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Multigrid method for the solution of EHL line contact with bio-based oils as lubricants

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Abstract

The paper presents Elastohydrodynamic lubrication line contact problem with bio-based oil as lubricants for an isothermal case. The fast convergence method for the solution of Elastohydrodynamic lubrication line contact problem with seed oil as lubricant is analyzed using Multigrid, Multilevel Multi-Integration with the influence of different load and speed. The result shows that the use of these oils has the potential to substitute the function of common lubricant so as to reduce dependence on conventional oil lubricants. The results obtained are comparable and the pressure spikes are smooth as compared to the earlier findings which are shown in terms of graphs and tables.

Keywords: Elastohydrodynamic lubrication; Line contact; Multigrid, Multilevel technique; Bio-based oil.
AMS 2010 codes: 35Q79, 92F05.

1 Introduction

The Elastohydrodynamic lubrication (EHL) is one of the important topics in Tribology, EHL problems have been studied by many researchers over the last three decades both theoretically and experimentally. EHL is commonly found in many machines components such as gear teeth contacts, cams and rolling element bearings during variations in load etc. There are generally two types of problems in EHL, line contact and point contact. In line contact EHL problem, the contact region of the elements are assumed to be infinitely long in one direction, that takes place as an infinitely long strips. In point contact EHL problem, the contact takes place within a finite elliptical region [1]. When two metals are in contact, the amount of asperities and interaction within the contact area increases, causing frictions which insist wear of the metal surfaces and generation of excessive heat. These

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friction, wear and excessive heat caused by the interaction between the surfaces are controlled by the inclusion of lubricant whose function is to reduce friction and wear to prevent oxidation and corrosion, acting as a coolant, facilitating heat dissipation from the engine.

The Bio-based oils are usually the extracts of various plants, animals, seeds, fruits etc. Bio based product plays an important role in lubrication [2–5]. Certain physical and chemical property of these oils enables it to be a suitable lubricant due to their liquidity at room temperature. In spite of these advantages it has several drawbacks for their use as lubricant. Chemical modification can overcome these drawbacks by adding additives in these oils. Among these, vegetable oils consists of long chain molecule structure which interacts strongly with metal surface reducing friction and wear, due to the availability of their small range of viscosities. These oils have been discussed by many researchers; Quinchia et al. [6] have successfully modified the viscosity of vegetable oils by using Ethylene-Vinyl Acetate copolymer (EVA) as additive material. There is a strong concern of awareness in biodegradable low-toxicity lubricants and biodegradability is obtained by using suitable biodegradable base fluid, by adding additives that is environmentally friendly [7]. There is a need to source out lubricants other than the computational ones obtained from biobased oils [8, 9], as an increasing world energy crisis.

Multigrid, Multilevel Multi-Integration (MLMI) with Full Approximation Scheme (FAS) are been discussed in [10, 11]. Multigrid technique has received immense attention from many investigators in EHL and is proved to be very efficient for solving EHL problems. Venner [12] examined FAS for the Reynolds equation, film thickness equation and the force balance equation by employing multigrid technique for the solution of EHL problems. Lubrecht et al. [13] presented its application in EHL and employed the reduced pressure in deriving the Reynolds equation. Brandt and Lubrecht [14] developed a MLMI algorithm for the fast calculation of elastic deformation. In the present work only the Gauss-Seidel relaxation method is adopted. The advantage of multigrid methods incorporate naturally and overcome the drawbacks of Newton-Raphson scheme, it enables to analyze EHL problems for higher loads required in non-conformal conjunctions such as rolling element bearings, gears etc., employing the Gauss-Seidel relaxation with small under-relaxation factors which illustrates that the Gauss-Seidal relaxation is a stable relaxation scheme with good smoothing properties. EHL problems have also been solved by many other different methods as in [15, 16], etc. Although many mathematical models of ordinary differential equations and partial differential equations [17, 18], etc., have been solved for its exact solution.

In this paper, we employ FAS for the solution of EHL line contact problem. This paper is organized as follows. Section 1 describes the introduction. Section 2 describes governing equation of the physical problem. Discretization of the governing equation is given in Section 3. In Section 4, Multigrid (FAS) method is presented. In Section 5, solution procedure is given in detail. Section 6 contains the results and discussions. Finally, the conclusion of the proposed work is given in Section 7.

2 Governing Equations

Let us consider the flow is laminar and the compressibility of the fluid is negligible. The isothermal EHL line contact problem models the lubricant flow contact between outer ring and cylindrical roller of roller element bearing under an applied load which consists of three equations: the Reynolds equation, the film thickness equation and the force balance equation. All these equations are presented in a dimensionless form as [19]:

$$\frac{\partial}{\partial X}\left(\varepsilon\frac{\partial P}{\partial X}\right)-\frac{\partial}{\partial X}(\overline{\rho}H), \tag{1}$$

where $\varepsilon=\frac{\overline{\rho}H^3}{\overline{\eta}K}$ and $K=\frac{3}{4}\left(\frac{\pi}{W}\right)^2 U$.

The non-dimensional parameters for magnitude analysis of Reynolds equation are

$$W=\frac{w}{E'R},\quad H=\frac{hR}{b^2},\quad U=\frac{\eta_0 u}{E'R},\quad G=\alpha E',\quad X=\frac{x}{b},\quad \overline{\eta}=\frac{\eta}{\eta_0},\quad \overline{\rho}=\frac{\rho}{\rho_0},\quad P=\frac{p}{p_H}.$$

The film thickness equation at any point on the contacted surface is [20]:

$$H(X) = H_0 + \frac{X^2}{2} - \overline{D},$$

where H_0 is the central offset film thickness, $\frac{X^2}{2}$ defines the un-deformed contact shape, and

$$\overline{D} = -\frac{1}{2\pi}\int_{X_{in}}^{X_{out}} \frac{dP}{dX'}(X-X')[\log(X-X')^2]dX' - \frac{1}{4}\left(R^2\frac{8W}{\pi}\right)$$

is the elastic deformation of the contacting surfaces. The equation for applied load in dimensionless form can be expressed as

$$\int_{X_{in}}^{X_{out}} Pdx = \frac{\pi}{2}.$$

The non-dimensional viscosity-pressure relation is given by [21] and pressure viscosity index as mentioned in [22]:

$$\overline{\eta} = exp\left\{[\log(\eta_0)+9.67]\left[-1+\left(1+\frac{p_H P}{p_0}\right)^z\right]\right\}$$
$$\alpha = z[5.1\times 10^-9\,(\log\eta_0 + 9.67)],$$

where η_0 is the absolute viscosity, p_0 is the ambient pressure and z is a constant characteristic of the liquid (pressure-viscosity index) given by

$$z = [7.81(H_{40} - H_{100}]^{1.5}F_{40},$$

where

$$F_{40} = (0.885 - 0.864), \quad H_{40} = \log(\log(\eta_{40}) + 1.200), \quad H_{100} = \log(\log(\eta_{100}) + 1.200).$$

The non-dimensional density-pressure relation is given by [1]:

$$\overline{\rho} = \left(1 + \frac{0.6p}{1+1.7p}\right).$$

The corresponding boundary conditions are

Inlet boundary condition at: $X = X_{in}, P = 0$.

Outlet boundary condition at: $X = X_{out}, P = \frac{\partial P}{\partial X} = 0$.

3 Discretization of the equations

Eq. (1) is discretized using second order finite differences with the grid size of $N = 65$ and the domain of interest initially is $[X_{in}, Xout] = [-4, 2.2]$. The discretized form of Reynolds equation is

$$\frac{\varepsilon_{i-\frac{1}{2}}P_{i-1} - (\varepsilon_{i-\frac{1}{2}} + \varepsilon_{1+\frac{1}{2}})P_i + \varepsilon_{i+\frac{1}{2}}P_{i+1}}{\Delta X^2} = \frac{\overline{\rho}_i H_i - \overline{\rho}_i H_{i-1}}{\Delta X},$$

where $\Delta X^2 = X_i - X_{i-1}$. The film thickness equation in discretized form is [16]:

$$H_i = H_0 + \frac{x_i^2}{2} - \frac{1}{\pi}\sum_{i=1}^{N} D_{i,j}P_j$$

where

$$D_{i,j} = -\left(i-j+\frac{1}{2}\right)\Delta X\left[\log\left(|i-j+\frac{1}{2}|\Delta X\right)-1\right]+\left(i-j+\frac{1}{2}\right)\Delta X\left[\log\left(|i-j-\frac{1}{2}|\Delta X\right)-1\right]$$

for $i = 0,1,2,\ldots,n$, $j = 0,1,2,\ldots,n$ and the force balance equation in discrete form is

$$\Delta X\sum_{i=1}^{\infty}\frac{P_i+P_{i+1}}{2}-\frac{\pi}{2}=0.$$

The boundary condition in the inlet $P(X_{in}) = 0$ and the boundary in the outlet can be obtained by setting the negative pressure equal to zero. The discrete form is

$$P(X_{out}) = 0 \quad \text{and} \quad \frac{P(X_{out})-P(X_{out}-1)}{\Delta X}=0.$$

The convergence criteria for pressure relation is:

$$\frac{\sum|P_i^{k+1}-P_i^k|}{\sum P_i^{k+1}} \le \varepsilon_1,$$

where ε_1 is the given error.

Table 1 Physical parameters of roller bearing.

Material (roller and disc)	Steel AISI 52100
Elastic modulus (roller and surface)	$E_r = E_d = 210\ GPa$
Poison's ratio	$\nu_r = \nu_d = 0.3$
Radius of roller	$R = 0.0114$

Table 2 Physical properties of tested oil at 40°C.

Oil name	Dynamic viscosity (Pa.s) at 40°C	α (m^2/N)	G
Sunflower oil	0.029	1.3008E-08	3002
Soyabean oil	0.031	1.3229E-08	3053
Castor oil	0.23	2.3283E-08	5373

4 Multigrid Method

The problem of solving the system of linear equations

$$Au = f,$$

where A is a square matrix and f is a column matrix, arising from discretization of a differential equation on grid Ω^h, where h is the step size. I_h^{2h} interpolation operator, the superscript denotes to the finer grid and the subscripts denotes to the coarser grid, and I_{2h}^h restriction operator. We define a multigrid method recursively, for the two-level V-cycle method, first relax a few steps on the finer grid Ω^h to get u^h, then compute the residual $r^h = f^h - A^h u^h$; restrict the residual to the coarser grid Ω^{2h}, $r^{2h} = I_h^{2h} r^h$; and solve the error equation $A^{2h}e^{2h} = r^{2h}$ on the coarser grid. Finally, set $u^h = u^h + I_h^{2h}e^{2h}$ and again relax a few steps on the finer grid. Based on this two level method, the V-cycle multigrid scheme is defined recursively, which are given in [12, 23, 24].

Algorithm (FAS). *The system of nonlinear equations can be written as*

$$L^h(u^h) = f^h,$$

where L is a non linear operator, u is the exact solution, f is a right hand side function and h is the mesh size of the uniform finer grid. Let v be an approximation to the exact solution u. Then

$$\begin{aligned} e^h &= u^h - v^h \quad \textit{(error)} \\ r^h &= f^h - L(v^h) \quad \textit{(residual)}. \end{aligned}$$

For simplicity, we shall consider a two grid correction scheme, as follows:

1. *Applying few number of nonlinear Gauss-Seidal iteration on fine grid with initial v^h.*
2. *Computation of the restricted residual and restricted current approximation obtained from step 1(finer to coarser) $r^{2h} = I_h^{2h}(f^h - L^h(v^h))$ and $v^{2h} = I_h^{2h} v^h$.*
3. *Solve the coarser grid problem $L^{2h}(u^2h) = L^{2h}(v^2h) + r^{2h}$.*
4. *Compute the error approximation of the coarser grid $e^{2h} = u^{2h} - v^{2h}$.*
5. *Interpolation of error approximation and correction (coarser to finer) $v^h \leftarrow v^h + I_h^{2h} e^{2h}$.*
6. *Applying few number of Gauss-Seidal iteration on finer grid.*

For the calculation of deformation surfaces, we employ MLMI. MLMI is a multiscale technique designed to speed up the evaluation of integrals. Apart from its use in EHL problems, this method has many applications in integral differentiation equation, integral equation, elasticity and acoustic problems, etc. The general formulation to solve the following integral is given by

$$v(x) = \int_{\Omega} K(x,y)u(y)dy, \tag{2}$$

where u is a given function and K is the kernel on a domain $\Omega = (a,b)$. The discrete form of the Eq. (2) is descretized on a regular mesh of N_x^h points, $\Delta x^h = \frac{(b-a)}{N_x^h - 1}$, where h is the grid level. The single grid routine for calculating multi summation at each node $x_i^h = a + (i-1)\Delta x^h$ for $i = 1, \ldots, N_x$ is

$$v_i^h = v^h(x_i^h) = \sum_{j=1}^{N_x^h} K_{i,j}^{hh} u_j^h,$$

where $K_{i,j}^{hh}$ is the discretized kernel and u_j^h is the approximation to $u^h(x_i)$. The discrete form of Eq. (2) represents a full matrix-vector multiplication having certain smoothness property, wherein the kernel is dense rather than a sparse matrix, the method is possible only when the kernel itself has sufficient smoothing properties. The discretization of the elastic deformation integrals for both EHL line and point contact is discussed by Venner and Luberchet [11].

5 Solution method

The process of multigrid includes the pressure correction and load balance by adjusting the rigid film thickness, which are carried out on the same grid. At any instant, the iteration procedure consists of the pressure iteration and the adjustment for H_0. The pressure must be relaxed on each grid level. If we denote $(LP)_i$ by

$L_i(P)$ and ignoring superscripts which represents the grid level, the algebraic equation for pressure on each level can be expressed as $L_i(P) = F_i(i = 1,2,\ldots,n-1)$. For an under-relaxation factor, the process for the pressure iteration can be described as:

$$\widetilde{P}_i = \overline{P}_i + c_i\delta_i,$$

where

$$\delta_i = (\partial L_i/\partial P_i)^{-1} r_i,$$

$$r_i = F_i[\varepsilon_{i-1/2}\widetilde{P}_{i-1} - (\varepsilon_{i-1/2}+\varepsilon_{i+1/2})\overline{P}_i + \varepsilon_{i+1/2}\overline{P}_{i+1}]/(\Delta X)^2 + (\overline{\rho}_i\overline{H}_i - \overline{\rho}_{i-1}\overline{H}_{i-1})/(\Delta X),$$

$$\partial L_i/\partial P_i = (\varepsilon_{i-1/2}+\varepsilon_{i+1/2})/(\Delta X)^2 + \tfrac{1}{\pi}(\overline{\rho}_i\overline{K}_{i,j} - \overline{\rho}_{i-1}\overline{K}_{i-1,j})/\Delta X,$$

and P_i, P_{i+1} are initial values of pressure in an iterative process and $\widetilde{P}_{i-1}$ is the new value of pressure obtained in the iteration. The load balance condition can be achieved by modifying the rigid film thickness $\overline{H}_0$ as follows.

$$\overline{H}_0 = \widetilde{H}_0 + c_2\left[G^{\Delta} - \frac{\Delta}{\pi}\sum_{j=1}^{N-1}(P_j + P_{j+1})\right],$$

where c_2 is the relaxation factor and G^{Δ} is the nondimensional load on the coarsest grid.

Following procedure is followed for the determination of pressure cavitation point [25]. Let X_c be a point in (X_{in}, X_{out}). If pressure gradient $\frac{dP}{dX}|_{X_c} = 0$, then X_c is the required pressure cavitation point. Otherwise, shift the point X_c on next interval to the left or right depending on $\frac{dP}{dX}|_{X_c} > 0$ or $\frac{dP}{dX}|X_c < 0$ respectively. Next approximate another cavitation point X_c' and solve EHL problem. If pressure gradient $\frac{dP}{dX}|_{X_c'}$ is zero, we get converged solution, otherwise verify the condition

$$\frac{dP}{dX}|_{X_c} \times \frac{dP}{dX}|_{X_c'} < 0. \tag{3}$$

If the condition Eq. (3) is not satisfied, repeat above process and find two approximate values X_c' and X_c'' for which Eq. (3) is satisfied. A new cavitation point X_{new} is obtained by interpolating between the two points X_c' and X_c'', which ensures pressure gradient nearly equal to zero at X_{new} and results into the required smooth solution of EHL problem.

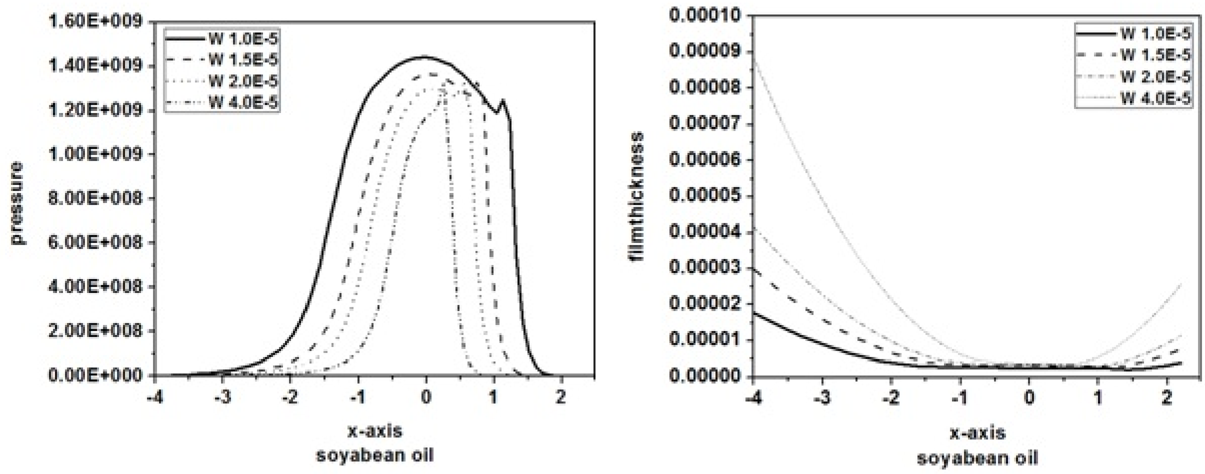

Fig. 1 Effects of load variation for Soyabean oil with $U = 1.0 \times 10^{-11}$.

W	U	Sunflower oil		Castor oil	
		Hmin	Pmax	Hmin	Pmax
1.0×10^{-5}	1.0×10^{-11}	$1.44093E+09$	$7.1574E-02$	$1.06166E+09$	$6.7714E-02$
1.5×10^{-5}	1.0×10^{-11}	$1.36376E+09$	$4.4856E-02$	$1.52957E+09$	$6.1424E-02$
2.0×10^{-5}	1.0×10^{-11}	$1.34270E+09$	$3.2407E-02$	$1.63780E+09$	$4.4377E-02$
4.0×10^{-5}	1.0×10^{-11}	$1.31519E+09$	$1.4807E-02$	$1.67202E+09$	$2.0276E-02$
2.045×10^{-5}	0.5×10^{-11}	$1.10563E+09$	$1.9451E-02$	$1.28860E+09$	$2.6636E-02$
2.045×10^{-5}	1.0×10^{-11}	$1.40761E+09$	$3.1599E-02$	$1.61174E+09$	$4.3270E-02$
2.045×10^{-5}	2.0×10^{-11}	$1.66680E+09$	$5.1333E-02$	$1.83806E+09$	$7.0293E-02$
2.045×10^{-5}	4.0×10^{-11}	$2.06199E+09$	$8.3390E-02$	$2.36530E+09$	0.11419154

Table 3 Effect of dimensional load and speed on dimensional pressure and filmthickness.

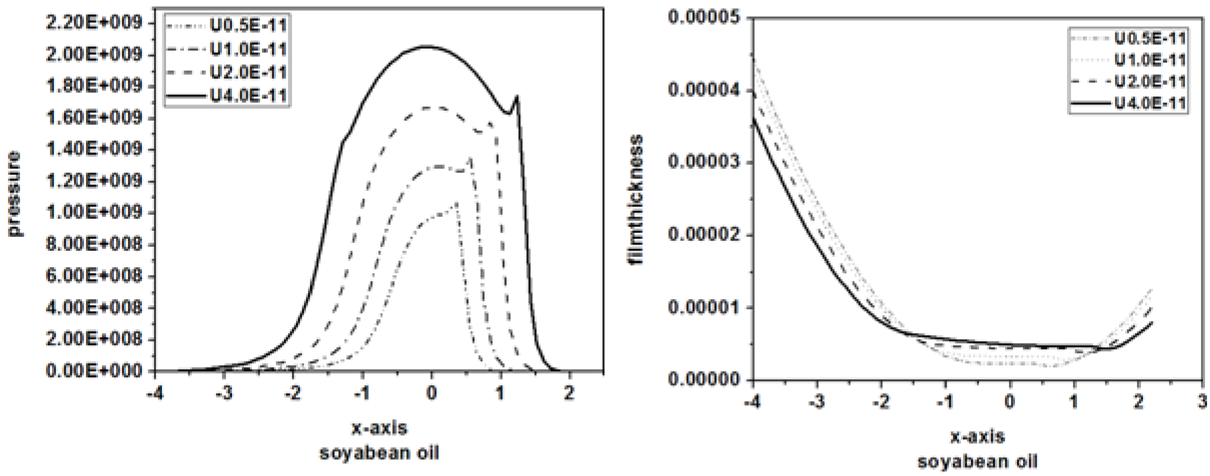

Fig. 2 Effects of speed variation for Soyabean oil with $W = 2.0452\times10^{-5}$.

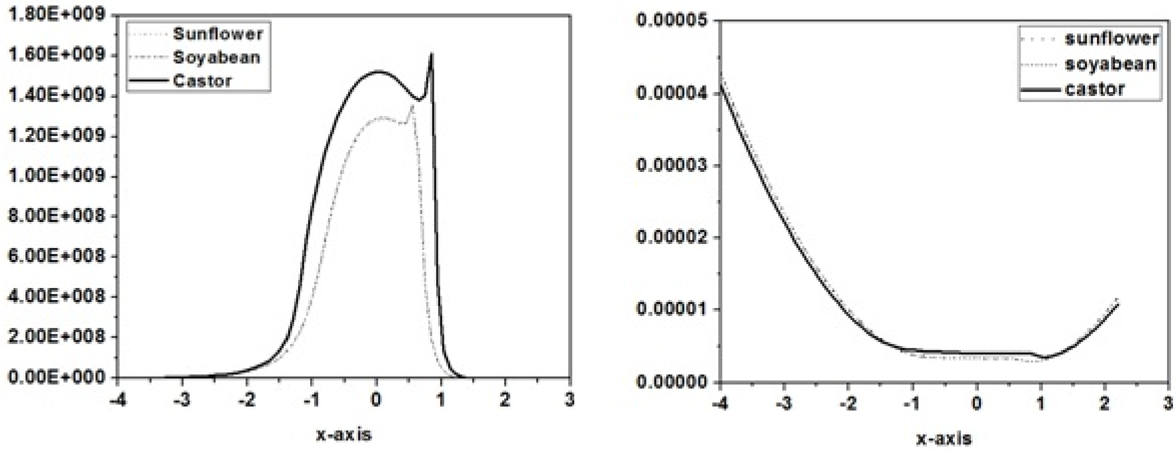

Fig. 3 EHL pressure and film thickness profile for all tested oils.$U = 1.0\times10^{-11}$ and $W = 2.0452\times10^{-5}$.

W	U	Soyabean oil		Dedi [19]	
		Hmin	Pmax	Hmin	Pmax
1.0×10^{-5}	1.0×10^{-11}	$1.44093E+09$	$7.1574E-02$	$1.06166E+09$	$6.7714E-02$
1.5×10^{-5}	1.0×10^{-11}	$1.36295E+09$	$4.5266E-02$	$1.03413E+09$	$4.3496E-02$
2.0×10^{-5}	1.0×10^{-11}	$1.33590E+09$	$3.2703E-02$	$0.97428E+09$	$3.1680E-02$
4.0×10^{-5}	1.0×10^{-11}	$1.32177E+09$	$1.4942E-02$	$0.78821E+09$	$1.4662E-02$
2.045×10^{-5}	0.5×10^{-11}	$1.06785E+09$	$1.9629E-02$	$0.85334E+09$	$1.9272E-02$
2.045×10^{-5}	1.0×10^{-11}	$1.35488E+09$	$3.1888E-02$	$0.97086E+09$	$3.0910E-02$
2.045×10^{-5}	2.0×10^{-11}	$1.67199E+09$	$5.1802E-02$	$1.09415E+09$	$4.9282E-02$
2.045×10^{-5}	4.0×10^{-11}	$2.05771E+09$	$8.4152E-02$	$1.20721E+09$	$7.7793E-02$

Table 4 Effect of dimensional load and speed on dimensional pressure and filmthickness.

6 Results and Discussion

The EHL line contact problem is studied with bio-based oil as lubricant, for an isothermal case with the influence of different loads and speed. The problem has been calculated to investigate the substantial variations in various parameters including its calculated physical properties as shown in Tables 3 and 4. Multigrid, multilevel techniques with FAS scheme is employed. The cavitation condition in the present analysis ranges from $X = -4$ to 2.2 with grid size of $N = 65$. The present computations are confined to the numerical parameters $c_1 = 0.037$ and $c_2 = 0.037$ used for the under-relaxation of the iterations P and H_0. The result of the simulation for the lubrication is as shown in Figs. 1-3 and are carried out for a different value of dimensionless speed (U) and load parameter (W) as specified in both Tables 3 and 4. Fig. 1 shows the effects of varying load for Sunflower, Soyabean, and Castor oils with constant speed $(U = 1.0 \times 10^{-11})$, where the load decreases as the value of pressure spike and film thickness increases and almost similar to hertzian pressure, film thickness. Fig. 2 shows the effects of varying speed at constant load $(W = 2.045 \times 10^{-5})$ as the speed increases the value of pressure spike and film thickness decreases. Fig. 3 shows the pressure and film thickness profiles of both Sunflower and Soyabean oils. They are almost the same due to their low temperature behavior as reported by Dow T. A. Dow et al. [26].

7 Conclusions

In this paper, the EHL line contact problem with seed oils as lubricant is analyzed by using multigrid, multilevel techniques in order to clarify the best performance of lubricants. The characteristics of EHL under operating condition shows that, at constant temperature 40°C, sunflower, soyabean and castor oil are able to replace other lubricants, as they have good pressure and film profile during simulation.

Nomenclature

α Pressure viscosity index (m^2/N).

η Viscosity of lubricant.

η_0 Viscosity at ambient pressure.

ν Poisson ratio.

$\overline{\eta}$ Dimensionless viscosity.

$\overline{\rho}$ Dimensionless density.

ρ Density of lubricant.

ρ_0 Density at ambient pressure.

b Half width of the Hertzian contact zone ($b = 4R\sqrt{W/2\pi}, (m)$).

E' Effective elastic modulus of roller and disc $\left(E' = \frac{1}{2}\left(\frac{1-v_r^2}{E_r} + \frac{1-v_r^2}{E_d}\right)\right)$.

G Non-dimensional material parameter.

H Dimensionless film thickness (m).

h Film thickness (m).

p Pressure (Pa).

p_0 Ambient pressure.

p_H Maximum Hertzian pressure ($p_H = E'b/4R$).

R Equivalent radius of the contact (m).

U Dimensionless speed parameter ($U = \frac{\eta_0 u_s}{E'R}$).

u_s Average entrainment speed.

W Dimensionless load parameter ($W = \frac{w}{E'R}$).

w Applied load per unit length (N/m).

x_0 Inlet co-ordinate.

x_e Outlet co-ordinate.

z Pressure viscosity parameter.

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