(F)$ (which always exists for all subsets of X), the set function $\mathscr{H}_n^s(F)$ described in Eq. (8) is not monotonic in $n\in\mathbb{N}$. This implies that $\mathscr{H}^s(F)$ does not exist in general. Consequently, it becomes necessary to consider again lower/upper limits in Eq. (9).

Interestingly, the problem consisting of the existence of the limit in Eq. (9) can be avoided whether the families $\mathscr{A}_n(F)$ are properly replaced by the following coverings of F by elements of a certain level of $\mathbf{\Gamma}$, instead:

$$\mathscr{A}_{n,3}(F)=\cup\{\mathscr{A}_m(F):m\geq n\}. \tag{11}$$

It is noteworthy that whether the families $\mathscr{A}_{n,3}(F)$ are considered to calculate the fractal dimension of F, then the arguments carried out above still remain valid.

5.2 Defining fractal dimension III

Next, we provide the key definition of fractal dimension for a fractal structure we shall explore along this section.

Definition 13. (c.f. [27, Definition 4.2]) Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ), F be a subset of X, and assume that $\delta(F,\Gamma_n)\to 0$. Moreover, consider

$$\mathscr{H}_{n,3}^s(F)=\inf\{\mathscr{H}_m^s(F):m\geq n\}, \tag{12}$$

where

$$\mathscr{H}_n^s(F)=\sum\{\operatorname{diam}(A)^s:A\in\mathscr{A}_n(F)\},$$

and

$$\mathscr{H}_3^s(F)=\lim_{n\to\infty}\mathscr{H}_{n,3}^s(F).$$

The fractal dimension III of F is defined as the following critical point:

$$\dim_{\mathbf{\Gamma}}^3(F)=\sup\{s\geq 0:\mathscr{H}_3^s(F)=\infty\}=\inf\{s\geq 0:\mathscr{H}_3^s(F)=0\}.$$

It is worth pointing out that the sequence $\{\mathscr{H}_{n,3}^s(F)\}_{n\in\mathbb{N}}$ provided in Eq. (12) can be also described throughout any of the equivalent expressions provided in the next remark.

Remark 10. (c.f. [27, Remark 4.3]) The following expressions are equivalent to calculate $\mathscr{H}_{n,3}^s(F)$ for all subset F of X and all natural number n:

1. $\inf\{\mathscr{H}_m^s(F):m\geq n\}$.
2. $\inf\{\sum_{A\in\mathscr{A}_m(F)}\operatorname{diam}(A)^s:m\geq n\}$.
3. $\inf\{\sum_{A\in\mathscr{B}}\operatorname{diam}(A)^s:\mathscr{B}\in\mathscr{A}_{n,3}(F)\}$, where $\mathscr{A}_{n,3}(F)$ was given previously in Eq. (11).

From Definition 13 of fractal dimension, it follows that the quantity $\mathcal{H}_3^s(F)$ can be described similarly to $\mathcal{H}_{\mathrm{H}}^s(F)$ in terms of $\dim_{\mathbf{\Gamma}}^3(F)$, instead (c.f. Eq. (4)):

$$\mathcal{H}_3^s(F) = \begin{cases} \infty \text{ if } s < \dim_{\mathbf{\Gamma}}^3(F) \\ 0 \text{ if } s > \dim_{\mathbf{\Gamma}}^3(F), \end{cases} \tag{13}$$

provided that $\delta(F,\Gamma_n) \to 0$.

Additionally, the next remark becomes quite useful for fractal dimension III calculation purposes, since it highlights that it is no longer necessary to consider lower/upper limits to define $\mathcal{H}_3^s(F)$.

Remark 11. (c.f. [27, Remark 4.4]) Since $\mathcal{H}_{n,3}^s(F)$ is the general term of a monotonic sequence in $n \in \mathbb{N}$, then the fractal dimension III of any subset F of X always exists.

5.3 Linking fractal dimension III to some fractal dimensions

Along this subsection, we contribute several results to theoretically connect fractal dimension III with the classical definitions of fractal dimension, namely, both Hausdorff and box dimensions, as well as with fractal dimension II, previously explored in Section 4 for fractal structures.

Theorem 20. *(c.f. [27, Theorem 4.5]) Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ) and F be a subset of X. In addition, assume that $\delta(F,\Gamma_n) \to 0$. The three following hold.*

1. $\dim_{\mathbf{\Gamma}}^3(F) \leq \underline{\dim}_{\mathbf{\Gamma}}^2(F) \leq \overline{\dim}_{\mathbf{\Gamma}}^2(F)$.
2. *If* $\operatorname{diam}(A) = \delta(F,\Gamma_n)$ *for all* $A \in \mathcal{A}_n(F)$, *then* $\underline{\dim}_B(F) \leq \dim_{\mathbf{\Gamma}}^3(F)$.
3. $\dim_{\mathrm{H}}(F) \leq \dim_{\mathbf{\Gamma}}^3(F)$.

The next corollary stands immediately from Theorem 20.

Corollary 21. *(c.f. [27, Corollary 4.6]) Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ) and F be a subset of X. In addition, assume that $\delta(F,\Gamma_n) \to 0$. The two following hold.*

1. $\dim_{\mathrm{H}}(F) \leq \dim_{\mathbf{\Gamma}}^3(F) \leq \underline{\dim}_{\mathbf{\Gamma}}^2(F) \leq \overline{\dim}_{\mathbf{\Gamma}}^2(F)$.
2. *If* $\operatorname{diam}(A) = \delta(F,\Gamma_n)$ *for all* $A \in \mathcal{A}_n(F)$, *then* $\dim_{\mathrm{H}}(F) \leq \underline{\dim}_B(F) \leq \dim_{\mathbf{\Gamma}}^3(F) \leq \underline{\dim}_{\mathbf{\Gamma}}^2(F) \leq \overline{\dim}_{\mathbf{\Gamma}}^2(F)$.

5.4 How to calculate the effective fractal dimension III

For a given subset $F \subseteq X$, the calculation of each term in the sequence $\mathcal{H}_n^s(F)$ (c.f. Eq. (8)) seems to be easier to be calculated than the corresponding in $\mathcal{H}_{n,3}^s(F)$ (as described in Eq. (12)). In addition, as Remark 11 points out, fractal dimension III always exists provided that the set function $\mathcal{H}_{n,3}^s$ is considered to deal with its effective calculation. Following the above, the next theoretical result we provide allows the calculation of fractal dimension III from easier Eqs. (8) and (9).

Theorem 22. *(c.f. [27, Theorem 4.7]) Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ) and F be a subset of X. In addition, assume that there exists $\mathcal{H}^s(F)$ and $\delta(F,\Gamma_n) \to 0$. The fractal dimension III of F is the unique critical point described as follows:*

$$\dim_{\mathbf{\Gamma}}^3(F) = \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\} = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\}.$$

5.5 Measure properties of $\mathscr{H}^s_{n,3}$

As well as the Hausdorff dimension definition is based on the $s-$dimensional Hausdorff measure $\mathscr{H}^s_H$, next we shall explore some measure properties regarding the set functions $\mathscr{H}^s_{n,3}, \mathscr{H}^s_3$, and $\mathscr{H}^s$ that allow the calculation of the fractal dimension III for any subset $F \subseteq X$.

To deal with, let $\mathscr{P}(X)$ denote the class of all subsets of a given space X. Recall that an *outer measure* is a set function $\mu : \mathscr{P}(X) \longrightarrow [0,\infty]$ satisfying the three following conditions (c.f., e.g., [38, Section 5.2]):

1. It assigns the value 0 to the empty set, namely, $\mu(\emptyset) = 0$.
2. It is monotonic increasing, i.e., if $E,F \in \mathscr{P}(X) : E \subseteq F$, then $\mu(E) \leq \mu(F)$.
3. It is *countably subadditive*, namely, it satisfies that

$$\mu\left(\cup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

for all sequence $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{P}(X)$.

Notice that $\mathscr{H}^s_n$ is an outer measure for all natural number n. Moreover, the following result concerning the set function $\mathscr{H}^s_{n,3}$ can be stated.

Proposition 23. *Let $\mathscr{H}^s_{n,3} : \mathscr{P}(X) \longrightarrow [0,\infty]$ be the set function defined by*

$$\mathscr{H}^s_{n,3}(F) = \inf\{\mathscr{H}^s_m(F) : m \geq n\}$$

(c.f. Eq. (12) or one of its equivalent expressions provided in Remark 10). Then $\mathscr{H}^s_{n,3}(F)$ is an outer measure for all $n \in \mathbb{N}$.

It is worth mentioning that though the two set functions $\mathscr{H}^s_n$ and $\mathscr{H}^s_{n,3}$ are outer measures for all natural number n, their limits as $n \to \infty$ are not, in general, as the following counterexample points out.

Counterexample 24. *(c.f. [27, Remark 4.8]) Neither $\mathscr{H}^s$ nor $\mathscr{H}^s_3$ are outer measures.*

5.6 Linking fractal dimension III to fractal dimensions I and II

Another issue naturally arising consists of determining some reasonable conditions on the elements in each level of a fractal structure to guarantee the equality among fractal dimension III and fractal dimensions I and II. In this way, the following result we provide allows the calculation of fractal dimension III from the fractal dimension I formula provided that fractal structures having and appropriate size (of $1/2^n-$order) are selected to deal with the calculations.

Theorem 25. *(c.f. [27, Theorem 4.10]) Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ) and F be a subset of X. Additionally, assume that $\delta(F,\Gamma_n) \to 0$ and $\mathscr{O}(\delta(F,\Gamma_n)) = \mathscr{O}(1/2^n)$. If there exists the fractal dimension I of F, then*

$$\dim^1_{\mathbf{\Gamma}}(F) = \dim^3_{\mathbf{\Gamma}}(F).$$

Regarding the existence of the fractal dimension I of F in previous Theorem 25, it is noteworthy that whether fractal dimension I does not exist, then Theorem 25 still throws the expected equality between (lower) fractal dimensions I and fractal dimension III. Next, we shall highlight that theoretical fact.

Remark 12. (c.f. [27, Remark 4.11]) Under the hypothesis of Theorem 25, suppose that fractal dimension I does not exist for a given subset $F \subseteq X$. Then we still have

$$\underline{\dim}^1_{\mathbf{\Gamma}}(F) = \dim^3_{\mathbf{\Gamma}}(F).$$

Next step is to find out appropriate conditions regarding the size of the elements in each level of a fractal structure to reach the equality between fractal dimensions II and III.

Theorem 26. *(c.f. [27, Theorem 4.12]) Let* $\mathbf{\Gamma}$ *be a fractal structure on a metric space* (X,ρ) *and* F *be a subset of* X*. Additionally, assume that* $\delta(F,\Gamma_n)\to 0$ *and there exists a natural number* n_0 *such that* $\operatorname{diam}(A)=\delta(F,\Gamma_n)$ *for all* $A\in\mathscr{A}_n(F)$ *and all* $n\geq n_0$*. If there exists the fractal dimension II of* F*, then*

$$\dim_{\mathbf{\Gamma}}^2(F)=\dim_{\mathbf{\Gamma}}^3(F).$$

Under the same hypothesis, a weaker result than Theorem 26 can be stated in the case that fractal dimension II does not exist. This is similar to Remark 12 allowing $\underline{\dim}_{\mathbf{\Gamma}}^1(F)=\dim_{\mathbf{\Gamma}}^3(F)$.

Remark 13. (c.f. [27, Remark 4.13]) Under the hypothesis of Theorem 26, assume that fractal dimension II does not exist for a given subset $F\subseteq X$. Then we have

$$\underline{\dim}_{\mathbf{\Gamma}}^2(F)=\dim_{\mathbf{\Gamma}}^3(F).$$

From both Remarks 12 and 13, we can state that fractal dimension III generalizes both fractal dimensions I and II for fractal structures having $1/2^n-$order elements in each level n.

Corollary 27. *(c.f. [27, Corollary 4.14]) Let* $\mathbf{\Gamma}$ *be a fractal structure on a metric space* (X,ρ) *and* F *be a subset of* X*. Additionally, assume that* $\operatorname{diam}(A)=\delta(F,\Gamma_n)$ *for all* $A\in\mathscr{A}_n(F)$ *and* $\mathscr{O}(\delta(F,\Gamma_n))=\mathscr{O}(1/2^n)$*. Then*

$$\underline{\dim}_{\mathbf{\Gamma}}^1(F)=\underline{\dim}_{\mathbf{\Gamma}}^2(F)=\dim_{\mathbf{\Gamma}}^3(F).$$

It is worth mentioning that Corollary 27 allows the calculation of the fractal dimension III of any subset with respect to a fractal structure $\mathbf{\Gamma}$ under the conditions provided therein via an easier box dimension type formula.

Moreover, the following result establishes that all these fractal dimensions are equal in the context of Euclidean GF-spaces equipped with their natural fractal structures. In other words, fractal dimension III generalizes all the box dimension type models for fractal dimension including the classical one.

Theorem 28. *(c.f. [27, Theorem 4.15]) Let* $\mathbf{\Gamma}$ *be the natural fractal structure on* $\mathbb{R}^d$ *and* $F\subseteq\mathbb{R}^d$*. Then the (lower/upper) box dimension of* F *equals the (lower/upper) fractal dimensions I, II, and III of* F*, namely:*

$$\underline{\dim}_B(F)=\underline{\dim}_{\mathbf{\Gamma}}^1(F)=\underline{\dim}_{\mathbf{\Gamma}}^2(F)=\dim_{\mathbf{\Gamma}}^3(F).$$

Previous Theorem 28 makes fractal dimension III to be understood as a hybrid approach to fractal dimension. In fact, though the analytical construction of fractal dimension III is based on a suitable discretization regarding the Hausdorff dimension, such a result states that fractal dimension III equals box dimension in the context of Euclidean subsets equipped with their natural fractal structures.

It is also worth pointing out that Theorem 28 also allows the calculation of fractal dimension III for Euclidean subsets throughout easier box dimension type expressions such as those provided in Sections 3 and 4.

5.7 Theoretical properties of fractal dimension III

Next, we collect several theoretical properties for fractal dimension III similarly to Proposition 5 for fractal dimension I and Proposition 10 for fractal dimension II.

Proposition 29. *(c.f. [27, Proposition 4.16]) Let* $\mathbf{\Gamma}$ *be a fractal structure on a metric space* (X,ρ) *and assume that* $\delta(F,\Gamma_n)\to 0$*. The following statements hold.*

1. *Fractal dimension III is monotonic.*
2. *Fractal dimension III is finitely stable.*

3. *There exist a countable subset F of X and a fractal structure $\mathbf{\Gamma}$ on X such that* $\dim_{\mathbf{\Gamma}}^3(F) \neq 0$.
4. *Fractal dimension III is not countably stable.*
5. *There exists a locally finite starbase fractal structure $\mathbf{\Gamma}$ defined on a certain subset $F \subseteq X$ such that* $\dim_{\mathbf{\Gamma}}^3(F) \neq \dim_{\mathbf{\Gamma}}^3(\overline{F})$.

5.8 An additional connection with box dimension

Recall that in Theorem 13, some properties regarding the elements in each level of a fractal structure were provided to reach the equality between fractal dimension II and box dimension. It is worth pointing out that box dimension may be also defined for metrizable spaces. The next result we provide has been carried out in the spirit of Theorem 13 and generalizes Theorem 28.

Theorem 30. *(c.f. [27, Theorem 4.17]) Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ), F be a subset of X, assume that $\delta(F,\Gamma_n) \to 0$, and suppose that $\mathbf{\Gamma}$ is under the $\kappa-$condition. If* $\operatorname{diam}(A) = \delta(F,\Gamma_n)$ *for all $A \in \mathscr{A}_n(F)$, and there exists* $\dim_{\mathrm{B}}(F)$, *then*

$$\dim_{\mathrm{B}}(F) = \dim_{\mathbf{\Gamma}}^3(F).$$

5.9 Fractal dimension III for IFS-attractors

As it was stated previously in Subsection 4.6, the issue concerning the calculation of the fractal dimension for IFS-attractors via algebraic expressions involving only a finite number of known quantities arises naturally for each new definition of fractal dimension. It is worth pointing out that this kind of theoretical results are inspired on classical Moran's Theorem (c.f. Theorem 16) and usually assume that the similarities that give rise to the IFS-attractor are under the OSC hypothesis (c.f. Definition 12). In fact, recall that this constitutes the main constraint required to an IFS to reach the equality between the Hausdorff and the box dimensions of its strict self-similar set (c.f. Theorem 16).

The OSC is a *strong* hypothesis requested to the self-similar copies of the whole IFS-attractor (sometimes called pre-fractals) to guarantee that they do not overlap "too much". In this way, Theorem 17 stands under the OSC for fractal dimension II. Interestingly, the fractal dimension III model allows the calculation of the fractal dimension of strict self-similar sets via a Moran's type equation (c.f. Eq. (7)) even if the similarities of the IFS do not lie under the OSC. This allows to generalize Moran's Theorem in the context of fractal structures. To prove such a theoretical result, both the natural fractal structure which any IFS-attractor can be always endowed with (c.f. Definition 2 or Remark 2) and Equivalent definition (2) in Remark 10 for $\mathscr{H}_{n,3}^s$ do play a relevant role herein.

Theorem 31. *(c.f. [27, Theorem 4.20]) Let X be a complete metric space, $\mathscr{F} = \{f_1,\ldots,f_k\}$ be an IFS whose IFS-attractor is $\mathscr{K}$, c_i be the similarity ratio associated with each similarity f_i on X, and $\mathbf{\Gamma}$ be the natural fractal structure on $\mathscr{K}$ as a self-similar set. Then*

$$\dim_{\mathbf{\Gamma}}^3(\mathscr{K}) = s : \quad \sum_{i=1}^{k} c_i^s = 1.$$

Additionally, for this value of s, it holds that $\mathscr{H}_3^s(\mathscr{K}) \in (0,\infty)$.

Next, we verify that Theorem 31 cannot be improved in the sense that the similarities $f_i \in \mathscr{F}$, that give rise to the IFS-attractor $\mathscr{K}$, cannot be weakened to merely contractions. To deal with, we provide an appropriate counterexample.

Counterexample 32. *(c.f. [27, Remark 4.21]) There exists a Euclidean IFS $\mathscr{F} = \{f_1,\ldots,f_k\}$ whose (non-strict) IFS-attractor $\mathscr{K}$, endowed with its natural fractal structure as a self-similar set, satisfies that*

$$\dim_{\mathbf{\Gamma}}^3(\mathscr{K}) \neq s : \quad \sum_{i=1}^{k} c_i^s = 1.$$

Proof. Let $I = \{1,\dots,8\}$ be a finite index set and $(\mathbb{R}^2, \mathscr{F} = \{f_i : i \in I\})$ be a Euclidean IFS whose attractor is $\mathscr{K} = [0,1] \times [0,1]$. Define the self-maps $f_i : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ as follows:

$$f_i(x,y) = \begin{cases} (\frac{-y}{2}, \frac{x}{4}) + (\frac{1}{2}, \frac{i-1}{4}) \text{ if } i = 1,\dots,4 \\ (\frac{-y}{2}, \frac{x}{4}) + (1, \frac{i-5}{4}) \text{ if } i = 5,\dots,8. \end{cases}$$

In addition, let $\mathbf{\Gamma}$ be the natural fractal structure on $\mathscr{K}$ as a self-similar set. First of all, notice that $\mathscr{K}$ is not a strict self-similar set. Further, observe that the contractions f_i are compositions of affine maps, including rotations, dilations (in the plane and with respect to one coordinate), and translations. Moreover, all the contractions f_i have a common ratio, equal to $1/2$. It is also clear that $s = 3$ is the solution of the equation $\sum_{i=1}^{k} c_i^s = 1$.

On the other hand, we affirm that $\dim_{\mathbf{\Gamma}}^3(\mathscr{K}) = 2$. To deal with, we shall calculate the fractal dimension III of $\mathscr{K}$ in the sense of Theorem 30. Consider all the even levels in $\mathbf{\Gamma}$. Thus, for all natural number n, each level $2n$ consists of squares with sides equal to $1/8^n$. Also, we have $\operatorname{diam}(A) = \delta(\mathscr{K}, \Gamma_{2n}) = \sqrt{2}/8^n$ for all $A \in \Gamma_{2n}$. Letting $n \to \infty$, it holds that $\delta(\mathscr{K}, \Gamma_{2n}) \to 0$. Next, we verify that $\mathbf{\Gamma}$ is under the $\kappa-$condition. We shall proceed by calculating the maximum number of elements in Γ_{2n} that are intersected by a subset $B : \operatorname{diam}(B) \leq \sqrt{2}/8^n$. Observe that the ratio between the diameter of each square in level $2n$ and its side is equal to $\sqrt{2} < 2$, then it holds that the number of elements in $\mathscr{A}_{2n}(B)$ is at most 3 in each direction for all subset $B : \operatorname{diam}(B) < \delta(K, \Gamma_{2n})$. Accordingly, $\kappa_1 = 9$ provides a suitable constant for all the levels of even order in $\mathbf{\Gamma}$. Similarly, notice that all the levels of odd order in $\mathbf{\Gamma}$ consist of rectangles with dimensions $\frac{1}{2} \cdot \frac{1}{8^n} \times \frac{1}{4} \cdot \frac{1}{8^n}$ for all $n \in \mathbb{N}$. It is worth noting that all the elements in each odd level $2n+1$ have the same diameter, equal to $\frac{1}{4} \cdot \frac{\sqrt{5}}{8^n}$. Hence, the sequence of diameters $\delta(\mathscr{K}, \Gamma_{2n+1}) \to 0$. Finally, to check the $\kappa-$condition, observe that the following ratios between each diameter and the corresponding sides of each rectangle stand:

$$\frac{\frac{1}{4} \cdot \frac{\sqrt{5}}{8^n}}{\frac{1}{2} \cdot \frac{1}{8^n}} = \frac{\sqrt{5}}{2} < 2, \qquad \frac{\frac{1}{4} \cdot \frac{\sqrt{5}}{8^n}}{\frac{1}{4} \cdot \frac{1}{8^n}} = \sqrt{5} < 3.$$

Therefore, each subset $A : \operatorname{diam}(A) \leq \delta(\mathscr{K}, \Gamma_{2n+1})$ meets at most to $\kappa_2 = 12$ elements in each level of odd order. Hence, the κ-condition is satisfied since $\kappa = \max\{\kappa_1, \kappa_2\} = 12$ is a valid constant for any level of $\mathbf{\Gamma}$. Accordingly,

$$\dim_{\mathrm{B}}(\mathscr{K}) = \dim_{\mathbf{\Gamma}}^3(\mathscr{K}) = 2.$$

As a consequence of Moran's Theorem and Theorem 31, we have that the fractal dimension III of any IFS-attractor (endowed with its natural fractal structure) equals both its Hausdorff and box dimensions provided that the corresponding IFS is under the OSC.

Corollary 33. *(c.f. [27, Corollary 4.22]) Let $\mathscr{F}$ be a Euclidean IFS under the OSC whose IFS-attractor is $\mathscr{K}$, c_i be the similarity ratio associated with each similarity $f_i \in \mathscr{F}$, and $\mathbf{\Gamma}$ be the natural fractal structure on $\mathscr{K}$ as a self-similar set. Then*

$$\dim_{\mathrm{H}}(\mathscr{K}) = \dim_{\mathrm{B}}(\mathscr{K}) = \dim_{\mathbf{\Gamma}}^3(\mathscr{K}).$$

Nevertheless, as the following counterexample highlights, the OSC hypothesis cannot be removed in previous Corollary 33.

Counterexample 34. *(c.f. [27, Remark 4.23]) There exists an IFS $\mathscr{F} = \{f_1,\dots,f_k\}$ whose IFS-attractor $\mathscr{K}$, endowed with its natural fractal structure as a self-similar set, satisfies that*

$$\dim_{\mathrm{H}}(\mathscr{K}) \neq \dim_{\mathbf{\Gamma}}^3(\mathscr{K}).$$

Proof. Let $I = \{1,2,3\}$ be a finite index set and $(\mathbb{R}, \{f_i : i \in I\})$ be an IFS whose attractor $\mathscr{K} = [0,1]$ satisfies the Hutchinson's equation $\mathscr{K} = \bigcup_{i \in I} f_i(\mathscr{K})$. Define also the contractions $f_i : \mathbb{R} \longrightarrow \mathbb{R}$ as follows:

$$f_i(x) = \begin{cases} \frac{x}{2} & \text{if } i = 1 \\ \frac{x+1}{2} & \text{if } i = 2 \\ \frac{2x+1}{4} & \text{if } i = 3. \end{cases}$$

Additionally, let $\mathbf{\Gamma}$ be the natural fractal structure on $\mathscr{K}$ as a self-similar set. Notice that $\mathscr{K} \subseteq \mathbb{R}$ is a strict self-similar set since all the contractions are similarities having a common similarity ratio, equal to $1/2$. Moeover, we have

$$\sum_{i\in I} c_i^{\dim_{\mathbf{\Gamma}}^3(\mathscr{K})} = 1,$$

due to Theorem 31. Hence, $\dim_{\mathbf{\Gamma}}^3(\mathscr{K}) = \frac{\log 3}{\log 2}$. On the other hand, both Theorems 15 and 25 lead to

$$\dim_{\mathbf{\Gamma}}^1(\mathscr{K}) = \dim_{\mathbf{\Gamma}}^2(\mathscr{K}) = \dim_{\mathbf{\Gamma}}^3(\mathscr{K}) = \frac{\log 3}{\log 2},$$

since all the elements in level n of $\mathbf{\Gamma}$ are of $1/2^n-$order. Finally, assume that $\mathscr{F}$ is under the OSC. Thus,

$$\dim_{\mathbf{\Gamma}}^3(\mathscr{K}) \text{ would equal } \dim_{\mathrm{H}}(\mathscr{K}) = 1,$$

by Corollary 33, a contradiction.

Following Remark 34, we conclude that fractal dimension III does not coincide, in general, with Hausdorff dimension. Similarly, next we state that fractal dimension III may be also different from both fractal dimensions I and II.

Remark 14. (c.f. [27, Remark 4.24]) There exists an IFS $\mathscr{F} = \{f_1,\ldots,f_k\}$ whose IFS-attractor $\mathscr{K}$, endowed with its natural fractal structure as a self-similar set, satisfies that

$$\dim_{\mathbf{\Gamma}}^1(\mathscr{K}) = \dim_{\mathbf{\Gamma}}^2(\mathscr{K}) \neq \dim_{\mathbf{\Gamma}}^3(\mathscr{K}).$$

Proof. Let $I = \{1,2,3\}$ be a finite index set and $(\mathbb{R},\{f_i : i\in I\})$ be an IFS whose attractor $\mathscr{K} = [0,1]$ satisfies the Hutchinson's equation $\mathscr{K} = \bigcup_{i\in I} f_i(\mathscr{K})$. Define also the self-maps $f_i : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f_i(x) = \begin{cases} \frac{x}{2} & \text{if } i = 1 \\ \frac{x+2}{4} & \text{if } i = 2 \\ \frac{x+3}{4} & \text{if } i = 3. \end{cases}$$

Further, let $\mathbf{\Gamma}$ be the natural fractal structure on $\mathscr{K}$ as a self-similar set. It is clear that $\mathscr{K}$ is a strict self-similar set since all the self-maps f_i are similarities. Moreover, the OSC is fulfilled. In fact, $\mathscr{V} = (0,1) \subset \mathbb{R}$ is a feasible open set. Hence, Corollary 33 leads to

$$\dim_{\mathrm{H}}(\mathscr{K}) = \dim_{\mathrm{B}}(\mathscr{K}) = \dim_{\mathbf{\Gamma}}^3(\mathscr{K}) = 1.$$

On the other hand, notice that each covering Γ_n of $\mathbf{\Gamma}$ contains 3^n subintervals of $[0,1]$. Since $\delta(\mathscr{K},\Gamma_n) = 1/2^n$,

$$\dim_{\mathbf{\Gamma}}^1(\mathscr{K}) = \dim_{\mathbf{\Gamma}}^2(\mathscr{K}) = \frac{\log 3}{\log 2},$$

by applying both Definition 8 and Theorem 15.

6 Hausdorff dimension type models for fractal structures

In this section, we study how to generalize the Hausdorff dimension throughout three new models of fractal dimension for a fractal structure: two of them will consist of an appropriate discretization regarding the Hausdorff dimension (fractal dimensions IV and V), whereas the remaining one will constitute a new continuous approach from a fractal structure viewpoint (fractal dimension VI). Several theoretical results to connect the three new definitions among them and also with fractal dimensions I, II, and III (introduced in previous sections) as well as with the classical definitions of fractal dimension will be provided. Moreover, we shall explore how the analytic construction regarding fractal dimension VI is based on a measure as it is the case of Hausdorff dimension. The main result in this section consists of a generalization of classical Hausdorff dimension in the context of Euclidean subspaces (endowed with their natural fractal structures) throughout both fractal dimensions V and VI. Finally, we shall contribute a result for IFS-attractors allowig to calculate these fractal dimensions via an easy equation only involving the similarity factors associated with the corresponding IFS.

6.1 Improving the accuracy of fractal dimension III

Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ), F be a subset of X, and assume that $\delta(F,\Gamma_n)\to 0$. It is worth mentioning that the topology induced by the fractal structure $\mathbf{\Gamma}$ usually coincides with the topology induced by the metric ρ on X. Thus, we shall always refer to the topology induced by $\mathbf{\Gamma}$, since both topologies do not have to be the same, in general (c.f. [28, Remark 3.1]).

On the other hand, recall that the calculation of $\mathscr{H}^s_{n,3}(F)$ basically consists of minimizing the sum of the $s-$powers of the diameters of all the elements in an appropriate $\delta(F,\Gamma_n)-$cover of F, say $\{A_i\}_{i\in I}$, where all the elements A_i belong to a same level of $\mathbf{\Gamma}$ deeper than or equal to level n. Such a condition can be mathematically described in the following terms: there exists $m\geq n$ such that $A_i\in\Gamma_m$ for all $i\in I$. A further consideration is to allow that given a level n of $\mathbf{\Gamma}$, each element A_i may lie in a level deeper than or equal to n, though not always being the same, necessarily. Formally, for all $i\in I$, there exists $m(i)\geq n$ such that $A_i\in\Gamma_{m(i)}$. Accordingly, let us define the following collection of $\delta(F,\Gamma_n)-$coverings of F:

$$\mathscr{B}_n(F)=\{\{A_i\}_{i\in I}:A_i\in\cup_{l\geq n}\Gamma_l, F\subseteq\cup_{i\in I}A_i\}. \tag{14}$$

Thus, let $s\geq 0$ and consider, additionally,

$$\mathscr{D}^s_n(F)=\inf\left\{\sum_{i\in I}\operatorname{diam}(A_i)^s:\{A_i\}_{i\in I}\in\mathscr{B}_n(F)\right\}. \tag{15}$$

If $\mathscr{D}^s(F)=\lim_{n\to\infty}\mathscr{D}^s_n(F)$, then it holds that the set function $\mathscr{D}^s$ behaves similarly to the $s-$dimensional Hausdorff measure. To justify that, let $t\geq 0:t>s$. Hence, $\sum\operatorname{diam}(A_i)^t\leq\delta(F,\Gamma_n)^{t-s}\cdot\sum\operatorname{diam}(A_i)^s$, where in the previous sums, $A_i\in\{A_i\}_{i\in I}\in\mathscr{B}_n(F)$. Therefore, $\mathscr{D}^t_n(F)\leq\delta(F,\Gamma_n)^{t-s}\cdot\mathscr{D}^s_n(F)$. Letting $n\to\infty$, we have $\mathscr{D}^t(F)\leq\mathscr{D}^s(F)\cdot\lim_{n\to\infty}\delta(F,\Gamma_n)^{t-s}$. Accordingly, if $\mathscr{D}^s(F)<\infty$, then $\mathscr{D}^s(F)=0$ since $t>s$ and $\delta(F,\Gamma_n)\to 0$, by hypothesis. This implies that the equality $\sup\{s:\mathscr{D}^s(F)=\infty\}=\inf\{s:\mathscr{D}^s(F)=0\}$ throws a critical (unique) value, which could be understood as a new fractal dimension for fractal structures consisting of a discretization regarding the classical Hausdorff dimension.

Similarly to Counterexample 19, the following result points out that the condition $\delta(F,\Gamma_n)\to 0$ becomes necessary for upcoming fractal dimension calculation purposes.

Counterexample 35. *(c.f. [40, Counterexample 5.2]) There exist a fractal structure $\mathbf{\Gamma}$ on a metric space (X,ρ) and a subset F of X with $\delta(F,\Gamma_n)\nrightarrow 0$, satisfying that*

$$\inf\{s\geq 0:\mathscr{D}^s(F)=0\}\neq\sup\{s\geq 0:\mathscr{D}^s(F)=\infty\}.$$

Proof. Indeed, let $F=(0,1]\subset\mathbb{R}$ and $\mathbf{\Gamma}=\{\Gamma_n:n\in\mathbb{N}\}$ be a fractal structure on $[0,1]$ whose levels are defined as follows:

$$\Gamma_n=\{[0,1]\}\cup\left\{\left[\frac{k}{2^n},\frac{k+1}{2^n}\right]:k=1,\ldots,2^n-1\right\}. \tag{16}$$

Since $\delta(F,\Gamma_n)=1$ for all $n\in\mathbb{N}$, then $\delta(F,\Gamma_n)\nrightarrow 0$. Additionally, the following identity stands:

$$\mathscr{D}^s(F)=\begin{cases}1 & \text{if } s\leq 1\\ 0 & \text{if } s>1.\end{cases}$$

In fact,

1. If $s \leq 1$, then $\mathscr{D}^s(F) = 1$, since one of the two following cases may occur.
   (a) Assume that the covering of F we choose for fractal dimension calculation purposes contains the interval $[0,1]$. Thus,
      i. If that covering is $\{[0,1]\}$, then $\mathscr{D}_n^s(F) = (\mathrm{diam}([0,1]))^s \leq 1$, and hence, $\mathscr{D}^s(F) \leq 1$.
      ii. Assume that the covering of F we choose, $\{A_i\}_{i\in I}$, contains $[0,1]$ as well as some elements of the natural fractal structure on $[0,1]$. Then $\sum_{i\in I}(\mathrm{diam}(A_i))^s \geq 1$. Accordingly, $\mathscr{D}^s(F) \leq 1$.
   (b) On the other hand, assume that the covering of F we select, $\{A_i\}_{i\in I}$, does not contain the closed unit interval $[0,1]$. Accordingly, such a covering will consist of some elements of the natural fractal structure on $[0,1]$. Since the fractal dimension of F with respect to that natural fractal structure is equal to 1, then $\sum_{i\in I}(\mathrm{diam}(A_i))^s \geq 1$. Hence, $\mathscr{D}^s(F) = 1$.

2. On the other hand, if $s > 1$, then $\mathscr{D}^s(F) = 0$. Indeed, if $\mathbf{\Sigma} = \{\Sigma_n : n \in \mathbb{N}\}$ denotes the natural fractal structure (induced) on $[0,1]$, then it becomes clear that $\mathscr{D}^s(F) = \mathscr{H}^s(F) = 0$, due to [28, Theorem 3.10]. Let $\varepsilon > 0$ be fixed but arbitrarily chosen. Thus, there exists a covering $\{A_i\}_{i\in I} \in \mathscr{B}_n(F)$ such that for all $i \in I$, it holds that $A_i \in \Sigma_k : k \geq n$ and it is satisfied that $\sum_{i\in I}\mathrm{diam}(A_i)^s < \varepsilon$. Therefore, one of the two following cases may occur.

   - $A_i \in \Gamma_k : k \geq n$, or
   - $A_i = \left[0, \frac{1}{2^k}\right] \notin \Gamma_k : k \geq n$. In this case, though, observe that

$$\left[0, \frac{1}{2^k}\right] = \bigcup_{\alpha\geq 1}\left[\frac{1}{2^{k+\alpha}}, \frac{1}{2^{k+\alpha-1}}\right].$$

   Accordingly, a new covering $\mathscr{B}$ of F can be constructed from all the elements in $\{A_i\}_{i\in I}$ but replacing the elements of the form $\left[0, \frac{1}{2^k}\right]$ by

$$\left\{\left[\frac{1}{2^{k+\alpha}}, \frac{1}{2^{k+\alpha-1}}\right] : \alpha \geq 1\right\},$$

   instead. In fact, for each element of the form $\left[0, \frac{1}{2^k}\right]$, we can write

$$\begin{aligned}\sum_{\alpha=1}^{+\infty}\frac{1}{(2^{k+\alpha})^s} &= \frac{1}{2^{ks}} \cdot \sum_{\alpha=1}^{+\infty}\frac{1}{(2^s)^\alpha} \\ &= \frac{1}{2^{ks}} \cdot \frac{1}{2^s - 1} \\ &< \frac{1}{2^{ks}} = \left(\mathrm{diam}\left(\left[0, \frac{1}{2^k}\right]\right)\right)^s.\end{aligned}$$

   Thus, $\sum_{B\in\mathscr{B}}\mathrm{diam}(B)^s \leq \sum_{i\in I}\mathrm{diam}(A_i)^s < \varepsilon$, which leads to $\mathscr{D}_k^s(F) < \varepsilon$. Hence, $\mathscr{D}^s(F) = 0$ for all $s > 1$.

Similarly to both Eqs. (14) and (15), the following expressions lead to a discrete fractal dimension for finite coverings, which will become especially appropriate to deal with empirical applications of fractal dimension [41]. Let us define

$$\mathscr{L}_n(F) = \{\{A_i\}_{i\in I} : A_i \in \cup_{l\geq n}\Gamma_l \text{ for all } i \in I, F \subseteq \cup_{i\in I}A_i, \mathrm{cd}(I) < \infty\},$$

as well as

$$\mathscr{K}_n^s(F) = \inf\left\{\sum_{i\in I}\mathrm{diam}(A_i)^s : \{A_i\}_{i\in I} \in \mathscr{L}_n(F)\right\}. \tag{17}$$

Thus, the asymptotic behavior of Eq. (17) plays a similar role to Hausdorff measure. Let $\mathscr{K}^s(F) = \lim_{n\to\infty} \mathscr{K}_n^s(F)$.

The following remark is analogous to Counterexample 35.

Counterexample 36. *(c.f. [40, Counterexample 5.2]) There exist a fractal structure $\mathbf{\Gamma}$ on a metric space (X,ρ) and a subset F of X with $\delta(F,\Gamma_n) \nrightarrow 0$, satisfying that*

$$\inf\{s \geq 0 : \mathscr{K}^s(F) = 0\} \neq \sup\{s \geq 0 : \mathscr{K}^s(F) = \infty\}.$$

Proof. Let $F = (0,1] \subset \mathbb{R}$ and $\mathbf{\Gamma}$ be a fractal structure whose levels are defined as in Eq. (16). Thus, any finite covering of F by elements of $\mathbf{\Gamma}$ must contain the closed unit interval $[0,1]$. This implies that $\mathscr{K}^s(F) = 1$ for all $s > 0$, and hence, $\{s : \mathscr{K}^s(F) = \infty\} = \{s : \mathscr{K}^s(F) = 0\} = \emptyset$.

Another fractal dimension model described in terms of fractal structures can be sketched in the following terms. Let (X,ρ) be a metric space, F be a subset of X, and $\delta > 0$. Moreover, let us define the next family of coverings of F:

$$\mathscr{G}_\delta(F) = \{\{A_i\}_{i\in I} : A_i \in \cup_{l\in\mathbb{N}} \Gamma_l \text{ for all } i \in I, \text{diam}\,(A_i) \leq \delta, F \subseteq \cup_{i\in I} A_i\},$$

as well as the expression that follows:

$$\mathscr{J}_\delta^s(F) = \inf\left\{\sum_{i\in I} \text{diam}\,(A_i)^s : \{A_i\}_{i\in I} \in \mathscr{G}_\delta(F)\right\}.$$

The asymptotic behavior of $\mathscr{J}_\delta^s(F)$ will be studied via the following expression:

$$\mathscr{J}^s(F) = \lim_{\delta\to 0} \mathscr{J}_\delta^s(F).$$

Let $t \geq 0$. Thus,

$$\sum_{i\in I} \text{diam}\,(A_i)^t \leq \delta^{t-s} \cdot \sum_{i\in I} \text{diam}\,(A_i)^s, \tag{18}$$

where the sums are considered on $\mathscr{G}_\delta(F)$. Taking infima in Eq. (18), we have

$$\mathscr{J}_\delta^t(F) \leq \delta^{t-s} \cdot \mathscr{J}_\delta^s(F).$$

Hence,

$$\mathscr{J}^t(F) \leq \mathscr{J}^s(F) \cdot \lim_{\delta\to 0} \delta^{t-s}.$$

Therefore, if $\mathscr{J}^s(F) < \infty$ and $\delta \to 0$ provided that $t > s$, then it holds that $\mathscr{J}^s(F) = 0$. Accordingly, the critical point where $\mathscr{J}^s(F)$ "jumps" from ∞ to zero throws a Hausdorff type dimension for F, namely,

$$\sup\{s : \mathscr{J}^s(F) = \infty\} = \inf\{s : \mathscr{J}^s(F) = 0\}.$$

The previous models are formalized along the upcoming section.

6.2 Hausdorff type dimensions for fractal structures

The fractal dimension definitions that we shall explore along this section are provided next.

Definition 14. (c.f. [28, Definition 3.2]) Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ), F be a subset of X, and assume that $\delta(F,\Gamma_n) \to 0$. Moreover, consider the following expression:

$$\mathscr{H}_{n,k}^s(F) = \inf\left\{\sum_{i\in I} \text{diam}\,(A_i)^s : \{A_i\}_{i\in I} \in \mathscr{A}_{n,k}(F)\right\},$$

where

$$\mathscr{A}_{n,k}(F) = \begin{cases} \{\{A_i\}_{i\in I} : A_i \in \cup_{l\geq n}\Gamma_l, F \subseteq \cup_{i\in I}A_i, \mathrm{cd}\,(I) < \infty\} & \text{if } k = 4 \\ \{\{A_i\}_{i\in I} : A_i \in \cup_{l\geq n}\Gamma_l, F \subseteq \cup_{i\in I}A_i\} & \text{if } k = 5, \end{cases}$$

and define

$$\mathscr{H}_k^s(F) = \lim_{n\to\infty} \mathscr{H}_{n,k}^s(F)$$

for $k = 4,5$. The fractal dimensions IV and V of F are defined, respectively, as the following critical points:

$$\dim_{\mathbf{\Gamma}}^k(F) = \inf\{s : \mathscr{H}_k^s(F) = 0\} = \sup\{s : \mathscr{H}_k^s(F) = \infty\}, \text{ for } k = 4,5.$$

In Definition 14 as well as in the next one, we shall assume that $\inf\, \emptyset = \infty$. For instance, if $\mathscr{A}_{n,4}(F) = \emptyset$, then $\dim_{\mathbf{\Gamma}}^4(F) = \infty$.

Definition 15. (c.f. [28, Definition 3.3]) Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ), F be a subset of X, $\delta > 0$, and assume that $\delta(F,\Gamma_n) \to 0$. Moreover, consider the following expression:

$$\mathscr{H}_{\delta,6}^s(F) = \inf\left\{\sum_{i\in I} \mathrm{diam}\,(A_i)^s : \{A_i\}_{i\in I} \in \mathscr{A}_{\delta,6}(F)\right\},$$

where

$$\mathscr{A}_{\delta,6}(F) = \{\{A_i\}_{i\in I} : A_i \in \cup_{l\in\mathbb{N}}\Gamma_l \text{ for all } i \in I, \mathrm{diam}\,(A_i) \leq \delta, F \subseteq \cup_{i\in I}A_i\},$$

and define

$$\mathscr{H}_6^s(F) = \lim_{\delta\to 0} \mathscr{H}_{\delta,6}^s(F).$$

The fractal dimension VI of F is given by the following critical point:

$$\dim_{\mathbf{\Gamma}}^6(F) = \inf\{s : \mathscr{H}_6^s(F) = 0\} = \sup\{s : \mathscr{H}_6^s(F) = \infty\}.$$

Equivalently, from both Definitions 14 and 15, it holds that

$$\mathscr{H}_k^s(F) = \begin{cases} \infty \text{ if } s < \dim_{\mathbf{\Gamma}}^k(F) \\ 0 \text{ if } s > \dim_{\mathbf{\Gamma}}^k(F), \end{cases} \tag{19}$$

for $k = 4,5,6$, provided that $\delta(F,\Gamma_n) \to 0$. From Eq. (19), it holds that fractal dimensions IV-VI do behave similarly to both Hausdorff measure and dimension (c.f. Eq. (4)) and fractal dimension III and its corresponding set function (c.f. Eq. (13)).

The next remark becomes especially useful, since it is not required to consider lower/upper limits (unlike it happens with box dimension) for $\mathscr{H}_k^s(F)$ $(k = 4,5,6)$ calculation purposes, which is also the case of both $\mathscr{H}_3^s(F)$ and $\mathscr{H}_H^s(F)$ (c.f. [2, Subsection 2.2]).

Remark 15.
1. Since $\mathscr{H}_{n,k}^s(F) : k = 4,5$ is the general term of a monotonic non-decreasing sequence in $n \in \mathbb{N}$, then the fractal dimensions IV and V of any subset F of X always exist.
2. Since $\mathscr{H}_{\delta,6}^s(F)$ is non-increasing for $s \geq 0$, then $\mathscr{H}_6^s(F)$ also is by definition, so the fractal dimension VI of any subset F of X always exists.

Recall that Hausdorff dimension constitutes the key reference for new definitions of fractal dimension to be mirrored in. In fact, Hausdorff dimension satisfies some desirable properties as a dimension function which can be found out in both Theorem 3 and Remark 4. Similarly to Propositions 5 (for fractal dimension I), Proposition 10 (for fractal dimension II), and Proposition 29 (for fractal dimension III), next we shall explore the behavior of fractal dimensions IV-VI as dimension functions throughout a pair of theoretical results. The first of them collects some analytical properties which stand for fractal dimension IV.

Proposition 37. *(c.f. [28, Proposition 3.4]) Let* $\mathbf{\Gamma}$ *be a fractal structure on a metric space* (X,ρ) *and assume that* $\operatorname{diam}(\Gamma_n) \to 0$. *The following statements hold.*

1. *Fractal dimension IV is monotonic.*
2. *There exist a countable subset* F *of* X *and a fractal structure* $\mathbf{\Gamma}$ *on* X *such that* $\dim_{\mathbf{\Gamma}}^4(F) \neq 0$.
3. *Fractal dimension IV is not countably stable.*
4. *Fractal dimension IV is finitely stable.*
5. $\dim_{\mathbf{\Gamma}}^4(F) = \dim_{\mathbf{\Gamma}}^4(\overline{F})$ *for all subset* F *of* X.

An analogous result to Proposition 37 stands for both fractal dimensions V and VI.

Proposition 38. *(c.f. [28, Proposition 3.4]) Let* $\mathbf{\Gamma}$ *be a fractal structure on a metric space* (X,ρ) *and assume that* $\operatorname{diam}(\Gamma_n) \to 0$. *The following statements hold.*

1. *Both fractal dimension V and fractal dimension VI are monotonic.*
2. *Both fractal dimension V and fractal dimension VI are countably stable.*
3. $\dim_{\mathbf{\Gamma}}^5(F) = \dim_{\mathbf{\Gamma}}^6(F) = 0$ *for all countable subset* F *of* X.
4. *There exists a locally finite starbase fractal structure* $\mathbf{\Gamma}$ *defined on a certain subspace* $F \subseteq X$ *such that* $\dim_{\mathbf{\Gamma}}^k(F) \neq \dim_{\mathbf{\Gamma}}^k(\overline{F})$ *for* $k = 5,6$.

Table 1 summarizes the behavior of all the fractal dimensions explored along this paper throughout the theoretical properties that each of them satisfy as dimension functions. It is worth mentioning that fractal dimensions V and VI behave more similarly to Hausdorff dimension than the other models, whereas fractal dimension I is the most similar definition to box dimension in this sense.

Theoretical properties	$\dim_B$	$\dim_{\mathbf{\Gamma}}^1$	$\dim_{\mathbf{\Gamma}}^2$	$\dim_{\mathbf{\Gamma}}^3$	$\dim_{\mathbf{\Gamma}}^4$	$\dim_{\mathbf{\Gamma}}^5$	$\dim_{\mathbf{\Gamma}}^6$	$\dim_H$
Monotonicity	1	1	1	1	1	1	1	1
Finite stability	1	1	0	1	1	1	1	1
Countable stability	0	0	0	0	0	1	1	1
Countable sets	0	0	0	0	0	1	1	1
Closure dimension	0	1	1	1	0	1	1	1

Table 1 The table above summarizes all the theoretical properties satisfied by each definition of fractal dimension explored throughout this paper. We set 1 to denote that the corresponding property is satisfied by each fractal dimension and 0 otherwise (c.f. [42]).

6.3 Linking fractal dimensions V and VI

In Section 6.2, we verified that fractal dimensions V and VI are quite close to classical Hausdorff dimension, at least from the viewpoint of the theoretical properties (as dimension functions) they satisfy. Following the above, the main goal in this section consists of going beyond so we can explore some conditions on a fractal structure to be able to connect fractal dimensions V and VI.

The first result we prove contains a first link between these fractal dimensions. It is worth mentioning that the only condition required therein concerns the sequence of diameters $\delta(F,\Gamma_n)$. In fact, we shall assume that it ibecomes smaller as deeper levels in the fractal structure are reached.

Lemma 39. *(c.f. [43, Lemma 5.3.6]) Let* $\mathbf{\Gamma}$ *be a fractal structure on a metric space* (X,ρ)*,* F *be a subset of* X*, and assume that* $\delta(F,\Gamma_n)\to 0$*. Then*

$$\dim_{\mathbf{\Gamma}}^{6}(F)\leq \dim_{\mathbf{\Gamma}}^{5}(F).$$

Next, we provide a sufficient condition on the elements in each level of a fractal structure to guarantee the equality between fractal dimensions V and VI.

Definition 16. (c.f. [28, Definition 3.6]) Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ). We shall understand that $\mathbf{\Gamma}$ is a diameter-positive fractal structure provided that the following condition stands:

$$\inf\{\operatorname{diam}(A) : A\in\Gamma_n^{+}\}>0,$$

where

$$\Gamma_n^{+}=\{A\in\Gamma_n : \operatorname{diam}(A)>0\}.$$

It is worth noting that wide families of fractal structures are diameter-positive. For instance, any finite fractal structure is diameter-positive. More specifically, next we provide several examples of wide families of fractal structures that are diameter-positive.

Remark 16. The following families of fractal structures are diameter-positive.

1. Any finite fractal structure.
2. The natural fractal structure for any Euclidean subset (c.f. Definition 4).
3. The natural fractal structure which any IFS-attractor can be always endowed with (c.f. Definition 2).

Under the diameter-positive condition for a fractal structure, it holds that fractal dimensions V and VI coincide.

Theorem 40. *(c.f. [28, Theorem 3.7]) Let* $\mathbf{\Gamma}$ *be a diameter-positive fractal structure on a metric space* (X,ρ)*,* F *be a subset of* X*, and assume that* $\delta(F,\Gamma_n)\to 0$*. Then*

$$\dim_{\mathbf{\Gamma}}^{5}(F)=\dim_{\mathbf{\Gamma}}^{6}(F).$$

6.4 Additional connections among fractal dimensions III, IV & V

Along this subsection, we shall connect fractal dimension V (and hence, fractal dimension VI, due to Lemma 39) with fractal dimension III, previously explored in Section 5. First of all, we state that the fractal dimension V is always $\leq$ than fractal dimension III.

Theorem 41. *(c.f. [43, Theorem 5.3.9]) Let* $\mathbf{\Gamma}$ *be a fractal structure on a metric space* (X,ρ)*,* F *be a subset of* X*, and assume that* $\delta(F,\Gamma_n)\to 0$*. Then*

$$\dim_{\mathbf{\Gamma}}^{6}(F)\leq \dim_{\mathbf{\Gamma}}^{5}(F)\leq \dim_{\mathbf{\Gamma}}^{3}(F).$$

The result provided below gathers several connections among Hausdorff dimension (c.f. Subsection 2.5) and fractal dimensions for fractal structures: II (c.f. Definition 8), III (see Definition 13), IV and V (both of them described in Definition 14), and VI (c.f. Definition 15).

Corollary 42. *(c.f. [28, Proposition 3.5]) Let* $\mathbf{\Gamma}$ *be a fractal structure on a metric space* (X,ρ)*,* F *be a subset of* X*, and assume that* $\delta(F,\Gamma_n)\to 0$*. The two following hold.*

1. $\dim_{\mathrm{H}}(F)\leq \dim_{\mathbf{\Gamma}}^{6}(F)\leq \dim_{\mathbf{\Gamma}}^{5}(F)\leq \dim_{\mathbf{\Gamma}}^{4}(F)$.
2. $\dim_{\mathrm{H}}(F)\leq \dim_{\mathbf{\Gamma}}^{6}(F)\leq \dim_{\mathbf{\Gamma}}^{5}(F)\leq \dim_{\mathbf{\Gamma}}^{3}(F)\leq \underline{\dim}_{\mathbf{\Gamma}}^{2}(F)\leq \overline{\dim}_{\mathbf{\Gamma}}^{2}(F)$.

Interestingly, an additional link between fractal dimensions III and IV can be stated in the context of finite fractal structures.

Theorem 43. *(c.f. [43, Lemma 5.3.11]) Let* $\mathbf{\Gamma}$ *be a finite fractal structure on a metric space* (X,ρ)*,* F *be a subset of* X*, and assume that* $\delta(F,\Gamma_n)\to 0$*. Then*

$$\dim_{\mathbf{\Gamma}}^4(F)\leq \dim_{\mathbf{\Gamma}}^3(F).$$

We shall conclude this subsection by the next result which involves all the fractal dimensions from II to VI.

Corollary 44. *(c.f. [43, Corollary 5.3.12]) Let* $\mathbf{\Gamma}$ *be a finite fractal structure on a metric space* (X,ρ)*,* F *be a subset of* X*, and assume that* $\delta(F,\Gamma_n)\to 0$*. The following chain of inequalities holds:*

$$\dim_{\mathrm{H}}(F)\leq \dim_{\mathbf{\Gamma}}^6(F)\leq \dim_{\mathbf{\Gamma}}^5(F)\leq \dim_{\mathbf{\Gamma}}^4(F)\leq \dim_{\mathbf{\Gamma}}^3(F)\leq \underline{\dim}_{\mathbf{\Gamma}}^2(F)\leq \overline{\dim}_{\mathbf{\Gamma}}^2(F).$$

6.5 Measure properties of $\mathscr{H}_6^s$

Our next goal is to show that the analytical construction regarding fractal dimension VI is also based on a measure. Indeed, unlike the set functions $\mathscr{H}^s$ and $\mathscr{H}_3^s$ that give rise to fractal dimension III, it holds that the set function $\mathscr{H}_6^s$ (which leads to fractal dimension VI) is an outer measure. To deal with, first, let us recall some concepts and results from probability and measure theories that will be useful for our purposes (c.f. [38, Sections 5.2 and 5.4]).

Let (X,ρ) be a metric space. A pair of subsets A,B of X are said to have positive separation provided that $\rho(A,B)>0$, namely, if there exists $r>0$ such that $\rho(x,y)\geq r$ for all $x\in A$ and all $y\in B$. Next, we recall a first approach to tackle with the construction of outer measures called as *Method I*. Let $\mathscr{A}$ be a family of subsets of X which covers it. Moreover, let $\mathbf{c}:\mathscr{A}\longrightarrow[0,\infty]$ be a set function. The following result is the so-called *Method I* (of construction of outer measures).

Theorem 45. *(Method I Theorem, c.f. [38, Theorem 5.2.2]) There exists a unique outer measure* $\overline{\mu}$ *on* X *satisfying the two following conditions:*

1. $\overline{\mu}(A)\leq \mathbf{c}(A)$ *for all* $A\in\mathscr{A}$*.*
2. *If* $\overline{\nu}$ *is any other outer measure on* X *such that* $\overline{\nu}(A)\leq \mathbf{c}(A)$ *for all* $A\in\mathscr{A}$*, then* $\overline{\nu}(B)\leq\overline{\mu}(B)$ *for all* $B\subseteq X$*.*

An outer measure $\mu:\mathscr{P}(X)\longrightarrow[0,\infty]$ is said to be a metric outer measure if $\mu(A\cup B)=\mu(A)+\mu(B)$ for any pair A,B of subsets with positive separation. The restriction of a metric outer measure to the class of its measurable sets is called a metric measure.

Since the Method I provided in Theorem 45 may fail to provide a measure for which the open sets are measurable, the so-called *Method II* will be applied to deal with this problem (c.f. [38, Subsection 5.4]).

Let $\mathscr{A}$ be a family of subsets of a metric space X and assume that for all $x\in X$ and all $\varepsilon>0$, there exists $A\in\mathscr{A}$ such that $x\in A:\mathrm{diam}\,(A)\leq\varepsilon$. Let $\mathbf{c}:\mathscr{A}\longrightarrow[0,\infty]$ be a set function. An outer measure will be constructed as follows. First of all, define $\mathscr{A}_\varepsilon=\{A\in\mathscr{A}:\mathrm{diam}\,(A)\leq\varepsilon\}$. In addition, let $\overline{\mu}_\varepsilon$ be the outer measure provided by Method I throughout the set function $\mathbf{c}$ and the collection $\mathscr{A}_\varepsilon$. For a given subset F of X, notice that the quantity $\overline{\mu}_\varepsilon(F)$ increases as the scale ε decreases. Accordingly, an outer measure is defined by

$$\overline{\mu}(F)=\lim_{\varepsilon\to 0}\overline{\mu}_\varepsilon(F)=\sup_{\varepsilon>0}\overline{\mu}_\varepsilon(F).$$

We shall refer μ to the restriction of $\overline{\mu}$ to the class of its measurable sets. In fact, that construction of an outer measure $\overline{\mu}$ from a set function $\mathbf{c}$ (and hence, a measure μ from $\overline{\mu}$) is called as Method II of construction of (outer) measures.

Theorem 46. *(c.f. [28, Theorem 3.8]) Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ) and assume that* $\operatorname{diam}(\Gamma_n) \to 0$*. Then the set function $\mathscr{H}_6^s : \mathscr{P}(X) \longrightarrow [0,\infty]$ given by $\mathscr{H}_6^s(F) = \lim_{\delta\to 0} \mathscr{H}_{\delta,6}^s(F)$ for all subset F of X is a metric outer measure.*

As an immediate consequence of Theorem 46, we have that $\mathscr{H}_6^s$ is actually a measure, as the following result points out. In this way, it is worth noting that the only (natural) constraint therein concerns the sequence of diameters in each level of the fractal structure.

Corollary 47. *(c.f. [28, Corollary 3.9]) Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ) and assume that* $\operatorname{diam}(\Gamma_n) \to 0$*. The restriction of the set function $\mathscr{H}_6^s : \mathscr{P}(X) \longrightarrow [0,\infty]$ to the class of all the Borel sets of X is a measure.*

6.6 Generalizing Hausdorff dimension

The main goal in this subsection is to explore some connections of fractal dimensions IV, V, and VI with classical Hausdorff dimension. With this aim, next, we provide one of the main results in this paper, where we shall guarantee that fractal dimension V generalizes Hausdorff dimension in the context of Euclidean subsets endowed with their natural fractal structures.

Theorem 48. *(c.f. [28, Theorem 3.10]) Let $\mathbf{\Gamma}$ be the natural fractal structure on $\mathbb{R}^d$ and $F \subseteq \mathbb{R}^d$. Then the fractal dimension V of F equals the Hausdorff dimension of F, namely:*

$$\dim_{\mathrm{H}}(F) = \dim_{\mathbf{\Gamma}}^5(F).$$

As a consequence of Theorem 48, we can state that fractal dimension VI also equals the Hausdorff dimension of Euclidean subsets endowed with their natural fractal structures.

Corollary 49. *(c.f. [28, Corollary 3.11]) Let $\mathbf{\Gamma}$ be the natural fractal structure on $\mathbb{R}^d$ and $F \subseteq \mathbb{R}^d$. Then both the fractal dimensions V and VI of F equal the Hausdorff dimension of F, namely,*

$$\dim_{\mathrm{H}}(F) = \dim_{\mathbf{\Gamma}}^6(F) = \dim_{\mathbf{\Gamma}}^5(F).$$

Interestingly, fractal dimension IV also generalizes Hausdorff dimension in the context of compact Euclidean subsets as the following results highlights.

Theorem 50. *(c.f. [28, Theorem 3.12]) Let $\mathbf{\Gamma}$ be the natural fractal structure on $\mathbb{R}^d$ and F be a compact subset of $\mathbb{R}^d$. Then the fractal dimensions IV, V, and VI of F equal the Hausdorff dimension of F, namely,*

$$\dim_{\mathrm{H}}(F) = \dim_{\mathbf{\Gamma}}^6(F) = \dim_{\mathbf{\Gamma}}^5(F) = \dim_{\mathbf{\Gamma}}^4(F).$$

In summary, fractal dimension V generalizes Hausdorff dimension for Euclidean subsets endowed with their natural fractal structures (c.f. Theorem 48). In addition, fractal dimension IV throws an upper bound to Hausdorff dimension in the same context as a consequence of Corollary 42 (1). The equality between Hausdorff dimension and fractal dimension IV has been reached for compact Euclidean subsets (c.f. Theorem 50). Going beyond, it becomes possible to weaken the hypothesis regarding the compactness of F to prove a further connection between fractal dimension IV and Hausdorff dimension. We shall deal with this issue along the forthcoming result.

Theorem 51. *(c.f. [28, Theorem 3.13]) Let $\mathbf{\Gamma}$ be the natural fractal structure on $\mathbb{R}^d$ and F be a bounded subset of $\mathbb{R}^d$. Then the fractal dimension IV of F equals the Hausdorff dimension of the closure of F, namely,*

$$\dim_{\mathbf{\Gamma}}^4(F) = \dim_{\mathrm{H}}(\overline{F}).$$

Interestingly, all the fractal dimension models we have explored along this paper can be theoretically connected among them in the context of Euclidean subsets endowed with their natural fractal structures.

Corollary 52. *(c.f. [28, Corollary 3.14]) Let* $\mathbf{\Gamma}$ *be the natural fractal structure on* $\mathbb{R}^d$ *and* F *be a subset of* $\mathbb{R}^d$*. Moreover, assume that there exists one of the following dimensions:* $\dim_B(F)$*,* $\dim_{\mathbf{\Gamma}}^1(F)$*, or* $\dim_{\mathbf{\Gamma}}^2(F)$*. The two following hold.*

1.
$$\dim_H(F) = \dim_{\mathbf{\Gamma}}^6(F) = \dim_{\mathbf{\Gamma}}^5(F) \leq \dim_{\mathbf{\Gamma}}^3(F) = \dim_{\mathbf{\Gamma}}^2(F) = \dim_{\mathbf{\Gamma}}^1(F) = \dim_B(F).$$

2. *If* F *is compact, then*
$$\dim_H(F) = \dim_{\mathbf{\Gamma}}^6(F) = \dim_{\mathbf{\Gamma}}^5(F) = \dim_{\mathbf{\Gamma}}^4(F) \leq \dim_{\mathbf{\Gamma}}^3(F) = \dim_{\mathbf{\Gamma}}^2(F) = \dim_{\mathbf{\Gamma}}^1(F) = \dim_B(F).$$

In particular, the fractal dimension IV model introduced in Definition 14 results especially appropriate from a theoretical viewpoint, since it becomes a middle dimension between classical fractal dimension definitions, namely, both box dimension and Hausdorff dimension, as we shall highlight along the next remark.

Counterexample 53. *(c.f. [28, Remark 3.15])*

1. *There exist a fractal structure* $\mathbf{\Gamma}$ *on* X *and a subset* F *of* X *such that* $\dim_{\mathbf{\Gamma}}^4(F) < \dim_B(F)$.
2. *There exist a fractal structure* $\mathbf{\Gamma}$ *on* X *and a subset* F *of* X *such that* $\dim_H(F) < \dim_{\mathbf{\Gamma}}^4(F)$.

To conclude this section, we would like to point out that fractal dimension IV can also be applied for computational purposes. In other words, it becomes possible to computationally approach the fractal dimension IV of a compact subset, which, by Theorem 51 equals its Hausdorff dimension. Therefore, this fractal dimension will lead us to computationally deal with the calculation of the Hausdorff dimension of compact Euclidean subsets.

Next, we provide a preliminary example about how to computationally calculate the Hausdorff dimension of the middle third Cantor set.

Example 54. *(c.f. [28, Example 1])* Let $\mathbf{\Gamma}$ be the natural fractal structure on $[0,1]$. Along this example, we shall consider only three levels of $\mathbf{\Gamma}$ to approach the fractal dimension IV of the middle third Cantor set $\mathscr{C}$. Firstly, let us denote
$$A_0 = \left[0, \frac{1}{2}\right],\ A_1 = \left[\frac{1}{2}, 1\right],\ A_{ij} = \left[\frac{2i+j}{4}, \frac{2i+j+1}{4}\right] : i,j \in \{0,1\},$$
and so on. Thus, we have $\Gamma_1 = \{A_0, A_1\}$, $\Gamma_2 = \{A_{00}, A_{01}, A_{10}, A_{11}\}, \ldots$, etc. On the other hand, we have carried out a suitable discretization of $\mathscr{C}$ taking the 2048 extremes of the intervals that appear in step 10 of its standard construction.
In this example, we shall apply the following algorithm for fractal dimension IV calculation purposes: given $s \geq 0$, let $\mathscr{H}_4^s(\mathscr{C}) = \infty$ if the infimum (a minimum in this case) of all the coverings by elements in each level $\Gamma_i : i = 1,2,3$ to calculate $\mathscr{H}_{1,4}^s(\mathscr{C})$ stands throughout a covering involving some element of Γ_1. Otherwise, we set $\mathscr{H}_4^s(\mathscr{C}) = 0$. For $s = 0.69$, we have found out that the minimum is reached by the covering $\{A_0, A_1\}$, whereas for $s = 0.7$, the minimum holds through the covering $\{A_{00}, A_{010}, A_{101}, A_{11}\}$. Hence, an approach of $\dim_{\mathbf{\Gamma}}^4(\mathscr{C})$ using only three levels of $\mathbf{\Gamma}$ lies between 0.69 and 0.7. If we use five levels of $\mathbf{\Gamma}$, then for $s = 0.63$, the minimum is obtained by the covering $\{A_0, A_1\}$ and for $s = 0.64$, the minimum is reached through the covering $\{A_{000}, A_{00111}, A_{010}, A_{101}, A_{11000}, A_{111}\}$. Accordingly, the estimation of $\dim_{\mathbf{\Gamma}}^4(\mathscr{C})$ (using only five levels of $\mathbf{\Gamma}$) lies between 0.63 and 0.64. It should be mentioned here that the real value of $\dim_{\mathbf{\Gamma}}^4(\mathscr{C})$ is $\frac{\log 2}{\log 3} \simeq 0.631$.

Example 54 has been provided to illustrate how to apply fractal dimension IV to deal with the effective calculation of the Hausdorff dimension of a compact Euclidean subset. It is worth mentioning that in [41], it was provided the first-known overall algorithm to calculate the Hausdorff dimension.

Acknowledgements

The author would like to express his gratitude to Prof. M.A. Sánchez-Granero for all these years of non-stop effort and joint top-quality research.

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